# On almost self-complementary graphs

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**Abstract**

A graph is called almost self-complementary if it is isomorphic to one of its almost complements $X^{\mathrm{c}} - \mathcal{I}$, where $X^{\mathrm{c}}$ denotes the complement of $X$ and $\mathcal{I}$ a perfect matching (1-factor) in $X^{\mathrm{c}}$. Almost self-complementary circulant graphs were first studied by Dobson and Šajna in 2004. In this paper we investigate some of the properties and constructions of general almost self-complementary graphs. In particular, we give necessary and sufficient conditions on the order of an almost self-complementary regular graph, and construct infinite families of almost self-complementary regular graphs, almost self-complementary vertex-transitive graphs, and non-cyclically almost self-complementary circulant graphs.

*Keywords:* Self-complementary graph, almost self-complementary graph, homogeneously almost self-complementary graph, non-cyclically almost self-complementary circulant graph, isomorphic factorization, homogeneous factorization.

## 1 Introduction

A graph $X$ is said to be *self-complementary* if it is isomorphic to its complement $X^{\mathrm{c}}$. Similarly, a graph $X$ on an even number of vertices is said to be *almost self-complementary* if it is isomorphic to a graph (called an *almost complement of* $X$) obtained from $X^{\mathrm{c}}$ by removing the edges of a 1-factor of $X^{\mathrm{c}}$. The study of almost self-complementary graphs was first suggested by Brian Alspach, who, encouraged by an analogous result for self-complementary circulant graphs [1, 6], proposed the determination of all possible orders of almost self-complementary circulant graphs. However, as it transpires from [3], where this problem was solved for a particularly "nice" subclass of almost self-complementary

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circulants (called cyclically almost self-complementary circulants), the structure of almost self-complementary graphs is much more complex than that of self-complementary graphs. The major difference between these two classes of graphs is best understood in a more general context of isomorphic factorizations of graphs.

An *isomorphic factorization* of a graph $Y = (V, E)$ is a partition $\mathcal{P}$ (the size of which is referred to as the *index*) of $E$ such that the factors $(V, E_i)$, $E_i \in \mathcal{P}$, are mutually isomorphic. Clearly, if $X$ is a self-complementary graph, then the edge sets of $X$ and $X^{\mathrm{C}}$ form an index-2 isomorphic factorization of a complete graph. Conversely, the factors of an index-2 isomorphic factorization of a complete graph are self-complementary graphs. Similarly, if $X$ is an almost self-complementary graph on $2n$ vertices and $X^*$ an isomorphic almost complement of $X$, then the edge sets of $X$ and $X^*$ form an index-2 isomorphic factorization of a graph isomorphic to the "cocktail party graph" $K_{2n} - nK_2$. (Note that this graph may not be unique; it depends on the choice of $X^*$.) Conversely, factors of an index-2 isomorphic factorization of a "cocktail party graph" are isomorphic mutual almost complements. Even though the above might suggest that there are no profound differences between self-complementary and almost self-complementary graphs, a very particular nature of almost self-complementary graphs becomes apparent when automorphisms of isomorphic factorizations are considered.

An *automorphism of an isomorphic factorization* $\mathcal{P}$ of a graph $Y$ is an automorphism of $Y$ which (when viewed as a permutation on the edge set of $Y$) preserves the partition $\mathcal{P}$. The group of all automorphisms of $\mathcal{P}$ acts on the set $\mathcal{P}$ in a natural way, permuting the factors. The kernel of this action is the intersection of the automorphism groups of all factors. In the case of a self-complementary graph $X$, every isomorphism from $X$ to its complement $X^{\mathrm{C}}$ is also an automorphism of the corresponding isomorphic factorization of a complete graph. Similarly, every automorphism of $X$ is also an automorphism of $X^{\mathrm{C}}$, and hence an element of the automorphism group of the corresponding isomorphic factorization of a complete graph that acts trivially on the set of factors.

This nice situation changes drastically in the case of almost self-complementary graphs. For example, there exists an almost self-complementary graph such that the automorphisms of all corresponding isomorphic factorizations act trivially when viewed as permutations of the factors (see Lemma 2.13 and Figure 4), that is, no isomorphism from this graph to an almost complement is an automorphism of the corresponding isomorphic factorization. Furthermore, the subgroup of automorphisms of an almost self-complementary graph $X$ that are also automorphisms of the corresponding isomorphic factorization may be much smaller than $\mathrm{Aut}(X)$; for example, the cycle on 6 vertices is an almost self-complementary graph with a vertex-transitive automorphism group, but the automorphism group of the corresponding isomorphic factorization is not vertex-transitive. Also, there exists an almost self-complementary graph on 14 vertices (arising from Construction 4.2) with the property that the automorphism group of one of the corresponding isomorphic factorizations (we do not know if it is unique) is trivial. It would be interesting to know if there exists an almost self-complementary graph such that this is true for all corresponding isomorphic factorizations (see Problem 7.1). These examples show that almost self-complementary graphs share many peculiarities with arbitrary isomorphic factorizations, and since they are in some sense the simplest kinds of "general" isomorphic factorizations, we find them worthwhile of study not only for their own sake, but also to gain some insight into the complexity of the general problem.

This article is organized as follows. In Section 2 we give precise definitions and prove some basic results. In Section 3 we present three binary operations on graphs that, when applied to appropriate graphs, result in almost self-complementary graphs. In most of the remaining sections, these operations are used to construct several infinite families of almost self-complementary graphs; in particular, regular almost self-complementary graphs are constructed in Section 4, vertex-transitive almost self-complementary graphs in Section 5, and non-cyclically almost self-complementary circulant graphs in Section 6. One of these constructions, for example, shows that regular almost self-complementary graphs exist for all even orders — in contrast with regular self-complementary graphs, which exist only for orders congruent to 1 modulo 4. In Section 5, homogeneously almost self-complementary graphs, that is, vertex-transitive graphs that give rise to a special kind of isomorphic factorizations called homogeneous factorizations (see [4, 5]), are distinguished. An in-depth analysis of these graphs is the topic of separate articles [10] and [11]. Finally, in Section 7, we pose some open questions about almost self-complementary graphs that we think are worth considering in future.

## 2 Preliminaries

We begin by reviewing some basic terms from graph theory and setting up the notation before introducing terminology specific to almost self-complementary graphs. For terminology and concepts from the theory of permutation groups, the reader is referred to [2]. We note, however, that superscript notation will be used here for group action. That is, if $G$ is a subgroup of the symmetric group $\mathrm{Sym}_\Omega$ acting on a set $\Omega$, then the image of a point $a \in \Omega$ by an element $\alpha \in G$ will be denoted $a^\alpha$. Similarly, $G^\varphi$ will denote the conjugate of the group $G$ by the element $\varphi \in \mathrm{Sym}_\Omega$, that is, $G^\varphi = \{\alpha^\varphi : \alpha \in G\}$ where $\alpha^\varphi = \varphi^{-1}\alpha\varphi$.

For a set $V$, let $V^{(2)}$ denote the set $\{\{u, v\} \mid u, v \in V, u \neq v\}$. A *graph* is an ordered pair $(V, E)$, where $V$ is any finite non-empty set and $E$ any subset of $V^{(2)}$. If $X = (V, E)$ is a graph, then the sets $V$ and $E$ are called the *vertex set* and *edge set* of $X$, and denoted by $V(X)$ and $E(X)$, respectively. The *complement* of a graph $X = (V, E)$ is the graph $(V, V^{(2)} \setminus E)$ denoted by $X^{\mathrm{c}}$. If $v$ is a vertex of a graph $X = (V, E)$, then the set of all vertices adjacent to $v$ is denoted by $X(v)$ and called the *neighbourhood* of $v$ in $X$. The size of the neighbourhood $X(v)$ is called the *valency* of $v$ in $X$ and denoted by $\mathrm{val}_X(v)$, where the subscript $X$ may be omitted if it is clear from the context what the graph $X$ is. A partition of a set $V$ into subsets of size 2 is called a *perfect matching on* $V$. If $X = (V, E)$ is a graph and $\mathcal{I}$ a perfect matching on $V$ with $\mathcal{I} \subseteq E$, then we say that $\mathcal{I}$ is a *perfect matching in* $X$, and $F = (V, \mathcal{I})$ is called a *1-factor in* $X$.

**Definition 2.1** Let $X$ be a graph with vertex set $V$ of even size and let $\mathcal{I}$ be a perfect matching in $X^{\mathrm{c}}$. An *almost complement of $X$ with respect to $\mathcal{I}$* , denoted by $\mathrm{AC}_{\mathcal{I}}(X)$, is the graph

with vertex set $V$ and edge set $V^{(2)} \setminus (E(X) \cup \mathcal{I})$. A graph is called an *almost complement* of $X$, if it is the almost complement of $X$ with respect to some perfect matching in $X^{\mathrm{c}}$.

**Definition 2.2** A graph $X$ is said to be *almost self-complementary with respect to a perfect matching* $\mathcal{I}$ in $X^{\mathrm{C}}$ if it is isomorphic to $\mathrm{AC}_{\mathcal{I}}(X)$. A graph $X$ is called *almost self-complementary* if there exists a perfect matching $\mathcal{I}$ in $X^{\mathrm{C}}$ such that $X \cong \mathrm{AC}_{\mathcal{I}}(X)$.

**Definition 2.3** An *antimorphism* of a graph $X = (V, E)$ is any permutation $\varphi \in \mathrm{Sym}_V$ such that $E^{\varphi} \cap E = \emptyset$. The set of all antimorphisms of a graph $X$ is denoted by $\mathrm{Ant}(X)$.

**Definition 2.4** For a graph $X$ and $\varphi \in \mathrm{Ant}(X)$ we define a function $f_{X,\varphi} : V(X) \to \mathbb{Z}$ by

$$f_{X,\varphi}(u) = \mathrm{val}_X(u) + \mathrm{val}_X(u^{\varphi}) \qquad \text{for all } u \in V(X).$$

**Definition 2.5** An antimorphism $\varphi$ of a graph $X$ is called an *$i$-antimorphism* of $X$ if $f_{X,\varphi}(u) = |V(X)| - 1 - i$ for all $u \in V(X)$. The set of all $i$-antimorphisms of a graph $X$ is denoted by $\mathrm{Ant}_i(X)$.

Clearly, a graph is self-complementary if and only if it admits a 0-antimorphism, and any antimorphism of a self-complementary graph is a 0-antimorphism. Observe also that any antimorphism of a $k$-regular graph of order $n$ is necessarily an $i$-antimorphism for $i = n - 1 - 2k$.

The following lemma introduces a simple characterization that will be used later in the paper to prove that our constructions yield almost self-complementary graphs.

**Lemma 2.6** *A graph $X = (V, E)$ is almost self-complementary if and only if it admits a 1-antimorphism. In particular, a regular graph on $2n$ vertices is almost self-complementary if and only if it has valency $n - 1$ and admits an antimorphism.*

PROOF. Suppose first that $X$ is a graph that is almost self-complementary with respect to perfect matching $\mathcal{I}$ in $X^{\mathrm{C}}$. Let $\varphi$ be an isomorphism from $X$ to $X^{\varphi} = \mathrm{AC}_{\mathcal{I}}(X)$. Then $E^{\varphi} = E(X^{\varphi})$, and since $E \cap E(X^{\varphi}) = \emptyset$, the permutation $\varphi$ is an antimorphism of $X$. Furthermore, since $V^{(2)}$ is a disjoint union of $E$, $E(X^{\varphi})$ and $\mathcal{I}$, we have $f_{X,\varphi}(v) = \mathrm{val}_X(v) + \mathrm{val}_X(v^{\varphi}) = \mathrm{val}_{X^{\varphi}}(v^{\varphi}) + \mathrm{val}_X(v^{\varphi}) = |X^{\varphi}(v^{\varphi}) \cup X(v^{\varphi})| = |V| - 2$ for every vertex $v \in V$. Hence $\varphi$ is a 1-antimorphism of $X$.

