Crossing graphs as joins of graphs and Cartesian products of median graphs

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January 26, 2005

Abstract

For a partial cube G its crossing graph G# is the graph with vertices representing Θ-classes of G, two classes being adjacent if they cross on some cycle in G. The following problem posed in [11, Problem 7.1] is considered: what can be said about the partial cube G if G# is the join A ⊕ B of not edge-less graphs A and B? It is proved that for arbitrary graphs A and B, where at least one of them contains an edge, there exists a Cartesian prime partial cube G such that G# = A ⊕ B. On the other hand, if G is a median graph, then G# = A ⊕ B if and only if G = H □ K, where H# = A and K# = B. Along the way some new facts about partial cubes are obtained.

2000 Mathematical Subject Classification: 05C75, 05C12.

Keywords: intersection graph, partial cube, median graph, Cartesian product of graphs, join of graphs

1 Introduction

The intersection concepts in graph theory have been extensively studied [12]. Although some of the intersection operations yield all graphs (for instance, every graph is the intersection graph of some set system), their importance is rather in characterizing particular classes of graphs, thus giving a deeper structural understanding. Here we study a nonstandard intersection operation where vertices of the intersection graph (called crossing graph) are equivalence classes of a certain equivalence relation Θ defined on the edge-set of a graph. Hence the edges of the crossing graph are not defined in the standard way (by intersections of subsets). The graphs that we are interested in

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are isometric subgraphs of hypercubes, and the relation $\Theta$ is of great importance for understanding the structure of these graphs. So before presenting the preliminary work on these graphs and the crossing graph operation, let us recall necessary definitions.

The distance $d_G(u,v)$ between vertices $u$ and $v$ of a graph $G$ is the length of a shortest $u,v$-path in $G$. A subgraph $U$ of $G$ is isometric if $d_U(u,v) = d_G(u,v)$ for all $u,v \in U$. Interval $I_G(u,v)$ is the set of vertices that lie on shortest paths between $u$ and $v$ in $G$. A subgraph $U$ is convex if $I_G(u,v) \subseteq U$ for all $u,v \in U$. (Indices in the above definitions are omitted when the graph is understood from the context.) Recall that the hypercube $Q_k$ or a $k$-cube is the graph with the vertex set $\{0,1\}^k$ where two vertices are adjacent whenever they differ in exactly one position.

Partial cubes are isometric subgraphs of hypercubes. A well-known characterizations of partial cubes involves the relation $\Theta$ on the edge set of a graph. Two edges $e = xy$ and $f = uv$ of a graph $G$ are in the Djoković-Winkler [4, 17] relation $\Theta_G$, $\Theta$ for short, if $d_G(x,u) + d_G(y,v) \neq d_G(x,v) + d_G(y,u)$. Winkler [17] proved that a bipartite graph is a partial cube if and only if $\Theta$ is transitive. Letting $R^*$ denoting the transitive closure of a relation $R$, Winkler’s result reads as: a connected bipartite graph $G$ is a partial cube if and only if $\Theta = \Theta^*$. Hence in partial cubes the relation $\Theta$ is an equivalence relation on $E(G)$, the classes of the corresponding partition will be called $\Theta$-classes.

For a partial cube $G$ its crossing graph $G^\#$ was introduced in [11] as follows. The vertices of $G^\#$ correspond to the $\Theta$-classes of $G$, two vertices being adjacent if the corresponding $\Theta$-classes meet (or cross) on some cycle (that is, there is a cycle $C$ that contains edges of both $\Theta$-classes).

In this paper we address the problem what can be said about the partial cube $G$ if $G^\# = A \oplus B$ and $A, B$ are not edge-less? Here $A \oplus B$ denotes the join of graphs $A$ and $B$, that is, the graph obtained from the disjoint union of $A$ and $B$ by joining every vertex of $A$ with every vertex of $B$ by an edge. In the next section we give preliminaries on the Cartesian product of graphs and median graphs that are needed later. In Section 3 we prove that for arbitrary graphs $A$ and $B$, where at least one of them contains an edge, there exists a Cartesian prime partial cube $G$ such that $G^\# = A \oplus B$. Then we restrict to median graphs and prove that the crossing graph of a median graph $G$ is the join of two graphs $A$ and $B$ if and only if $G$ is a Cartesian product graph. In due course we also characterize partial cubes of radius two and observe that a partial cube contains no nontrivial convex subgraph that meets all of its $\Theta$-classes.

