APPROXIMATION ALGORITHMS
for
SPREADING POINTS

SERGIO CABELLO
INFN, LJUBLJANA, SLOVENIA

research done at Utrecht University
THE ABSTRACT PROBLEM

- \((X, d)\) metric space
  
  \(S_1, \ldots, S_n \subseteq X\)

- \(t\)-distant representatives: \([\text{Fiala et al. '02}]

\[
\begin{align*}
X_1, \ldots, X_n \\
S_1, \ldots, S_n \\
\end{align*}
\]

\(\text{s.t. } d(x_i, x_j) \geq t \quad \forall i \neq j\)

- Optimization problem:
  
  maximize \(t\) that admits \(t\)-representatives
OUR PROBLEMS

- \((\mathbb{R}^2, L_\infty)\)
  \(S_1, \ldots, S_n\) disks (squares in \(L_\infty\))
  Choose \(p_i \in S_i\) maximizing the distance of the closest pair.

- \((\mathbb{R}^2, L_2)\)
  \(S_1, \ldots, S_n\) (congruent) disks
  Choose \(p_i \in S_i\) maximizing the distance of c.p.
PREVIOUS/RELATED WORK

- Fiala et. al mo
- Baur & Fekete mo \implies NP-hard to get PTAS

- Baur & Fekete: Choose K points inside a polygonal region maximizing the distance of c.p. NP-hard to get a PTAS. \( \frac{3}{2} \)-approximation in \( L_\infty \)

- Packing problems - Map labelling problems.
## Our Results

<table>
<thead>
<tr>
<th>Space</th>
<th>Regions</th>
<th>Result</th>
</tr>
</thead>
<tbody>
<tr>
<td>((\mathbb{R}^2, L_\infty))</td>
<td>disks</td>
<td>2-approximation, (O(n \sqrt{n} \log^2 n)) *</td>
</tr>
<tr>
<td>((\mathbb{R}^2, L_2))</td>
<td>congruent disks</td>
<td>(\lceil \frac{8}{3} \rceil)-approximation, (O(n^2))</td>
</tr>
</tbody>
</table>

* today's aim
OUTLINE

- The abstract problem / Our problem
- Previous / Related work
- Our results
  - Approximate placement algorithm
  - Efficiency of approximate placement
  - Decision no Optimization

\[ L_\infty \times 2\text{-approximation} \]
PLACEMENT ALGORITHM (t)

Idea: Try to place points at $t \mathbb{Z}^2$.

$\mathbb{Z}^2 \setminus \text{waste}$

Disks $5$

$G(t)$

Matching

\begin{align*}
(4, 2) & \rightarrow (4, 3) \\
(4, 3) & \rightarrow (4, 2) \\
(4, 4) & \rightarrow (4, 4) \\
(2, 4) & \rightarrow (4, 4) \\
(2, 5) & \rightarrow (2, 5)
\end{align*}
Placement Algorithm (t)

- \( \text{WASTE} := \emptyset \);
- \( \text{if } S_e \cap n(t, Z^2) = \emptyset, \text{ two cases} \)

\[
G(t) = \left( \{ S_1, \ldots, S_n \} \cup (t, Z^2 \setminus \text{WASTE}), \{ (p, S_i) \mid p \in S_i \} \right)
\]

- matching in \( G(t) \)

\( \rightarrow \) Placement.
Placement Algorithm \((t)\)

Let \(t^*\) be the optimal solution.

\underline{Lemma}: If \(2t \leq t^*\) then:

- \(G(t)\) has a matching so we get placement;
- c.p. of placement is \(t\) apart.

We may have a huge graph \(G(t)\).\[\checkmark\]

\(\rightarrow\) Modify algorithm
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\[ L_\infty \quad 2\text{-approximation} \]
EFFICIENCY Placement (t)

\[ \text{G}(t) \text{ looks like } \begin{array}{c}
\square \\
\vdots \\
\square \end{array} \begin{array}{c}
t \cdot \mathbb{Z}^2 \\
\vdots \\
\square \end{array} \begin{array}{c}
n \text{ disks} \\
\vdots \\
\square \end{array} \begin{array}{c}
\text{Many points} \\
\vdots \\
\square \end{array} \]

\(\uparrow\) Each disk needs degree \(\leq n\)
\(\uparrow\) For each disk, take \(5n\) points \((|\text{WASTE}| \leq 4n)\)
\(\uparrow\) \(\text{G}(t)\) has \(n + O(n^2)\) vertices.

Lemma: Placement \(\in P\).
**Efficiency Placement (t)**

**Lemma:** Placement (t) can be done in $O(n\sqrt{n \log n})$ time.

**Proof:** Two steps:
1. Considering $O(n\sqrt{n})$ points in total
2. Geometry helps for matching $T$ disks, $P$ points $\implies O(P \log P + \sqrt{T} \cdot T \cdot \log P)$

$\implies O(n\sqrt{n \log n})$ time
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  - $L_\infty$
  - $O(2)$-approximation
DECISION OPTIMIZATION

**Lemma:** If $2t \leq t^*$, Placement $(t)$ ✓

**Corollaries:** How to prove Placement$(t)$ is $Z$-approximated

- If Placement$(t)$ ✓ and Placement$(t)$ X after translation $\Rightarrow t > \frac{t^*}{2}$
- If $t > t'$, Placement$(t)$ ✓, Placement$(t')$ X $\Rightarrow t > \frac{t^*}{2}$
- If Placement$(t)$ ✓ and Placement$(t + \varepsilon)$ X for infinitesimal $\varepsilon > 0$ $\Rightarrow t \geq \frac{t^*}{2}$
**DECISION OPTIMIZATION**

Main idea: binary search at 3 levels

![Diagram](image)

Invariant: \( t_1 < t_2 \) s.t. \( \text{Placement}(t_1) \checkmark \)

Objective: \( t_1 < t_2 \) like above and \( G(t_1 + \varepsilon) \approx G(t_2) \)

\[ \Rightarrow t_1 \text{ is a } 2\text{-approximation} \]
Theorem: In $(\mathbb{R}^2, L_\infty)$, regions are disks, we have a 2-approximation in $O(n\sqrt{n} \log^2 n)$.

Proof: $3 \times O(\log n) \times O(n\sqrt{n} \log n)$ time we find $t_1 < t_2$ s.t.

$\text{Placement}(t_1) \checkmark$

$\text{Placement}(t_2) \times$

$G(t_1 + \varepsilon) \preceq G(t_2)$
OUTLINE

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• Our results
• Approximate placement algorithm \( L_\infty \) 2-approximation
• Efficiency of approximate placement
• Decision no Optimization
OPEN PROBLEM

• If $S_i$'s disjoint no CENTERS is 2-approximation

Improve it?

Get a $T$-approximation with $T < 2$ for disjoint disks.