Suppose now that $X$ is a graph with a 1-antimorphism $\varphi$, that is, such that $f_{X,\varphi}(v) = \mathrm{val}(v) + \mathrm{val}(v^{\varphi}) = |V| - 2$ for every vertex $v \in V$. Since by the definition of an antimorphism $E \cap E^{\varphi} = \emptyset$, the antimorphism $\varphi$ is an isomorphism from $X$ to $X^{\varphi} = (V, E^{\varphi})$, which is a subgraph of $X^{\mathrm{C}}$. We just need to show that $\mathcal{I} = V^{(2)} \setminus (E \cup E^{\varphi})$ is a perfect matching. Take any $v \in V$. Then $\mathrm{val}_X(v) + \mathrm{val}_{X^{\varphi}}(v) = \mathrm{val}_X((v^{\varphi^{-1}})^{\varphi}) + \mathrm{val}_{X^{\varphi}}((v^{\varphi^{-1}})^{\varphi}) = \mathrm{val}_X((v^{\varphi^{-1}})^{\varphi}) + \mathrm{val}_X(v^{\varphi^{-1}}) = f_{X,\varphi}(v^{\varphi^{-1}}) = |V| - 2$, whence $(V, \mathcal{I})$ is a regular graph of valency 1 and $\mathcal{I}$ is a perfect matching. Hence $X^{\varphi} = \mathrm{AC}_{\mathcal{I}}(X)$ and $X$ is almost self-complementary. ■

In the next few definitions we introduce terminology that will allow us to examine the action of a 1-antimorphism of an almost self-complementary graph on the edges of its corresponding perfect matching. An overview of smallest almost self-complementary graphs follows.

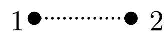
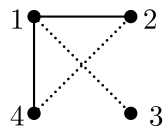
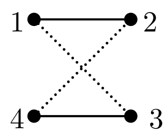
| Graph $X$ with a perfect matching $\mathcal{I}$ in $X^{\mathrm{C}}$ | $\mathcal{I}$-fair 1-antimorphism | $\mathcal{I}$-unfair 1-antimorphism |
|---|---|---|
|  | $\varphi = \mathrm{id}$ | none |
|  | $\varphi = (1, 3)$ | none |
|  | $\varphi = (1, 2, 3, 4)$ | $\psi = (2, 4, 3)$ |

Figure 1: Almost self-complementary graphs of order at most 4.

**Definition 2.7** Let $X = (V, E)$ be an almost self-complementary graph, $\varphi \in \mathrm{Ant}_1(X)$, and $\mathcal{I}$ a perfect matching in $X^{\mathrm{C}}$. By $\mathcal{I}_\varphi$ we shall denote the perfect matching $V^{(2)} \setminus (E \cup E^\varphi)$ corresponding to $\varphi$, and by $\mathrm{Ant}(X, \mathcal{I})$ the set of all 1-antimorphisms $\psi \in \mathrm{Ant}_1(X)$ with $E^\psi = V^{(2)} \setminus (E \cup \mathcal{I})$.

**Definition 2.8** A 1-antimorphism $\varphi$ of a graph $X = (V, E)$ is called *fair* if it is an automorphism of the graph $(V, E \cup E^\varphi)$, and *unfair* if it is not fair.

The following observation is immediate.

**Lemma 2.9** *Let $X = (V, E)$ be an almost self-complementary graph, $\mathcal{I}$ a perfect matching in $X^{\mathrm{C}}$, and $\varphi$ a 1-antimorphism of $X$. Then $\varphi$ is a fair 1-antimorphism of $X$ and $\mathcal{I} = \mathcal{I}_\varphi$ if and only if $\mathcal{I}^\varphi = \mathcal{I}$.*

PROOF. If $\varphi$ is a fair 1-antimorphism of $X$, then clearly $\mathcal{I}_\varphi^\varphi = \mathcal{I}_\varphi$.

Conversely, suppose $\mathcal{I}^\varphi = \mathcal{I}$. Since $\mathcal{I} \cap E = \emptyset$ and $(\mathcal{I}_\varphi \cup E^\varphi)^\varphi = \mathcal{I}_\varphi \cup E$, we have that $\mathcal{I}^\varphi \subseteq \mathcal{I}_\varphi$, whence $\mathcal{I} = \mathcal{I}_\varphi$. It is now easy to see that $\varphi$ preserves $E \cup E^\varphi$, that is, that it is fair. ■

The previous lemma allows for the following convenient terminology. We shall say that a 1-antimorphism $\varphi$ of an almost self-complementary graph $X$ is *$\mathcal{I}$-fair ($\mathcal{I}$-unfair)* to mean that it is fair (unfair) and $\mathcal{I}$ is a perfect matching in $X^{\mathrm{C}}$ such that $\mathcal{I} = \mathcal{I}_\varphi$. Note that the perfect matching $\mathcal{I}$ with respect to which $\varphi$ is fair or unfair is uniquely determined by $\varphi$.

**Definition 2.10** The set of all $\mathcal{I}$-fair 1-antimorphisms of $X$ is denoted by $\mathrm{Ant}_{\mathcal{I}}(X)$.

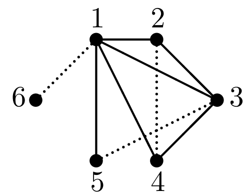
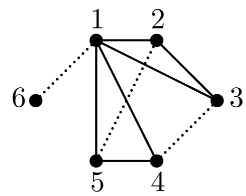
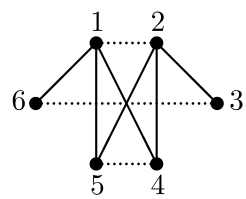
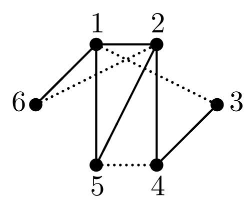
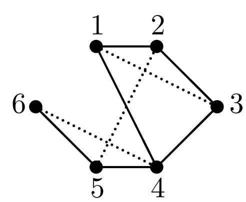
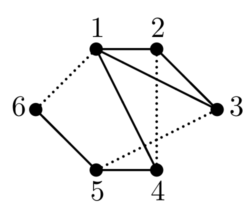
| Graph $X$ with a perfect matching $\mathcal{I}$ in $X^{\mathrm{c}}$ | $\mathcal{I}$-fair 1-antimorphism | $\mathcal{I}$-unfair 1-antimorphism |
|---|---|---|
|  | $\varphi = (1,6)(3,5)$ | none |
|  | $\varphi = (1,6)(2,3,5,4)$ | $\psi = (1,6)(3,4,5)$ |
|  | $\varphi = (1,6,2,3)$ | none |
|  | $\varphi = (1,3)(2,6)$ | none |
|  | $\varphi = (2,5)(4,6)$ | none |
|  | $\varphi = (1,6)(2,4)$ | $\psi = (1,6)(2,3,4)$ |

Figure 2: Almost self-complementary graphs of order 6 (continued in Figure 3).

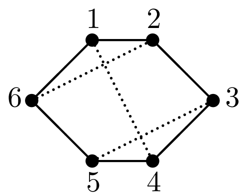
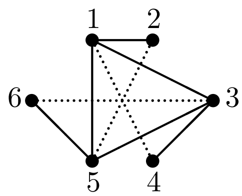
| Graph $X$ with a perfect matching $\mathcal{I}$ in $X^{\mathrm{C}}$ | $\mathcal{I}$-fair 1-antimorphism | $\mathcal{I}$-unfair 1-antimorphism |
|---|---|---|
|  | $\varphi = (2,3,6,5)$ | $\psi = (1,2,4,5,3)$ |
|  | $\varphi = (1,2,3,4,5,6)$ | $\psi = (1,4,3,2,5,6)$ |

Figure 3: Almost self-complementary graphs of order 6 (continued from Figure 2).

**Definition 2.11** An automorphism $\alpha$ of $X$ is called *$\mathcal{I}$-fair* if $\mathcal{I}^\alpha = \mathcal{I}$. The subgroup of all $\mathcal{I}$-fair automorphisms of $X$ is denoted by $\mathrm{Aut}_{\mathcal{I}}(X)$.

Note that $\mathrm{Ant}_{\mathcal{I}}(X) \subseteq \mathrm{Ant}(X,\mathcal{I}) \subseteq \mathrm{Ant}_1(X) \subseteq \mathrm{Ant}(X)$ and $\mathrm{Aut}_{\mathcal{I}}(X) \leq \mathrm{Aut}(X)$.

**Definition 2.12** Let $X$ be an almost self-complementary graph and $\mathcal{I}$ a perfect matching in $X^{\mathrm{C}}$. Then $X$ is called *$\mathcal{I}$-fairly ($\mathcal{I}$-unfairly) almost self-complementary* if it admits an $\mathcal{I}$-fair ($\mathcal{I}$-unfair) 1-antimorphism. A graph is called *fairly (unfairly) almost self-complementary* if it is $\mathcal{I}$-fairly ($\mathcal{I}$-unfairly) almost self-complementary for some perfect matching $\mathcal{I}$ in $X^{\mathrm{C}}$.

The list of all almost self-complementary graphs of order at most 6 is presented in Figures 1, 2, and 3. Up to isomorphism, each graph $X$ on the list admits exactly one perfect matching $\mathcal{I}$ (depicted in dotted lines) in the complement $X^{\mathrm{C}}$ such that the corresponding almost complement $\mathrm{AC}_{\mathcal{I}}(X)$ is isomorphic to $X$. The table also indicates whether the graph is fairly and/or unfairly almost self-complementary. Figure 4 shows an unfairly but not fairly almost self-complementary graph of smallest order (see Lemma 2.13). From these small examples we shall construct infinite families of graphs that are fairly and unfairly almost self-complementary, fairly but not unfairly almost self-complementary, and unfairly but not fairly almost self-complementary (see Corollary 3.3).

**Lemma 2.13** *The graph $X$ in Figure 4 is a smallest unfairly but not fairly almost self-complementary graph.*

Proof. First observe that $\psi = (1,5,4,2,3,6)$ is an unfair 1-antimorphism of $X$, whence $X$ is unfairly almost self-complementary.