2 Cartesian products and median graphs

The Cartesian product $G \square H$ of graphs $G$ and $H$ is the graph with the vertex set $V(G) \times V(H)$ in which vertices $(a,x)$ and $(b,y)$ are adjacent whenever $ab \in E(G)$ and $x = y$, or $a = b$ and $xy \in E(H)$. The Cartesian product is associative and commutative with $K_1$ as its unit. It is easy to see that the Cartesian product of $k$ copies of $K_2$ is the hypercube $Q_k$. A graph $G$ is called prime (with respect to the Cartesian product) if it cannot be represented as the product of two nontrivial graphs, that is, $G = G_1 \square G_2$
implies that \( G_1 \) or \( G_2 \) is the one-vertex graph \( K_1 \).

The well-known prime factorization theorem, proved independently by Sabidussi [15] and Vizing [16], states that every connected graph has a unique prime factor decomposition with respect to the Cartesian product. This decomposition can be made explicit in the following way. Edges \( uv \) and \( uw \) are said to be in relation \( \tau_G \), or \( \tau \) for short, if \( u \) is the unique common neighbor of \( v \) and \( w \). Feder [5] proved (cf. also [7, Theorem 4.8] and [9]) that \( (\Theta \cup \tau)^* \) is the Cartesian product relation of a connected graph. This actually means that the equivalence classes of the relation \( (\Theta \cup \tau)^* \) determine the prime factor decomposition of a graph—any equivalence class yields one factor of the decomposition. The following consequence of this theorem will be useful for us.

**Corollary 1** A connected graph \( G \) is prime if and only if \( (\Theta_G \cup \tau_G)^* = E(G) \).

We will also need the following (in a way part of the folklore) result on the Cartesian product, see [2].

**Lemma 2** A subgraph \( C \) of the Cartesian product \( G_1 \square \cdots \square G_m \) of connected graphs is convex if and only if \( C = p_1(C) \square \cdots \square p_m(C) \), where \( p_i(C) \) is convex in \( G_i \), \( 1 \leq i \leq m \). (Here \( p_i \) is the projection map from \( G \) onto \( G_i \).

The most important subclass of partial cubes are median graphs. They have been rediscovered several times, and a rich structure theory on these graphs and related structures has been developed, cf. the survey [10]. The most common definition is the following. \( G \) is a median graph if for every triple of vertices \( u, v, w \in V(G) : I(u, v) \cap I(u, w) \cap I(v, w) \) consists of precisely one vertex (which is called the median of the triple \( u, v, w \)). One of the most well-known characterizations of median graphs involves a certain expansion procedure, a result due to Mulder [13]. (By the way, it inspired Chepoi [3] to prove a similar characterization of partial cubes.) In this note we will make use of a variation of the expansion procedure that involves peripheral subgraphs of a median graph [14], see also [1].

Let \( G \) be a connected graph and \( G_0 \) its convex subgraph. Then the peripheral expansion of \( G \) is the graph \( G' \) obtained as follows. Take the disjoint union of a copy of \( G \) and a copy of \( G_0 \). Join each vertex \( u \) in the copy of \( G_0 \) with the vertex that corresponds to \( u \) in the copy of \( G \) (actually in the subgraph \( G_0 \) of \( G \)). We say that the resulting graph \( G' \) is obtained by a (peripheral) expansion from \( G \) along \( G_0 \). We also say that we expand \( G_0 \) in \( G \) to obtain \( G' \). Note that in a peripheral expansion one new \( \Theta \)-class appears. It is easy to prove that expanding a convex subgraph of a median graph yields again a median graph. It is more surprising that the converse is also true, as proved by Mulder in [14]:

**Theorem 3** A graph \( G \) is a median graph if and only if it can be obtained from \( K_1 \) by a sequence of peripheral expansions.

Hence each median graph contains a peripheral subgraph, that is a subgraph \( H \) of vertices which are all incident with a particular \( \Theta \)-class \( F \) in \( G \), and such that \( H \) is a
connected component of \( G - F \) (the graph obtained from \( G \) by removal of edges from \( F \)). Even more is known \[14\]:

**Proposition 4** Let \( G \) be a median graph and \( F \) any \( \Theta \)-class in \( G \). Then both connected components of \( G - F \) contain a peripheral subgraph.

It is easy to see that median graphs are closed for the Cartesian product, and conversely, if a median graph is not prime, all the factors must also be median graphs.