Suppose $X$ admits a fair 1-antimorphism $\varphi$. Partition $V(X)$ into sets $A, B, C$ of vertices with equal valencies, namely, $A = \{1,3,4\}$, $B = \{2,5,6\}$, and $C = \{7,8\}$. Then $C^\varphi = C$

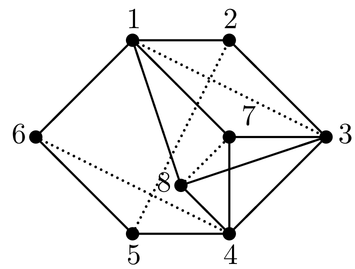



Figure 4: A smallest unfairly but not fairly almost self-complementary graph $X$ with a perfect matching $\mathcal{I}$ (dotted lines) in $X^{\mathrm{C}}$ and an $\mathcal{I}$-unfair 1-antimorphism $\psi = (1, 5, 4, 2, 3, 6)$.

while $A^\varphi = B$ (and $B^\varphi = A$). Since $X[A]$ and $X[B]$ are both isomorphic to $K_2 + K_1$, the perfect matching $\mathcal{I}_\varphi$ must have an edge in $A$ and an edge in $B$ (since all edges of $X^\varphi$ are images of edges of $X$). Observe that if $\mathcal{I}_\varphi \cap X[A \cup B]^{\mathrm{C}} = \{\{1, 4\}, \{2, 5\}, \{3, 6\}\}$, then $\varphi$ can not be extended to a 1-antimorphism of $X$. Hence it is not difficult to see that $\mathcal{I}_\varphi \cap X[A \cup B]^{\mathrm{C}}$ is either $\{\{1, 3\}, \{2, 5\}, \{4, 6\}\}$ or $\{\{1, 4\}, \{2, 6\}, \{3, 5\}\}$. In the first case, $\varphi$ clearly interchanges the vertices 4 and 6, but the restriction of $\varphi$ to the set $\{1, 2, 3, 5\}$ can not be a required 1-antimorphism of the induced graph $P_2 + K_1$. The second case is shown to be impossible in a similar way. Thus, $X$ admits no fair 1-antimorphisms.

Since all almost self-complementary graphs of order at most 6 are fairly almost self-complementary (see Figures 1, 2, and 3), we conclude that $X$ is an unfairly but not fairly almost self-complementary graph of minimum order. ■

We next present a few observations on the relationship between the automorphisms and 1-antimorphisms of an almost self-complementary graph. The easy proof of the first is left to the reader.

**Lemma 2.14** *Let $X$ be an almost self-complementary graph and $\varphi \in \mathrm{Ant}(X, \mathcal{I})$. Then:*

(i) $\mathrm{Aut}_{\mathcal{I}}(X) = \mathrm{Aut}_{\mathcal{I}}(X^\varphi) = \mathrm{Aut}(X) \cap \mathrm{Aut}(X^\varphi) = \mathrm{Aut}(X) \cap \mathrm{Aut}(X)^\varphi$.

(ii) $\mathrm{Ant}(X, \mathcal{I}) = \mathrm{Aut}(X)\varphi$. *In particular, if* $\mathrm{Aut}(X)$ *is trivial, then* $\mathrm{Ant}(X, \mathcal{I}) = \{\varphi\}$.

(iii) *If $\varphi$ is $\mathcal{I}$-fair, then* $\mathrm{Ant}_{\mathcal{I}}(X) = \mathrm{Aut}_{\mathcal{I}}(X)\varphi$.

**Lemma 2.15** *Let $X = (V, E)$ be an almost self-complementary graph of order at least 4 and $\mathcal{I}$ a perfect matching in $X^{\mathrm{C}}$.*

(i) *If $M \leq \mathrm{Aut}(X)$ and $\varphi \in \mathrm{Ant}(X, \mathcal{I})$ such that $M$ is a normal subgroup of index 2 in $\langle M, \varphi \rangle$, then $M \leq \mathrm{Aut}_{\mathcal{I}}(X)$ and $\varphi \in \mathrm{Ant}_{\mathcal{I}}(X)$.*

(ii) *If $\varphi$ is an $\mathcal{I}$-fair 1-antimorphism of $X$, then $\mathrm{Aut}_{\mathcal{I}}(X)$ is a normal subgroup of index 2 in $\langle \mathrm{Aut}_{\mathcal{I}}(X), \varphi \rangle$.*

PROOF. To prove (i), assume $M \leq \mathrm{Aut}(X)$ and $\varphi \in \mathrm{Ant}(X,\mathcal{I})$ such that $M$ is a normal subgroup of index 2 in $\langle M, \varphi\rangle$. Then $\varphi^2 \in M$ and $M^\varphi = M$. The former implies $(E^\varphi)^\varphi = E$, whence $\mathcal{I}^\varphi = \mathcal{I}$ and $\varphi \in \mathrm{Ant}_\mathcal{I}(X)$. Since $\varphi$ normalizes $M$, for any $\alpha \in M$ we have $E^{\varphi\alpha\varphi^{-1}} = E$, whence $(E^\varphi)^\alpha = E^\varphi$ and therefore $\mathcal{I}^\alpha = \mathcal{I}$, showing that $\alpha \in \mathrm{Aut}_\mathcal{I}(X)$. The conclusion follows easily.

To prove statement (ii), assume $\varphi$ is an $\mathcal{I}$-fair 1-antimorphism of $X$. Hence $\mathcal{I}^\varphi = \mathcal{I}$ and $(E^\varphi)^\varphi = E$. Thus $\varphi^2 \in \mathrm{Aut}_\mathcal{I}(X)$, and for any $\alpha \in \mathrm{Aut}_\mathcal{I}(X)$ we have $\varphi\alpha \in \mathrm{Aut}_\mathcal{I}(X)\varphi$ since $\varphi\alpha\varphi^{-1} \in \mathrm{Aut}_\mathcal{I}(X)$. Therefore $\mathrm{Aut}_\mathcal{I}(X)$ is a normal subgroup of index 2 in $\langle \mathrm{Aut}_\mathcal{I}(X), \varphi\rangle$, and (ii) follows. ■

**Proposition 2.16** *Let the graph $X = (V, E)$ be almost self-complementary graph with respect to a perfect matching $\mathcal{I}$ in $X^{\mathrm{C}}$. Consider the following two statements:*

$S_1$**:** $\mathrm{Ant}(X,\mathcal{I}) = \mathrm{Ant}_\mathcal{I}(X)$ *and*

$S_2$**:** $\mathrm{Aut}(X) = \mathrm{Aut}_\mathcal{I}(X)$.

*Then $S_1$ implies $S_2$. Conversely, $S_2$ implies $S_1$ if at least one of the following conditions is satisfied:*

$A$**:** *$X$ is $\mathcal{I}$-fairly almost self-complementary.*

$B$**:** *Each orbit of $\mathrm{Aut}(X)$ on $E$ has size greater than $\frac{|V|}{2}$.*

$C$**:** *$X$ is edge-transitive.*

PROOF. ($S_1 \Rightarrow S_2$): Take any $\alpha \in \mathrm{Aut}(X)$ and any antimorphism $\varphi \in \mathrm{Ant}(X,\mathcal{I}) = \mathrm{Ant}_\mathcal{I}(X)$. Then $\alpha\varphi$ is an antimorphism in $\mathrm{Ant}(X,\mathcal{I})$ by Lemma 2.14, and so $\mathcal{I}^{\alpha\varphi} = \mathcal{I}$ by the assumption. But since $\mathcal{I}^{\varphi^{-1}} = \mathcal{I}$, we must have $\mathcal{I}^\alpha = \mathcal{I}$. Thus $\mathcal{I}^{\mathrm{Aut}(X)} = \mathcal{I}$ and $\mathrm{Aut}(X) = \mathrm{Aut}_\mathcal{I}(X)$.

($S_2 \wedge A \Rightarrow S_1$): Assume that $\mathrm{Aut}(X) = \mathrm{Aut}_\mathcal{I}(X)$ and $\varphi$ is an $\mathcal{I}$-fair 1-antimorphism of $X$. If $S_1$ does not hold, then there exists $\psi \in \mathrm{Ant}(X,\mathcal{I}) \setminus \mathrm{Ant}_\mathcal{I}(X)$. But then $\alpha = \varphi\psi^{-1} \in \mathrm{Aut}(X)$ is not $\mathcal{I}$-fair, a contradiction. Hence $S_1$ follows.

($S_2 \wedge B \Rightarrow S_1$): Let $X^* = \mathrm{AC}_\mathcal{I}(X) = (V, E^*)$. Assume that $\mathrm{Aut}(X) = \mathrm{Aut}_\mathcal{I}(X)$ and that the size of each orbit of $\mathrm{Aut}(X)$ on $E$ is greater than $\frac{|V|}{2}$. Hence by Lemma 2.14, $\mathrm{Aut}(X) = \mathrm{Aut}(X^*)$, and so the size of each orbit of $\mathrm{Aut}(X)$ on $E^*$ is also greater than $\frac{|V|}{2}$. Take any antimorphism $\varphi \in \mathrm{Ant}(X,\mathcal{I})$. Suppose $\mathcal{I}^\varphi \neq \mathcal{I}$. Then there exists $e \in E^*$ such that $e^\varphi \in \mathcal{I}$. Since the orbit of $\mathrm{Aut}(X)$ containing $e$ has size greater than $|\mathcal{I}|$, there exists $\alpha \in \mathrm{Aut}(X)$ such that $(e^\alpha)^\varphi \notin \mathcal{I}$. Hence $(e^\varphi)^{\varphi^{-1}\alpha\varphi} \notin \mathcal{I}$. But $\varphi^{-1}\alpha\varphi \in \mathrm{Aut}(X^*) = \mathrm{Aut}(X)$, whence $\mathcal{I}^{\varphi^{-1}\alpha\varphi} = \mathcal{I}$, contradicting $e^\varphi \in \mathcal{I}$. We conclude that $\mathcal{I}^\varphi = \mathcal{I}$ for all antimorphisms $\varphi \in \mathrm{Ant}(X,\mathcal{I})$, and $S_1$ follows.

Since condition $C$ implies condition $B$, statements $S_2$ and $C$ also imply $S_1$. ■

**Corollary 2.17** *If $X$ is an $\mathcal{I}$-fairly almost self-complementary graph, then $X$ is $\mathcal{I}$-unfairly almost self-complementary if and only if $X$ admits an automorphism that is not $\mathcal{I}$-fair.*

Proposition 2.16 also shows that statements $S_1$ and $S_2$ are not equivalent for a graph $X$ that is almost self-complementary graph $X$ with respect to a perfect matching $\mathcal{I}$ in $X^{\mathrm{C}}$ if and only if $X$ is not $\mathcal{I}$-fairly almost self-complementary and $\mathrm{Aut}(X) = \mathrm{Aut}_{\mathcal{I}}(X)$. An example of such a graph $X$ is found in Figure 4. Lemma 2.13 shows that $X$ is $\mathcal{I}$-unfairly but not $\mathcal{I}$-fairly almost self-complementary, and it is not difficult to see that $\mathrm{Aut}(X) = \mathrm{Aut}_{\mathcal{I}}(X) = \{(7, 8)\}$.

# 3 Constructions of almost self-complementary graphs

In this section we present general constructions for almost self-complementary graphs via three binary operations: skew joins, partial joins, and lexicographic products. In most constructions we also determine the conditions on component graphs to yield fairly and/or unfairly almost self-complementary graphs.

The following terminology will be used in this and the remaining sections. An *ordered bipartition* of a set $S$ is an ordered pair $(A, B)$ of subsets of $S$ such that $\{A, B\}$ is a partition of $S$. If $(A_1, A_2)$ is an ordered bipartition of a set $S$ and $\alpha_i$ a permutation on $A_i$, $i = 1, 2$, then $(\alpha_1, \alpha_2)$ will denote the permutation on $S$ defined by $x^{(\alpha_1,\alpha_2)} = x^{\alpha_i}$ for all $x \in A_i$, $i = 1, 2$.