### 3 Partial cubes whose crossing graphs are joins

Crossing graphs of Cartesian products have a simple structure \[11, Proposition 6.1\]:

**Proposition 5** Let \( H \) and \( K \) be partial cubes. Then \( (H \square K)^\# = H^\# \oplus K^\# \).

Let \( A \) and \( B \) be graphs. Clearly, \( A \oplus B \) is a complete bipartite graph if and only if both \( A \) and \( B \) are edge-less. In \[11\] it has also been proved that \( G^\# \) is a complete bipartite graph if and only if \( G \) is the Cartesian product of two trees. In this section we show, a bit surprisingly, that any other join of graphs can be realized as the crossing graph of a partial cube that is prime with respect to the Cartesian product. We begin with the following lemma. (Recall that the radius of a connected graph \( G \) is defined as \( \min_{u \in V(G)} \max_{v \in V(G)} d_G(u, v) \).)

**Lemma 6** Let \( G \) be a bipartite graph of radius 2. Then \( G \) is a partial cube if and only if \( G \) is \( K_{2,3} \)-free.

**Proof.** We only need to show that if \( G \) is bipartite of radius 2 and \( K_{2,3} \)-free, then \( G \) is a partial cube. Let \( u \) be a vertex that realizes the radius of \( G \) and let \( v_1, \ldots, v_k \) be its neighbors. As \( G \) is bipartite, \( v_1, \ldots, v_k \) induce an edge-less subgraph of \( G \). Let \( w_1, \ldots, w_r \) be the remaining vertices of \( G \), then they are all at distance 2 from \( u \). Again, there is no edge between \( w_i \) and \( w_j \).

Note that a graph is a partial cube if and only if the graph obtained from it by removing a pendant vertex is a partial cube. Hence we may without loss of generality assume that \( G \) has no pendant vertex. Since \( G \) is \( K_{2,3} \)-free it follows that every vertex \( w_i \) is of degree 2. Moreover, no two vertices \( w_i \) and \( w_j, i \neq j \), have the same pair of neighbors. Therefore every edge of the form \( w_i v_j \) lies in precisely one square.

No two edges \( w_i w_j \) and \( w_i w_j, i \neq j \), are in relation \( \Theta \). We claim that \( G \) isometrically embeds into \( Q_k \) and construct edge subsets \( E_1, \ldots, E_k \) of \( E(G) \) as follows. For \( i = 1, \ldots, k \) put \( w_i v_j \) in \( E_i \). Consider an edge \( w_i v_j \) and let \( w_i v_j w \ell \) be the unique square containing this edge. Then \( w_i v_j \) is in relation \( \Theta \) with \( w \ell \). Put \( w_i v_j \in E_\ell \). We claim that \( E_1, \ldots, E_k \) form the \( \Theta = \Theta^* \)-classes of \( G \).

Clearly, \( E_1, \ldots, E_k \) is a partition of \( E(G) \). Suppose \( w_i v_j, w_i v_j' \in E_\ell \). Since \( G \) is \( K_{2,3} \)-free, we infer \( i \neq i' \) and \( j \neq j' \). Then \( w_i v_\ell \in E(G) \) and \( w_i' v_\ell \in E(G) \) which implies that \( w_i v_j \) is in relation \( \Theta \) with \( w_i' v_j' \). So all pairs of edges from \( E_\ell \) are in relation \( \Theta \).
Now assume \( w_i v_j \in E_\ell \) and \( w_i v_j' \in E_{\ell'} \), where \( \ell \neq \ell' \). If \( i = i' \) or \( j = j' \) then clearly \( w_i v_j \) and \( w_i v_j' \) are not in relation \( \Theta \). Next, if \( \ell = j' \) then \( d(w_i, w_{i'}) + d(v_{j}, v_{j'}) = 2 + 2 \) is equal to \( d(w_i, v_{j}) + d(v_{j}, v_{j'}) = 1 + 3 \), hence they are again not in relation \( \Theta \) (the case \( \ell' = j \) is analogous). Otherwise we get \( d(w_i, w_{i'}) + d(v_{j}, v_{j'}) = 4 + 2 = 3 + 3 = d(w_i, v_{j}) + d(v_{j}, v_{j'}) \). Hence we conclude that \( \Theta = \Theta^* \) and thus \( G \) is a partial cube by Winkler’s theorem.

**Theorem 7** Let \( A \) and \( B \) be arbitrary graphs, where at least one of them contains an edge. Then there exists a Cartesian prime partial cube \( G \) such that \( G^\# = A \oplus B \).

**Proof.** For a graph \( H \) let \( \tilde{H} \) be the graph obtained from \( H \) by subdividing all edges of \( H \) and adding a new vertex \( u \) joined to all the original vertices of \( H \). (This construction has been introduced in [8] to establish a connection between median graphs and triangle-free graphs.) We claim that \( G = A + B \) does the job.

Let \( V(A) = \{a_1, \ldots, a_n\} \) and \( V(B) = \{b_1, \ldots, b_m\} \), so that in \( G \) the vertex \( u \) is adjacent to \( a_1, \ldots, a_n \) and to \( b_1, \ldots, b_m \). Let \( x_{ij} \) be the vertex of \( G \) obtained by subdividing the edge \( a_ib_j, 1 \leq i \leq n, 1 \leq j \leq m \).

We first observe that \( G \) is a partial cube by Lemma 6. Let \( E_i \) be the \( \Theta \)-classes of \( G \) with the representative \( ua_i, 1 \leq i \leq n \), and let \( F_i \) be the \( \Theta \)-classes of \( G \) with the representative \( ub_i, 1 \leq i \leq m \). Consider the square \( u a_i x_{ij} b_j \) to infer that \( E_i \) and \( F_j \) cross. Similarly, \( E_i \) and \( E_j \) (resp. \( F_i \) and \( F_j \)) cross if and only if \( a_i a_j \in E(A) \) (resp. \( b_i b_j \in E(B) \)). Hence \( G^\# = A \oplus B \).

It remains to show that \( G \) is prime with respect to the Cartesian product. Assume without loss of generality that \( n \geq 2 \) and that \( a_1 a_2 \in E(A) \). Let \( a_i, a_j, i \neq j \), be arbitrary vertices of \( A \) and \( b_k \) a vertex of \( B \). Then we have \( x_{ik} b_k \in E_i \) and \( x_{jk} b_k \in E_j \). By the construction of \( G \) (recall that \( x_{ik} \) and \( x_{jk} \) are of degree 2) we infer that the edges \( x_{ik} b_k \) and \( x_{jk} b_k \) are in relation \( \tau \). As \( i \) and \( j \) were arbitrary it follows that \( E_1, \ldots, E_n \) belong to the same equivalence class of \( (\Theta_G \cup \tau_G)^* \). Analogously, \( F_1, \ldots, F_m \) belong to the same equivalence class of \( (\Theta_G \cup \tau_G)^* \). Let \( y \) be the vertex of \( G \) obtained by subdividing the edge \( a_1 a_2 \). Then we have \( a_1 y \in E_2 \) and \( a_1 x_{11} \in F_1 \). Moreover, \( a_1 y \) is in relation \( \tau \) with \( a_1 x_{11} \) which implies that \( (\Theta_G \cup \tau_G)^* \) consists of a single equivalence class. By Corollary 1 we conclude that \( G \) is Cartesian prime graph.

As we already mentioned, \( G^\# \) is a complete bipartite graph if and only if \( G \) is the Cartesian product of two trees. Let \( \overline{K}_n \) be the graph on \( n \) vertices and no edges. Then it is not difficult to see that the above construction gives:

\[
\overline{K}_n \oplus \overline{K}_m = K_{1,n} \sqcup K_{1,m}.
\]

In particular, \( \overline{K}_2 \oplus \overline{K}_2 = P_3 \sqcup P_3 \).

Other constructions that give joins of graphs as crossing graphs can also be obtained. Let \( A \) be a graph and let \( G \) be the graph that is obtained from \( A \) by the Chepoi expansion (cf. [3]) with covering sets \( A \) and the star induced by \( u \) and its neighbors.
Then $G$ is a partial cube with $G^# = K_1 \circ A$. This construction is illustrated in Fig. 1 for the case when $A$ is the graph on four vertices and five edges. The new $\Theta$-class of $G$ that yields the $K_1$ in the join decomposition is denoted with thick lines.

![Figure 1: Expanding $\tilde{A}$ into $G$, so that $G^# = K_1 \oplus A$](image)

4 The case of median graphs

Crossing graphs of median graphs are easier to study than those of general partial cubes, since if two $\Theta$-classes of a median graph cross on some cycle then there exists a square in which they cross. This fact can be easily seen by using the expansion procedure and induction.

In [11] it is proved that every graph is the crossing graph of some median graph. However, it was wrongly mentioned that there are prime median graphs whose crossing graphs are joins of two graphs. The graph presented there on Fig. 7.2 is a Cartesian product graph, namely $P_3 \Box G$, where $G$ is the graph obtained from $C_4$ and another vertex joined to one of the vertices of $C_4$. In this section we prove that the above remark is indeed wrong by proving that a median graph whose crossing graph is the join of two graphs is necessarily the Cartesian product of two graphs. Note that this is in a surprising contrast to the situation from the previous section. We will need the following lemma that might be of independent interest. It follows from the Convexity Lemma from [6] which asserts that an induced connected subgraph $H$ of a bipartite graph $G$ is convex if and only if no edge with one endvertex in $H$ and the other not in $H$ is in relation $\Theta$ to an edge in $H$.