## 3.1 Skew joins

**Definition 3.1** Let $X$ and $Y$ be graphs with disjoint vertex sets, and let $(U_1, U_2)$ be an ordered bipartition of $V(X)$. The *skew join* $(X, (U_1, U_2)) \blacktriangleleft Y$ of $X$ and $Y$ with respect to the ordered bipartition $(U_1, U_2)$ of $V(X)$ is the graph with vertex set $V(X) \cup V(Y)$ and edge set $E(X) \cup E(Y) \cup \{xy : x \in U_1, y \in V(Y)\}$. We shall write shortly $X \blacktriangleleft Y$ for $(X, (U_1, U_2)) \blacktriangleleft Y$ if it is understood what the ordered bipartition of $V(X)$ is.

**Lemma 3.2** *Let $X$ and $Y$ be almost self-complementary graphs of orders $2m$ and $2n$, respectively. Let $(U_1, U_2)$ be an ordered bipartition of $V(X)$ such that $|U_1| = |U_2|$, and suppose $X$ admits a 1-antimorphism $\varphi$ with $U_1^{\varphi} = U_2$. Then the skew join $X \blacktriangleleft Y$ with respect to the ordered bipartition $(U_1, U_2)$ of $V(X)$ is an almost self-complementary graph. Furthermore, if $\max\{\mathrm{val}_X(x) : x \in U_2\} + \max\{\mathrm{val}_Y(y) : y \in V(y)\} < m + 2n - 2$, then:*

(i) *$X \blacktriangleleft Y$ is fairly almost self-complementary if and only if $X$ admits a fair 1-antimorphism $\vartheta$ with $U_1^{\vartheta} = U_2$ and $Y$ is fairly almost self-complementary.*

(ii) *$X \blacktriangleleft Y$ is unfairly almost self-complementary if and only if $X$ admits an unfair 1-antimorphism $\vartheta$ with $U_1^{\vartheta} = U_2$ or $Y$ is unfairly almost self-complementary.*

(iii) *If $X$ admits a fair 1-antimorphism $\vartheta$ with $U_1^{\vartheta} = U_2$, then $X \blacktriangleleft Y$ is fairly (unfairly) almost self-complementary if and only if $Y$ is fairly (unfairly) almost self-complementary.*

Proof. Let $\varphi$ be a 1-antimorphism of $X$ with $U_1^{\varphi} = U_2$. If $\psi$ is any 1-antimorphism of the graph $Y$, then clearly $\Psi = (\varphi, \psi)$ is an antimorphism of $X \blacktriangleleft Y$. Since

$$f_{X \blacktriangleleft Y,\Psi}(y) = \mathrm{val}_{X \blacktriangleleft Y}(y) + \mathrm{val}_{X \blacktriangleleft Y}(y^{\Psi}) = \mathrm{val}_Y(y) + m + \mathrm{val}_Y(y^{\psi}) + m = 2m + 2n - 2$$

for all $y \in V(Y)$, and

$$f_{X \blacktriangleleft Y, \Psi}(x) = \mathrm{val}_{X \blacktriangleleft Y}(x) + \mathrm{val}_{X \blacktriangleleft Y}(x^{\Psi}) = \mathrm{val}_X(x) + 2n + \mathrm{val}_X(x^{\varphi}) = 2m + 2n - 2$$

for all $x \in V(X)$, we have that $\Psi$ is a 1-antimorphism of $X \blacktriangleleft Y$, and $X \blacktriangleleft Y$ is almost self-complementary by Lemma 2.6.

Let $\Delta = \max\{\mathrm{val}_Y(y) : y \in V(y)\}$, $\delta = \min\{\mathrm{val}_Y(y) : y \in V(y)\}$, and $\Delta_i = \max\{\mathrm{val}_X(x) : x \in U_i\}$, $\delta_i = \min\{\mathrm{val}_X(x) : x \in U_i\}$ for $i \in \{1, 2\}$. Then $\Delta + \delta = 2n - 2$, and $\Delta_1 + \delta_2 = \Delta_2 + \delta_1 = 2m - 2$ by the existence of $\varphi$ and Lemma 2.6. Assume for the rest of the proof that $\Delta_2 + \Delta < m + 2n - 2$.

First we show that any 1-antimorphism of $X \blacktriangleleft Y$ fixes $V(X)$ and $V(Y)$ setwise. Take any $\Psi \in \mathrm{Ant}_1(X \blacktriangleleft Y)$. For any $y \in V(Y)$ we have that $m+\delta \leq \mathrm{val}_{X \blacktriangleleft Y}(y) = m+\mathrm{val}_Y(y) \leq m + \Delta$, whence

$$m + 2n - 2 - \Delta \leq \mathrm{val}_{X \blacktriangleleft Y}(y^{\Psi}) \leq m + \Delta$$

as $\mathrm{val}_{X \blacktriangleleft Y}(y^{\Psi}) = 2m + 2n - 2 - \mathrm{val}_{X \blacktriangleleft Y}(y)$ by Lemma 2.6, and since $\delta = 2n - 2 - \Delta$. Using the assumption $\Delta_2 + \Delta < m + 2n - 2$, we similarly obtain

$$\mathrm{val}_{X \blacktriangleleft Y}(x_1) = 2n + \mathrm{val}_X(x_1) \geq 2n + \delta_1 = 2m + 2n - 2 - \Delta_2 > m + \Delta$$

for all $x_1 \in U_1$, and

$$\mathrm{val}_{X \blacktriangleleft Y}(x_2) = \mathrm{val}_X(x_2) \leq \Delta_2 < m + 2n - 2 - \Delta$$

for all $x_2 \in U_2$. Thus

$$\mathrm{val}_{X \blacktriangleleft Y}(x_2) < \mathrm{val}_{X \blacktriangleleft Y}(y^{\Psi}) < \mathrm{val}_{X \blacktriangleleft Y}(x_1)$$

for all $y \in V(Y), x_i \in U_i$. We conclude that $y^{\Psi} \in V(Y)$ for all $y \in V(Y)$ and therefore $x^{\Psi} \in V(X)$ for all $x \in V(X)$. Note that this also implies $U_1^{\Psi} = U_2$ by the definition of a skew join.

Now suppose that $X \blacktriangleleft Y$ is fairly almost self-complementary. Let $\Psi$ be a fair 1-antimorphism of $X \blacktriangleleft Y$. By the observation above, there exist 1-antimorphisms $\vartheta$ of $X$ and $\psi$ of $Y$ such that $\Psi = (\vartheta, \psi)$ and $U_1^{\vartheta} = U_2$. Moreover, since $\Psi$ is fair, the 1-antimorphisms $\vartheta$ and $\psi$ will be fair. Thus $X$ admits a fair 1-antimorphism $\vartheta$ with $U_1^{\vartheta} = U_2$ and $Y$ is fairly almost self-complementary.

Conversely, suppose $X$ admits a fair 1-antimorphism $\vartheta$ with $U_1^{\vartheta} = U_2$ and $Y$ is fairly almost self-complementary. Then clearly, for any fair 1-antimorphism $\psi$ of $Y$, the 1-antimorphism $(\vartheta, \psi)$ of $X \blacktriangleleft Y$ will be fair. Hence $X \blacktriangleleft Y$ will be a fairly almost self-complementary graph.

To prove statement (ii), assume first that $X \blacktriangleleft Y$ is unfairly almost self-complementary and let $\Psi$ be an unfair 1-antimorphism. Again, we rely on the observation that there exist 1-antimorphisms $\vartheta$ of $X$ and $\psi$ of $Y$ such that $\Psi = (\vartheta, \psi)$ and $U_1^{\vartheta} = U_2$. Since $\Psi$ is unfair, at least one of $\vartheta$ and $\psi$ is unfair. The right side of the equivalence (ii) follows.

Finally, suppose that $X$ admits an unfair 1-antimorphism $\vartheta$ with $U_1^{\vartheta} = U_2$ or $Y$ is unfairly almost self-complementary. In the first case, $(\vartheta, \psi)$ will be an unfair 1-antimorphism of

$X \blacktriangleleft Y$ for any 1-antimorphism $\psi$ of $Y$. In the second case, if $\psi$ is an unfair 1-antimorphism of $Y$, then $(\varphi, \psi)$, where $\varphi$ is a 1-antimorphism of $X$ with $U_1^{\varphi} = U_2$, will be an unfair 1-antimorphism of $X \blacktriangleleft Y$. Thus $X \blacktriangleleft Y$ is unfairly almost self-complementary in either case.

Statement (iii) of the lemma easily follows from statements (i) and (ii). ∎

The easy corollary below gives constructions of fairly/unfairly almost self-complementary graphs of all possible orders.

**Corollary 3.3** *An almost self-complementary graph of order $2k$ exists for every $k \geq 1$. Moreover,*

(i) *there exists a fairly but not unfairly almost self-complementary graph of order $2k$ if and only if $k \geq 1$;*

(ii) *there exists a fairly and unfairly almost self-complementary graph of order $2k$ if and only if $k \geq 2$;*

(iii) *there exists an unfairly but not fairly almost self-complementary graph of order $2k$ if and only if $k \geq 4$.*

Proof. Let $X$ be the graph $2K_1$ of order $2m$ for $m = 1$. Then $X$ is almost self-complementary with a natural ordered bipartition $(U_1, U_2)$ and a 1-antimorphism $\varphi$ such that $U_1^{\varphi} = U_2$. Furthermore, for any almost self-complementary graph $Y_n$ of order $2n$, we have $\max\{\mathrm{val}_X(x) : x \in U_2\} + \max\{\mathrm{val}_Y(y) : y \in V(y)\} = \max\{\mathrm{val}_Y(y) : y \in V(y)\} \leq 2n - 2 < m + 2n - 2$. Hence by Lemma 3.2, $X \blacktriangleleft Y$ is fairly (unfairly) almost self-complementary whenever $Y_n$ is.

To construct a fairly but not unfairly almost self-complementary graph $Y_k$ of order $2k$ for all $k \geq 1$, we recursively define $Y_1 = 2K_1$ and $Y_k = X \blacktriangleleft Y_{k-1}$ for $k \geq 2$.

To construct a fairly and unfairly almost self-complementary graph $Y_k$ of order $2k$ for all $k \geq 2$, we start with $Y_2 = 2K_2$ and recursively define $Y_k = X \blacktriangleleft Y_{k-1}$ for $k \geq 3$.

To construct an unfairly but not fairly almost self-complementary graph $Y_k$ of order $2k$ for all $k \geq 4$, we let $Y_4$ be the graph from Figure 4 and Lemma 2.13, and recursively define $Y_k = X \blacktriangleleft Y_{k-1}$ for $k \geq 5$.

Figure 1 and Lemma 2.13 show that there is no fairly and unfairly almost self-complementary graph of order 2, and no unfairly but not fairly almost self-complementary graph of order at most 6, respectively, whence the conditions on the orders are also necessary. ∎

### 3.2 Partial joins

A "balanced" variation on skew joins, partial joins will be used in subsequent sections to construct regular and vertex-transitive almost self-complementary graphs.