Lemma 8 Let $G$ be a partial cube and $H$ a convex subgraph of $G$. If $H$ intersects all $\Theta$-classes of $G$ then $H = G$.

Proof. Suppose $H$ is a proper subgraph of $G$. Then, since $H$ is convex and hence induced, there exists an edge $uv$ of $G$ such that $u \in H$ and $v \notin H$. By the Convexity Lemma, $uv$ is in relation $\Theta$ to no edge of $H$. But then $H$ does not intersect the $\Theta$-class of $uv$, a contradiction. \qed
We can now state the main result of this section.

**Theorem 9** Let $G$ be a median graph. Then $G^\# = A \oplus B$ if and only if $G = H \square K$, where $H^\# = A$ and $K^\# = B$.

**Proof.** By Proposition 5 one direction is proved: the crossing graph of the Cartesian product of median graphs is the join of the crossing graphs of the factors. Hence we only need to prove the converse of this statement, for which we will use the induction on the number of $\Theta$-classes of a median graph $G$. Clearly the smallest graph which is the join of two graphs and the crossing graph of a median graph is $K_2$. It is obvious that the only median graph with exactly two $\Theta$-classes that cross is $C_4$, and $C_4 = K_2 \square K_2$, showing the basis of induction.

Assume the statement holds for median graphs with less than $k$ $\Theta$-classes. Let $G$ be a median graph with $k$ $\Theta$-classes and $G^\# = A \oplus B$. By Theorem 3, $G$ can be obtained from a median graph $M$ by a peripheral expansion. As $M$ has one less $\Theta$-class than $G$, $M^\#$ is an induced subgraph of $G^\#$. More precisely $M^\# = G^\# - u$ where $u$ corresponds to the peripheral $\Theta$-class of $G$. Without loss of generality we may assume that $u \in A$.

Assume first that $|A| = 1$, and let $E'$ denote the $\Theta$-class of $G$ that corresponds to $u$. By Proposition 4 both connected components of $G - E'$ contain a peripheral subgraph. One component is the peripheral subgraph by definition of the peripheral expansion, and let $F$ be the $\Theta$-class that induces a peripheral subgraph $P$ in the other component of $G - E'$. Denote by $v$ the vertex of $G^\#$ that corresponds to $F$. If $F \neq E'$ then $F$ and $E'$ do not cross, for otherwise $P$ would lie in both components of $G - E'$. Hence, in $G^\#$ vertices $u$ and $v$ are not adjacent which means that they must both be in $A$, but this is a contradiction with $|A| = 1$. The remaining case is $E' = F$, hence also the other component is a peripheral subgraph, namely $P$. But then $G = K_2 \square P$, where $P^\# = B$.

Let now $|A| > 1$. Hence $M^\# = (A - u) \oplus B$, and by the induction hypothesis $M = U \square K$, where $U^\# = A - u$ and $K^\# = B$. Note that $\Theta$-classes of $M$ consist of $\Theta$-classes of $U$ and of $\Theta$-classes of $K$. More precisely, if $F$ is a $\Theta$-class of $U$ (resp. $K$), then $F \times V(K)$ (resp. $V(U) \times F$) is a $\Theta$-class of $U \square K$, cf. [7, Lemma 4.3]. Denote by $u_1, \ldots, u_p$ vertices of $A - u$ that correspond to $\Theta$-classes of $U$, and by $v_1, \ldots, v_r$ vertices of $B$ that correspond to $\Theta$-classes of $K$. (Observe that the role of sets $A$ and $B$ could be changed in $G^\#$ by means of some automorphism of $G^\# - u$, but we use the notation of vertices accordingly. That is by $v_1, \ldots, v_r$ are denoted the vertices that correspond to $B$ in $G^\#$ and are thus adjacent to $u$ in $G^\#$.) Note that $G$ is obtained from $M$ by expanding a convex subgraph $C$ of $M$. By Lemma 2, $C = U' \square K'$, where $U'$ is a convex subgraph of $U$ and $K'$ is a convex subgraph of $K$.

Suppose $K'$ is a proper subgraph of $K$. By Lemma 8 there exists a $\Theta$-class of $K$ that does not cross with the new $\Theta$-class which corresponds to $u$. This implies that there is a vertex $v_i \in B$ which is not adjacent to $u \in A$, a contradiction. Hence $K' = K$, and $C = U' \square K$, where $U'$ is a convex subgraph of $U$. We deduce that $G = H \square K$ where $H$ is the graph obtained from $U$ by expanding $U'$. Clearly $H^\# = A$ and $K^\# = B$ which completes the proof. □
References