**Definition 3.4** Let $X$ and $Y$ be graphs with disjoint vertex sets, and let $(U_1, U_2)$ and $(W_2, W_2)$ be ordered bipartitions of $V(X)$ and $V(Y)$, respectively. The *partial join* $(X, (U_1, U_2)) \blacklozenge (Y, (W_1, W_2))$ of $X$ and $Y$ with respect to ordered bipartitions $(U_1, U_2)$ and $(W_1, W_2)$ is the graph with vertex set $V(X) \cup V(Y)$ and edge set $E(X) \cup E(Y) \cup \{u_i w_i \mid u_i \in U_i, w_i \in W_i, i \in \{1,2\}\}$. We shall write shortly $X \blacklozenge Y$ for $(X,(U_1,U_2)) \blacklozenge (Y,(W_1,W_2))$ if it is understood what the two ordered bipartitions are.

**Lemma 3.5** *Let $X$ and $Y$ be almost self-complementary graphs with disjoint vertex sets admitting ordered bipartitions $(U_1,U_2)$ of $V(X)$ and $(W_1,W_2)$ of $V(Y)$, and 1-antimorphisms $\varphi \in \mathrm{Ant}_1(X)$ and $\psi \in \mathrm{Ant}_1(Y)$ such that $U_1^{\varphi} = U_1$ and $W_1^{\psi} = W_2$. Then the partial join $X \blacklozenge Y$ with respect to the given ordered bipartitions is an almost self-complementary graph. Morever:*

(i) *If both $X$ and $Y$ are regular graphs and $|U_1| = |U_2|$, then $X \blacklozenge Y$ is also regular.*

(ii) *If $\varphi$ and $\psi$ are fair 1-antimorphisms of $X$ and $Y$, respectively, then $X \blacklozenge Y$ is fairly almost self-complementary.*

(iii) *If at least one of $\varphi$ and $\psi$ is unfair, then $X \blacklozenge Y$ is unfairly almost self-complementary.*

Proof. Since $\varphi \in \mathrm{Ant}_1(X)$ and $\psi \in \mathrm{Ant}_1(Y)$ are 1-antimorphisms such that $U_1^{\varphi} = U_1$ and $W_1^{\psi} = W_2$, it is not difficult to see that $\Psi = (\varphi, \psi)$ is an antimorphism of $X \blacklozenge Y$. Since $|W_1| = |W_2|$, we have

$$\begin{aligned} f_{X\blacklozenge Y,\Psi}(x) &= \mathrm{val}_{X\blacklozenge Y}(x) + \mathrm{val}_{X\blacklozenge Y}(x^{\Psi}) = \mathrm{val}_X(x) + |W_i| + \mathrm{val}_X(x^{\varphi}) + |W_i| \\ &= |V(X)| - 2 + |V(Y)| = |V(X\blacklozenge Y)| - 2 \end{aligned}$$

for all $x \in U_i$, $i \in \{1,2\}$, and

$$\begin{aligned} f_{X\blacklozenge Y,\Psi}(w) &= \mathrm{val}_{X\blacklozenge Y}(w) + \mathrm{val}_{X\blacklozenge Y}(w^{\Psi}) = \mathrm{val}_Y(w) + |U_i| + \mathrm{val}_Y(w^{\psi}) + |U_{3-i}| \\ &= |V(Y)| - 2 + |V(X)| = |V(X\blacklozenge Y)| - 2 \end{aligned}$$

for all $w \in W_i$, $i \in \{1,2\}$. Therefore $\Psi$ is a 1-antimorphism of $X \blacklozenge Y$ and $X \blacklozenge Y$ is almost self-complementary by Lemma 2.6.

If $X$ and $Y$ are regular almost self-complementary graphs, then their valencies must be $\frac{1}{2}|V(X)| - 1$ and $\frac{1}{2}|V(Y)| - 1$, respectively. If, in addition, $|U_1| = |U_2|$, then $\mathrm{val}_{X\blacklozenge Y}(x) = \frac{1}{2}|V(X)| - 1 + \frac{1}{2}|V(Y)|$ for all $x \in V(X)$, and $\mathrm{val}_{X\blacklozenge Y}(w) = \frac{1}{2}|V(Y)| - 1 + \frac{1}{2}|V(X)|$ for all $w \in V(Y)$, whence $X \blacklozenge Y$ is regular.

Finally, if $\varphi$ and $\psi$ are both fair, then clearly $\Psi$ is fair, and if at least one of $\varphi$ and $\psi$ is unfair, then $\Psi$ is unfair. Hence the last two statements of the lemma. ■

### 3.3 Wreath products

It has been noted in [3] that the wreath (or lexicographic) product $X \wr K_2^{\mathrm{c}}$, where $X$ is a self-complementary circulant, is an almost self-complementary circulant. We extend this method in Theorem 3.8 to construct general almost self-complementary graphs. But first we need to review a few facts about wreath products of groups and graphs.

**Definition 3.6** Let $U = \{u_1, u_2, \ldots, u_m\}$ and $V$ be finite sets of cardinalities $m$ and $n$, respectively, and let $H \leq \mathrm{Sym}_U$ and $K \leq \mathrm{Sym}_V$ be permutation groups on the sets $U$ and $V$, respectively. The *imprimitive wreath product* $H \wr K$ of $K$ by $H$ is the permutation group with elements of the form $(h, k_1, \ldots, k_m)$ for $h \in H$ and $k_1, \ldots, k_m \in K$, acting on the set $U \times V$ according to the rule

$$(u_i, v)^{(h,k_1,\ldots,k_m)} = ({u_i}^h, v^{k_i})$$

for all $i \in \{1, 2, \ldots, m\}$ and $v \in V$. If $S$ and $T$ are subsets of the groups $H$ and $K$, respectively, then we similarly define the subset $S \wr T$ of $H \wr K$.

The subgroups $B(H \wr K) = \{(1_H, k_1, \ldots, k_m) : k_i \in K, i = 1, 2, \ldots, m\}$ and $\Delta(H \wr K) = \{(h, k, k, \ldots, k) : h \in H, k \in K\}$ of $H \wr K$ are referred to as the *base group* and the *diagonal group* of $H \wr K$, respectively.

Note that the product in the group $H \wr K$ obeys the rule $(h, k_1, \ldots, k_m)(h', k'_1, \ldots, k'_m) = (hh', k_1 k'_{1^h}, \ldots, k_m k'_{m^h})$, where for $i \in \{1, \ldots m\}$, the index $i^h$ is defined by ${u_i}^h = u_{i^h}$. Observe also that the diagonal group $\Delta(H \wr K)$ is isomorphic to the direct product $H \times K$ (in its natural action on $U \times V$), while the base group $B(H \wr K)$ is isomorphic to the group $K^m$. We also remark that the product $H \wr K$ as defined here is often denoted by $K \,\mathrm{wr}\, H$ or also by $K \wr H$ (see for example [2]). The motivation for our notation lies in the following lemma, the proof of which is left to the reader. Recall that the *wreath* (or *lexicographic*) *product* $X \wr Y$ of graphs $X$ and $Y$ is the graph with vertex set $V(X \wr Y) = V(X) \times V(Y)$ and edge set $E(X \wr Y) = \{\{(x_1, y_1), (x_2, y_2)\} : \{x_1, x_2\} \in E(X),$ or $x_1 = x_2$ and $\{y_1, y_2\} \in E(Y)\}$.

**Lemma 3.7** *Let $X$ and $Y$ be graphs. If $G$ and $H$ are subgroups of the automorphism groups $\mathrm{Aut}(X)$ and $\mathrm{Aut}(Y)$, respectively, then $G \wr H \leq \mathrm{Aut}(X \wr Y)$. In particular, if $G$ is a regular subgroup of $\mathrm{Aut}(X)$ and if $H$ is a regular subgroup of $\mathrm{Aut}(Y)$, then the diagonal group $\Delta(G \wr H) \cong G \times H$ is a regular subgroup of $\mathrm{Aut}(X \wr Y)$.*

**Theorem 3.8** *Let $X$ be a self-complementary graph and $Y$ a graph that is almost self-complementary with respect to a perfect matching $\mathcal{I}$ in $Y^{\mathrm{C}}$. Let $\mathcal{I}^*$ denote the perfect matching ${K_{V(X)}}^{\mathrm{C}} \wr \mathcal{I}$ in $(X \wr Y)^{\mathrm{C}}$. Then:*

(i) *$X \wr Y$ is almost self-complementary with respect to the perfect matching $\mathcal{I}^*$.*

(ii) $\mathrm{Ant}(X) \wr \mathrm{Ant}(Y, \mathcal{I}) \subseteq \mathrm{Ant}(X \wr Y, \mathcal{I}^*)$.

(iii) *If $Y$ is $\mathcal{I}$-fairly ($\mathcal{I}$-unfairly) almost self-complementary, then $X \wr Y$ is $\mathcal{I}^*$-fairly ($\mathcal{I}^*$-unfairly) almost self-complementary.*

*If* $\mathrm{Aut}(X \wr Y) = \mathrm{Aut}(X) \wr \mathrm{Aut}(Y)$, *then:*

(iv) $\mathrm{Ant}(X) \wr \mathrm{Ant}(Y, \mathcal{I}) = \mathrm{Ant}(X \wr Y, \mathcal{I}^*)$ *and*

(v) *$Y$ is $\mathcal{I}$-fairly ($\mathcal{I}$-unfairly) almost self-complementary if and only if $X \wr Y$ is $\mathcal{I}^*$-fairly ($\mathcal{I}^*$-unfairly) almost self-complementary.*

Proof. Denote the vertex sets of $X$ and $Y$ by $U$ and $V$, respectively, where $U = \{u_1, u_2, \ldots, u_m\}$ and $|V| = n$. First notice that $\mathrm{Ant}(X) \wr \mathrm{Ant}(Y, \mathcal{I})$ is a non-empty set since $\mathrm{Ant}(X)$ and $\mathrm{Ant}(Y, \mathcal{I})$ are non-empty sets. Take any $\Phi \in \mathrm{Ant}(X) \wr \mathrm{Ant}(Y, \mathcal{I})$. Then $\Phi = (\varphi, \psi_1, \psi_2, \ldots, \psi_m)$ for some $\varphi \in \mathrm{Ant}(X)$ and $\psi_1, \psi_2, \ldots, \psi_m \in \mathrm{Ant}(Y, \mathcal{I})$, and $\Phi$ acts as

$$(u_i, v)^\Phi = (u_i, v)^{(\varphi, \psi_1, \psi_2, \ldots, \psi_m)} = (u_i^\varphi, v^{\psi_i})$$

for all $u_i \in U$, $v \in V$. We will show that $\Phi$ is a 1-antimorphism of $X \wr Y$.

Let $u_i, u_j \in U$ and $v, w \in V$ be such that $\{(u_i, v), (u_j, w)\} \in E(X \wr Y)$. We need to show that $\{(u_i, v)^\Phi, (u_j, w)^\Phi\} \in E(X^{\mathrm{C}} \wr \mathrm{AC}_{\mathcal{I}}(Y))$. We have that either $\{u_i, u_j\} \in E(X)$, or $u_i = u_j$ and $\{v, w\} \in E(Y)$. In the first case, $\{u_i^\varphi, u_j^\varphi\} \in E(X^{\mathrm{C}})$ and $u_i^\varphi \neq u_j^\varphi$ since $\varphi$ is an antimorphism of $X$, so $\{(u_i^\varphi, v^{\psi_i}), (u_j^\varphi, w^{\psi_j})\} \in E(X^{\mathrm{C}} \wr \mathrm{AC}_{\mathcal{I}}(Y))$. In the second case, $u_i^\varphi = u_j^\varphi$ but $\{v^{\psi_i}, w^{\psi_j}\} \in E(\mathrm{AC}_{\mathcal{I}}(Y))$ since $\psi_i = \psi_j \in \mathrm{Ant}(Y, \mathcal{I})$, and so $\{(u_i^\varphi, v^{\psi_i}), (u_j^\varphi, w^{\psi_j})\} \in E(X^{\mathrm{C}} \wr \mathrm{AC}_{\mathcal{I}}(Y))$. Therefore, $\{(u_i, v)^\Phi, (u_j, w)^\Phi\} \in E(X^{\mathrm{C}} \wr \mathrm{AC}_{\mathcal{I}}(Y))$. Since $X^{\mathrm{C}} \wr \mathrm{AC}_{\mathcal{I}}(Y)$ is a subgraph of $(X \wr Y)^{\mathrm{C}} = X^{\mathrm{C}} \wr Y^{\mathrm{C}}$ and is edge-disjoint from $\mathcal{I}^* = {K_U}^{\mathrm{C}} \wr \mathcal{I}$, we conclude that $X^{\mathrm{C}} \wr \mathrm{AC}_{\mathcal{I}}(Y) = \mathrm{AC}_{\mathcal{I}^*}(X \wr Y)$ and $\Phi \in \mathrm{Ant}(X \wr Y, \mathcal{I}^*)$. Thus statements (i) and (ii) follow.

Since $\mathcal{I}^* = {K_U}^{\mathrm{C}} \wr \mathcal{I}$, it is not difficult to see that $\{v, w\}^{\psi_j} = \{v^{\psi_j}, w^{\psi_j}\} \in \mathcal{I}$ for all $\{v, w\} \in \mathcal{I}$ and all $j = 1, 2, \ldots, m$ if and only if $\{(u_i, v), (u_i, w)\}^\Phi = \{(u_i^\varphi, v^{\psi_i}), (u_i^\varphi, w^{\psi_i})\} \in \mathcal{I}^*$ for all $\{(u_i, v), (u_i, w)\} \in \mathcal{I}^*$. Hence $\Phi = (\varphi, \psi_1, \psi_2, \ldots, \psi_m)$ is an $\mathcal{I}^*$-fair 1-antimorphism of $X \wr Y$ if and only if all $\psi_j$, for $j = 1, 2, \ldots, m$, are $\mathcal{I}$-fair 1-antimorphisms of $Y$. Statement (iii) now naturally follows from (ii) and, similarly, statement (v) will follow from (iv).

Assume now that $\mathrm{Aut}(X \wr Y) = \mathrm{Aut}(X) \wr \mathrm{Aut}(Y)$. Take any $\Psi \in \mathrm{Ant}(X \wr Y, \mathcal{I}^*)$. By Lemma 2.14, we have $\mathrm{Ant}(X \wr Y, \mathcal{I}^*) = \mathrm{Aut}(X \wr Y)\Phi$ for a 1-antimorphism $\Phi$ as above, and hence by the assumption there exist $\alpha \in \mathrm{Aut}(X)$ and $\beta_i \in \mathrm{Aut}(Y)$, for $i = 1, 2, \ldots, m$, such that $\Psi = (\alpha, \beta_1, \beta_2, \ldots, \beta_m)\Phi = (\alpha, \beta_1, \beta_2, \ldots, \beta_m)(\varphi, \psi_1, \psi_2, \ldots, \psi_m) = (\alpha\varphi, \beta_1\psi_{1^\alpha}, \beta_2\psi_{2^\alpha}, \ldots, \beta_m\psi_{m^\alpha})$, where $i^\alpha$ is defined by $u_{i^\alpha} = u_i^\alpha$ for all $i = 1, 2, \ldots, m$. Since $\alpha\varphi \in \mathrm{Ant}(X)$ and, by Lemma 2.14, $\beta_i\psi_{i^\alpha} \in \mathrm{Ant}(Y, \mathcal{I})$ for all $i = 1, 2, \ldots, m$, we have that $\Psi \in \mathrm{Ant}(X) \wr \mathrm{Ant}(Y, \mathcal{I})$. This, together with statement (ii), then implies (iv). ∎

We remark that Sabidussi [13] has shown that $\mathrm{Aut}(X \wr Y) = \mathrm{Aut}(X) \wr \mathrm{Aut}(Y)$ if and only if the graph $X$ contains no two vertices $x$ and $y$ with $X(x) \setminus \{y\} = X(y) \setminus \{x\}$.

## 4 Regular almost self-complementary graphs

Since a wreath product of regular graphs is a regular graph, Theorem 3.8 can be used for constructing regular almost self-complementary graphs. However, regular almost self-complementary graphs of some orders can not be constructed in this way. In this section we present two constructions — one direct and one via partial joints — resulting in regular almost self-complementary graphs of all possible orders. A short discussion on the "fairness" of these graphs follows at the end of the section.

Keep in mind that every antimorphism of a regular almost self-complementary graph is in fact a 1-antimorphism (see Lemma 2.6).

**Construction 4.1** For each positive integer $n$ we recursively construct a regular almost self-complementary graph $X_n$ on $2n$ vertices admitting an antimorphism $\varphi_n \in \mathrm{Ant}(X_n)$ with 2 orbits of size 1 and $n-1$ orbits of size 2.

Let $X_1$ be the totally disconnected graph ${K_2}^{\mathrm{C}}$ and $\varphi_1$ the identity permutation on $V({K_2}^{\mathrm{C}})$. Clearly, $\varphi_1 \in \mathrm{Ant}(X_1)$ and has 2 orbits of size 1.

Now suppose that we have already constructed $X_n$ and $\varphi_n \in \mathrm{Ant}(X_n)$ with the above properties. If $n$ is odd, let $U_1$ be a union of $(n-1)/2$ $\varphi_n$-orbits of size 2 and one $\varphi_n$-orbit of size 1. Similarly, if $n$ is even, let $U_1$ be a union of $n/2$ orbits of size 2. In both cases, let $U_2 = V(X_n) \setminus U_1$ and observe that $(U_1, U_2)$ is an ordered bipartition of $V(X_n)$ into two $\varphi_n$-invariant subsets of equal size. Next, let $Y = {K_2}^{\mathrm{C}}$, let $(W_1, W_2)$ be the partition of $V(Y)$ into singletons, and let $\psi$ be the non-identity antimorphism of $Y$. By Lemma 3.5, the partial join $X_{n+1} = (X, (U_1, U_2))\blacklozenge(Y, (W_1, W_2))$ is a regular almost self-complementary graph. Furthermore, it admits an antimorphism $\varphi_{n+1} = (\varphi_n, \psi)$ acting as $\varphi_n$ on $V(X_n)$ and as $\psi$ on $V(Y)$. Hence, it has two orbits of size 1 and $n$ orbits of size 2.

Since $\varphi_1$ and $\psi$ are both fair antimorphisms, by Lemma 3.5, each $\varphi_n$ is a fair antimorphism and $X_n$ is a fairly almost self-complementary graph. Note, however, that $X_2 \cong 2K_2$ and $X_3 \cong C_6$ are also unfairly almost self-complementary as they admit antimorphisms other than $\varphi_2$ and $\varphi_3$, respectively. It would be interesting to determine precisely which of the graphs $X_n$ from this construction are unfairly almost self-complementary.

**Construction 4.2** We shall construct a regular almost self-complementary graph $X = (V, E)$ of order $2n$ for every integer $n \geq 1$. The details will depend on the congruency class of $n$ modulo 4, in all cases, however, we let $V = \mathbb{Z}_{2n}$ and use the permutation $\rho = (0, 1, \ldots, 2n-1)$ as an antimorphism of $X$. We shall define $E$ so that $\rho^2$ preserves the adjacency of all or "almost all" pairs of vertices.

First assume $n \equiv 1$ or $2 \pmod 4$. Then there exists a partition $\{A_0^0, A_0^1, A_1\}$ of $\{1, 2, \ldots, n-1\}$ such that $|A_0^0| = |A_0^1| = \lfloor \frac{n-1}{4} \rfloor$, $|A_1| = \lceil \frac{n-1}{2} \rceil$, $A_0^0 \cup A_0^1$ contains only even integers, and $A_1$ contains only odd integers. We define the edge set of $X$ as

$$\begin{aligned} E &= \big\{\{2i, 2i+\ell\} : \ell \in A_1 \cup A_0^0, i \in \{0, 1, \ldots, n-1\}\big\} \\ &\quad \cup \big\{\{2i+1, 2i+1+\ell\} : \ell \in A_0^1, i \in \{0, 1, \ldots, n-1\}\big\}. \end{aligned}$$

Then $\rho$ is a 1-antimorphism of $X$ with the corresponding perfect matching

$$\mathcal{I}_\rho = \{\{i, i+n\} : i \in \{0, 1, \ldots, n-1\}\}.$$

Next, let $n \equiv 0 \pmod 4$. Now we can find a partition $\{A_0^0, A_0^1, A_1\}$ of $\{1, 3, 4, 5, , \ldots, n-1\}$ such that $|A_0^0| = |A_0^1| = \frac{n-4}{4}$, $|A_1| = \frac{n}{2}$, $A_0^0 \cup A_0^1$ contains only even integers, and $A_1$ contains only odd integers. Then let

$$\begin{aligned} E &= \big\{\{2i, 2i+\ell\} : \ell \in A_1 \cup A_0^0, i \in \{0, 1, \ldots, n-1\}\big\} \\ &\quad \cup \big\{\{2i+1, 2i+1+\ell\} : \ell \in A_0^1, i \in \{0, 1, \ldots, n-1\}\big\} \\ &\quad \cup \Big\{\{2i, 2i+n\} : i \in \{0, 1, \ldots, \frac{n}{2}-1\}\Big\} \\ &\quad \cup \Big\{\{4i+1, 4i+3\} : i \in \{0, 1, \ldots, \frac{n}{2}-1\}\Big\}, \end{aligned}$$

whence $\rho$ is a 1-antimorphism of $X$ with the corresponding perfect matching

$$\mathcal{I}_\rho = \Big\{\{4i+3, 4i+5\}, \{4i, 4i+2\} : i \in \{0, 1, \ldots, \frac{n}{2} - 1\}\Big\}.$$

Finally, let $n \equiv 3 \pmod 4$. There exist a partition $\{A_0^0, A_0^1, A_1\}$ of $\{1, 2, \ldots, n-2\}$ such that $|A_0^0| = |A_0^1| = \frac{n-3}{4}$, $|A_1| = \frac{n-1}{2}$, and $A_0^0 \cup A_0^1$ contains only even integers while $A_1$ contains only odd integers. We define

$$\begin{aligned} E &= \big\{\{2i, 2i+\ell\} : \ell \in A_1 \cup A_0^0, i \in \{0, 1, \ldots, n-1\}\big\} \\ &\quad \cup \big\{\{2i+1, 2i+1+\ell\} : \ell \in A_0^1, i \in \{0, 1, \ldots, n-1\}\big\} \\ &\quad \cup \Big\{\{2i, 2i+n-1\} : i \in \{1, 2, \ldots, \frac{n-1}{2}\}\Big\} \\ &\quad \cup \Big\{\{2i+1, 2i+n\} : i \in \{\frac{n+1}{2}, \frac{n+3}{2}, \ldots, n-1\}\Big\} \cup \{\{0, n\}\} \end{aligned}$$

and observe that $\rho$ is a 1-antimorphism of $X$ with the corresponding perfect matching

$$\mathcal{I}_\rho = \{\{i, i+n\} : i \in \{2, 3 \ldots, n-1\}\} \cup \{\{1, n\}\{n+1, 0\}\}.$$

**Lemma 4.3** *A graph $X$ of order $2n$ obtained in Construction 4.2 is an almost self-complementary regular graph. Moreover, it is $\mathcal{I}_\rho$-fairly almost self-complementary whenever $n \not\equiv 3 \pmod 4$, and is $\mathcal{I}_\rho$-unfairly almost self-complementary whenever $n \equiv 0$ or $3 \pmod 4$.*

Proof. It is easy to see that the graphs from Construction 4.2 are regular and almost self-complementary with an antimorphism $\rho$. It is also clear that in the case $n \equiv 1$ or $2 \pmod 4$ the antimorphism $\rho$ preserves $\mathcal{I}_\rho$ while in the cases $n \equiv 0$ or $3 \pmod 4$ it does not. In the case $n \equiv 0 \pmod 4$, however, it is not difficult to check that the permutation $\varphi$ on $\mathbb{Z}_{2n}$ defined by $\varphi(i) = -i + 1$ if $i$ is odd, and $\varphi(i) = n - i + 1$ if $i$ is even (with the arithmetic modulo $2n$) is an $\mathcal{I}_\rho$-fair antimorphism of $X$. ■

Note that in the case $n \equiv 3 \pmod 4$ of Construction 4.2, the antimorphism $\rho$ is not fair. Computer-aided calculation shows that for $n \leq 30$ these graphs have a trivial automorphism group, whence, by Lemma 2.14, they do not admit an $\mathcal{I}_\rho$-fair antimorphism.

The two constructions above allow for some interesting conclusions.

**Corollary 4.4** *There exists a regular fairly almost self-complementary graph of order $m$ if and only if $m$ is even. There exists a regular fairly and unfairly almost self-complementary graph of order $m$ whenever $m \equiv 0 \pmod 8$.*

## 5 Vertex-transitive almost self-complementary graphs

In this section we shall introduce vertex-transitive almost self-complementary graphs. We shall construct two infinite families, one via partial joints and the other via wreath products. As a sensible analogue of vertex-transitive self-complementary graphs, the subfamily of homogeneously almost self-complementary vertex-transitive graphs (to be defined below) will be distinguished.

If $X$ is a vertex-transitive self-complementary graph and $\varphi$ any antimorphism (necessarily a 0-antimorphism) of $X$, then $\mathrm{Aut}(X)$ is a normal subgroup of index 2 in $\langle \mathrm{Aut}(X), \varphi \rangle$. By contrast, there exist almost self-complementary vertex-transitive graphs such that for any vertex-transitive subgroup $M$ of $\mathrm{Aut}(X)$ and any antimorphism (necessarily a 1-antimorphism) $\varphi$ of $X$, the group $M$ is not normalized by $\varphi$; the smallest example is the cycle of length 6. Although general almost self-complementary vertex-transitive graphs may be interesting on their own, to obtain structural results similar to those on self-complementary vertex-transitive graphs we shall focus on homogeneously almost self-complementary graphs. In Lemma 2.15, we have seen that for any almost self-complementary graph $X$, if $M \leq \mathrm{Aut}(X)$ and $\varphi \in \mathrm{Ant}(X, \mathcal{I})$ such that $M$ is a normal subgroup of index 2 in $\langle M, \varphi \rangle$, then $\varphi \in \mathrm{Ant}_{\mathcal{I}}(X)$ and $M \leq \mathrm{Aut}_{\mathcal{I}}(X)$, and if $\varphi \in \mathrm{Ant}_{\mathcal{I}}(X)$, then $\mathrm{Aut}_{\mathcal{I}}(X)$ is necessarily a normal subgroup of index 2 in $\langle \mathrm{Aut}_{\mathcal{I}}(X), \varphi \rangle$, provided $X \not\cong {K_2}^{\mathrm{C}}$. These observations suggest the following definition.

**Definition 5.1** Let $X$ be a graph on at least 4 vertices that is $\mathcal{I}$-fairly almost self-complementary with respect to a perfect matching $\mathcal{I}$ in $X^{\mathrm{C}}$. Then $X$ is said to be *$\mathcal{I}$-homogeneously almost self-complementary* if $\mathrm{Aut}_{\mathcal{I}}(X)$ acts transitively on $V(X)$. A graph $X$ is called *homogeneously almost self-complementary* if it is $\mathcal{I}$-homogeneously almost self-complementary for some perfect matching $\mathcal{I}$ in $X^{\mathrm{C}}$.

In the introductory section we mentioned the correspondence between almost self-complementary graphs of order $2n$ and index-2 isomorphic factorizations of the graph $K_{2n} - nK_2$. Similarly, a homogeneously almost self-complementary graph of order $2n$ corresponds to a homogeneous index-2 factorization of the graph $K_{2n} - nK_2$; whence our choice of the term. Homogeneous factorizations of complete graphs were first defined in [8] and later generalized to arbitrary graphs and digraphs [4, 5]. A *homogeneous factorization* of a graph $Y$ is thus defined as a quadruple $(M, G, Y, \mathcal{P})$, where $M \leq G \leq \mathrm{Aut}(Y)$, $\mathcal{P}$ is a $G$-invariant partition of the edge set of $Y$, $G$ acts transitively on $\mathcal{P}$, and $M$ acts transitively on the vertex set of $Y$ (and is usually assumed to be the kernel of the action of $G$ on $\mathcal{P}$ – note that $M$ is then normal in $G$). As with general isomorphic factorizations, the *index* of a homogeneous factorization is the size of the partition $\mathcal{P}$. Hence, if $X = (V, E)$ is an $\mathcal{I}$-homogeneously almost self-complementary graph, then for any $\varphi \in \mathrm{Ant}_{\mathcal{I}}(X)$ we have that $(M, G, Y, \mathcal{P})$ is an index-2 homogeneous factorization of the graph $Y = K_V - \mathcal{I}$ for $M = \mathrm{Aut}_{\mathcal{I}}(X)$, $G = \langle M, \varphi \rangle$, and partition $\mathcal{P} = \{E, E^{\varphi}\}$.

While the following results serve only as an introduction to homogeneously almost self-complementary graphs, an in-depth study can be found in [10, 11].

**Lemma 5.2** *Let $X = (V, E)$ and $X' = (V', E')$ be isomorphic almost self-complementary graphs with an isomorphism $\iota : X \to X'$. Let $(U_1, U_2)$ be an ordered bipartition of $V$. Suppose $X$ admits an automorphism $\alpha$ and a 1-antimorphism $\varphi$ such that $U_1^{\alpha} = U_2$ and $U_1^{\varphi} = U_1$. Then the partial join $X \blacklozenge X'$ with respect to ordered bipartitions $(U_1, U_2)$ of $X$ and $(U_1^{\iota}, U_2^{\iota})$ of $X'$ is an almost self-complementary graph with the ordered bipartition $(V, V')$*

*of its vertex set admitting an automorphism $\beta$ and a 1-antimorphism $\psi$ such that $V^\beta = V'$ and $V^\psi = V$. Moreover:*

(i) *If $\alpha$ is an $\mathcal{I}$-fair automorphism and $\varphi$ is an $\mathcal{I}$-fair ($\mathcal{I}$-unfair) 1-antimorphism of $X$, then $\beta$ is an $(\mathcal{I} \cup \mathcal{I}^\iota)$-fair automorphism and $\psi$ is an $(\mathcal{I} \cup \mathcal{I}^\iota)$-fair ($(\mathcal{I} \cup \mathcal{I}^\iota)$-unfair) 1-antimorphism of $X \blacklozenge X'$.*

(ii) *If $X$ is $\mathcal{I}$-homogeneously almost self-complementary, $\varphi$ is an $\mathcal{I}$-fair antimorphism of $X$, and $U_1$ is a block of imprimitivity for a transitive subgroup of $\mathrm{Aut}_{\mathcal{I}}(X)$, then $X \blacklozenge X'$ is $(\mathcal{I} \cup \mathcal{I}^\iota)$-homogeneously almost self-complementary, $\psi$ is an $(\mathcal{I} \cup \mathcal{I}^\iota)$-fair antimorphism of $X \blacklozenge X'$, and $V$ is a block of imprimitivity for a transitive subgroup of $\mathrm{Aut}_{\mathcal{I}}(X \blacklozenge X')$.*

PROOF. By the definition of a partial join we have $V(X \blacklozenge X') = V \cup V'$ and $E(X \blacklozenge X') = E \cup E' \cup \{uv^\iota : u, v \in U_i, i = 1, 2\}$. Since $\varphi$ is a 1-antimorphism of $X$ with $U_1^\varphi = U_1$ and $\varphi' = \iota^{-1}\alpha\varphi\iota$ is a 1-antimorphism of $X'$ with $(U_1^\iota)^{\varphi'} = U_2^\iota$, we have that $X \blacklozenge X'$ is an almost self-complementary graph by Lemma 3.5.

Define a permutation $\beta$ on $V \cup V'$ by $u^\beta = u^\iota$ and $(u^\iota)^\beta = u$ for all $u \in V$. Then it is clear that $\beta$ is an automorphism of $X \blacklozenge X'$ such that $V^\beta = V'$.

Next define a permutation $\psi$ on $V \cup V'$ as $\psi = (\varphi, \varphi')$, that is, $u^\psi = u^\varphi$ and $(u^\iota)^\psi = (u^{\alpha\varphi})^\iota$ for all $u \in V$. Observe that $V^\psi = V$, and that $\psi|_V$ and $\psi|_{V'}$ are 1-antimorphisms of $X$ and $X'$, respectively. Since $u \in U_i$ and $v^\iota \in U_i^\iota$ if and only if $u^\psi \in U_i$ and $(v^\iota)^\psi \in U_{3-i}^\iota$, the permutation $\psi$ maps edges between $V$ and $V'$ to non-edges. Thus $\psi$ is the required 1-antimorphism of $X \blacklozenge X'$ with $V^\psi = V$.

If $\alpha \in \mathrm{Aut}_{\mathcal{I}}(X)$ and $\varphi \in \mathrm{Ant}(X, \mathcal{I})$, then for $\beta$ and $\psi$ as defined above, we clearly have $\beta \in \mathrm{Ant}_{\mathcal{I} \cup \mathcal{I}^\iota}(X \blacklozenge X')$ and $\psi \in \mathrm{Ant}(X \blacklozenge X', \mathcal{I} \cup \mathcal{I}^\iota)$. Moreover, $\psi$ is $(\mathcal{I} \cup \mathcal{I}^\iota)$-fair if and only if $\varphi$ is $\mathcal{I}$-fair.

Finally, assume that $X$ is $\mathcal{I}$-homogeneously almost self-complementary, $\varphi$ is an $\mathcal{I}$-fair antimorphism and $U_1$ is a block of imprimitivity for a transitive subgroup $G$ of $\mathrm{Aut}_{\mathcal{I}}(X)$. Then for any $\gamma \in G$, either $U_1^\gamma = U_1$ or $U_1^\gamma = U_2$, whence the permutation $(\gamma, \iota^{-1}\gamma\iota)$, which acts as $u^{(\gamma, \iota^{-1}\gamma\iota)} = u^\gamma$ and $(u^\iota)^{(\gamma, \iota^{-1}\gamma\iota)} = (u^\gamma)^\iota$ for all $u \in V$, is an $(\mathcal{I} \cup \mathcal{I}^\iota)$-fair automorphism of $X \blacklozenge X'$. Since $G$ is transitive on $V$, we have that $H = \langle \{(\gamma, \iota^{-1}\gamma\iota) : \gamma \in G\}, \beta \rangle$ is a transitive subgroup of $\mathrm{Aut}_{\mathcal{I} \cup \mathcal{I}^\iota}(X \blacklozenge X')$ and $V$ is a block of imprimitivity for $H$. Since $\varphi \in \mathrm{Ant}_{\mathcal{I}}(X)$ implies $\psi \in \mathrm{Ant}_{\mathcal{I} \cup \mathcal{I}^\iota}(X \blacklozenge X')$ as seen above, we conclude that $X \blacklozenge X'$ is $(\mathcal{I} \cup \mathcal{I}^\iota)$-homogeneously almost self-complementary with the desired properties. ■

Self-complementary vertex-transitive graphs have received a considerable amount of attention in the last two decades. For example, Rao [12] showed that there exists a vertex-transitive self-complementary graph of order $n$ for every positive odd integer $n$ satisfying the condition $C(n)$, where

$$C(n): \quad \text{if } p^\ell \text{ is the highest power of an odd prime } p \text{ dividing } n, \text{ then } p^\ell \equiv 1 \pmod 4.$$

(Note that this condition is defined for both odd and even positive integers.) Muzychuk [9] later showed that Rao's condition is also necessary. With the aid of Rao's result we shall construct our two infinite families of homogeneously almost self-complementary graphs, thus proving the following.

**Theorem 5.3** *There exists a homogeneously almost self-complementary graph of order $2n$ for every positive integer $n$ satisfying $C(n)$.*

**Construction 5.4** Let $n = 2^k m$, where $k \geq 0$ is an integer and $m \geq 1$ is an odd integer satisfying $C(m)$. Then, by [12], there exists a vertex-transitive self-complementary graph $X$ of order $m$. Let $Y_0 = X \wr {K_2}^{\mathrm{C}}$, where we take $V({K_2}^{\mathrm{C}}) = \{1,2\}$. Then $(U_1, U_2)$ for $U_i = \{(u,i) : u \in V(X)\}$, $i \in \{1,2\}$, is an ordered bipartition of $V(Y_0)$. Let $\delta$ be the non-trivial automorphism of ${K_2}^{\mathrm{C}}$ and $\psi$ any antimorphism of $X$. Then $\alpha = (1, \delta, \delta, \ldots, \delta)$ and $\varphi = (\psi, 1, 1, \ldots, 1)$ are an $\mathcal{I}$-fair automorphism and $\mathcal{I}$-fair antimorphism of $Y_0$ for the perfect matching $\mathcal{I} = \{\{(u,1),(u,2)\} : u \in V(X)\}$, and $U_1^{\alpha} = U_2$ whereas $U_1^{\varphi} = U_1$. Moreover, for every $\gamma \in \mathrm{Aut}(X)$, the permutation $\gamma' = (\gamma, 1, 1, \ldots, 1)$ is an $\mathcal{I}$-fair automorphism of $Y_0$ fixing $U_1$ set-wise, whence $\langle\{\gamma' : \gamma \in \mathrm{Aut}(X)\}, \alpha\rangle = \mathrm{Aut}(X) \wr \mathrm{Sym}_2$ is a transitive subgroup of $\mathrm{Aut}_{\mathcal{I}}(Y_0)$ admitting $U_1$ as a block of imprimitivity. Hence $Y_0$ satisfies all conditions of the last paragraph of Lemma 5.2, whence the partial join $Y_1 = Y_0 \blacklozenge Y_0'$ with respect to $(U_1, U_2)$ and $(U_1', U_2')$, where $Y_0$ is an isomorphic copy of $Y_0$ with the isomorphic bipartition $(U_1', U_2')$, will satisfy the same conditions. We can therefore recursively define $Y_i = Y_{i-1} \blacklozenge Y_{i-1}'$, where for $i = 2, \ldots, k$, the graph $Y_{i-1}'$ is an isomorphic copy of $Y_{i-1}$, and the partial join is taken with respect to the natural bipartition $(V(Y_{i-2}), V(Y_{i-2}'))$ of $V(Y_{i-1})$ and the isomorphic bipartition of $V(Y_{i-1}')$. By Lemma 5.2, each of these graphs satisfies the conditions of the last paragraph of Lemma 5.2, whence $Y_k$ is a homogeneously almost self-complementary graph of order $2n = 2^{k+1}m$.

It is natural to ask if the sufficient condition on the order of a homogeneously almost self-complementary graph in Theorem 5.3 is also necessary. Our extended study of homogeneously almost self-complementary graphs [10, 11] answers this question in the negative way while indicating that a complete classification of the orders of homogeneously almost self-complementary graphs is a difficult problem. We mention that a homogeneously almost self-complementary graph of order $2(p^k + 1)$ can be constructed for every prime $p$ and positive integer $k$ such that $p^k \equiv 1 \pmod 4$, and of course, $p^k + 1$ need not satisfy the condition $C(p^k + 1)$.

In our second construction, described in Theorem 5.5 below, new homogeneously almost self-complementary graphs will be obtained from old via lexicographic products with vertex-transitive self-complementary graphs. Since $2K_2$ is both homogeneously almost self-complementary and unfairly almost self-complementary, this theorem together with Theorem 3.8 and Rao's result [12] mentioned above shows existence of an infinite family of graphs that are both homogeneously and unfairly almost self-complementary.

**Theorem 5.5** *Let $X$ be a vertex-transitive self-complementary graph and $Y$ an almost self-complementary graph. If $Y$ is homogeneously almost self-complementary, then so is $X \wr Y$. Conversely, if $\mathrm{Aut}(X \wr Y) = \mathrm{Aut}(X) \wr \mathrm{Aut}(Y)$ and $X \wr Y$ is homogeneously almost self-complementary, then $Y$ is homogeneously almost self-complementary.*

Proof. If $Y$ is almost self-complementary with respect to a perfect matching $\mathcal{I}$, let $\mathcal{I}^*$ denote the perfect matching ${K_{V(X)}}^{\mathrm{C}} \wr \mathcal{I}$ in $(X \wr Y)^{\mathrm{C}}$. It follows from Theorem 3.8 that $X \wr Y$ is almost self-complementary with respect to $\mathcal{I}^*$, that $\mathrm{Ant}(X) \wr \mathrm{Ant}_{\mathcal{I}}(Y) \subseteq \mathrm{Ant}_{\mathcal{I}^*}(Y)$, and $\mathrm{Aut}(X) \wr \mathrm{Aut}_{\mathcal{I}}(Y) \le \mathrm{Aut}_{\mathcal{I}^*}(X \wr Y)$, and of course, the wreath product of transitive permutation groups is transitive. Hence, if $Y$ is $\mathcal{I}$-homogeneously almost self-complementary, then $X \wr Y$ is $\mathcal{I}^*$-homogeneously almost self-complementary.

If $\mathrm{Aut}(X \wr Y) = \mathrm{Aut}(X) \wr \mathrm{Aut}(Y)$, then the proof of Theorem 3.8 shows that $\mathrm{Ant}(X) \wr \mathrm{Ant}_{\mathcal{I}}(Y) = \mathrm{Ant}_{\mathcal{I}^*}(Y)$ and $\mathrm{Aut}(X) \wr \mathrm{Aut}_{\mathcal{I}}(Y) = \mathrm{Aut}_{\mathcal{I}^*}(X \wr Y)$. Hence, if $X \wr Y$ is $\mathcal{I}^*$-homogeneously almost self-complementary, then $Y$ is $\mathcal{I}$-homogeneously almost self-complementary. ■

# 6 Non-cyclically almost self-complementary circulant graphs

Recall that a graph $X$ is called *circulant* if $\mathrm{Aut}(X)$ contains a cyclic group acting regularly on the vertex set; that is, if $X$ is isomorphic to a Cayley graph on a cyclic group.

As we have mentioned in Section 1, almost self-complementary graphs were first studied in [3]. The paper focuses on circulant graphs, in particular, cyclically almost self-complementary circulants. A circulant graph $X$ is defined to be *cyclically almost self-complementary* with respect to a perfect matching $\mathcal{I}$ in $X^{\mathrm{C}}$ if $X$ is almost self-complementary with respect to $\mathcal{I}$ and the group $\mathrm{Aut}_{\mathcal{I}}(X)$ of $\mathcal{I}$-fair automorphisms of $X$ contains a regular cyclic subgroup. A circulant is called non-cyclically almost self-complementary with respect to $\mathcal{I}$ if it is almost self-complementary but not cyclically almost self-complementary with respect to $\mathcal{I}$. Finally, a circulant is *cyclically (non-cyclically) almost self-complementary* if it is cyclically (non-cyclically) almost self-complementary with respect to some perfect matching $\mathcal{I}$ in $X^{\mathrm{C}}$. In [3], the cycle of length 6 is given as the only known example of a non-cyclically almost self-complementary circulant, and a question is posed about possible existence of others.

We first remark that the authors of [3] have overlooked the smallest non-cyclically almost self-complementary circulant, namely, $2K_2$. In our last result, we construct infinite families of non-cyclically almost self-complementary circulants from the two smallest examples $2K_2$ and $C_6$, and from self-complementary circulants via lexicographic products.

**Theorem 6.1** *Let $X$ be a self-complementary circulant of order $m$ and $Y$ an almost self-complementary circulant of order $2n$. If $m$ and $n$ are relatively prime, then the graph $X \wr Y$ is an almost self-complementary circulant of order $2mn$.*

*Moreover, if $p|n$ for some prime $p \not\equiv 1 \pmod 4$, then $X \wr Y$ is a non-cyclically almost self-complementary circulant.*

Proof. From Theorem 3.8 it follows that $X \wr Y$ is an almost self-complementary graph. It just remains to show that it is circulant. Let $G$ and $H$ be regular cyclic subgroups of $\mathrm{Aut}(X)$ and $\mathrm{Aut}(Y)$, respectively. By Lemma 3.7, the graph $X \wr Y$ admits a regular action of the diagonal group $\Delta(G \wr H)$. Since $m$ is odd and relatively prime to $n$, the group $\Delta(G \wr H)$, isomorphic to $G \times H$, is cyclic of order $2mn$.

By [3], a cyclically almost self-complementary circulant of order $2k$ exists if and only if every prime divisor of $k$ is congruent to 1 modulo 4. Hence, if $p|n$ for some prime $p \not\equiv 1 \pmod 4$, then $X \wr Y$ can not be cyclically almost self-complementary, and the last statement of the theorem follows. ■

**Corollary 6.2** *Let $n$ be any positive integer all of whose prime divisors are congruent to 1 modulo 4. There exist non-cyclically almost self-complementary circulants of orders $4n$ and $6n$.*

PROOF. [1, 6] show that a self-complementary circulant $X$ of order $n$ exists if and only if every prime divisor of $n$ is congruent to 1 modulo 4. The corollary then follows immediately from Theorem 6.1 by taking $Y = 2K_2$ and $Y = C_6$, respectively. ∎

The results of [3] show that every cyclically almost self-complementary circulant is a homogeneously almost self-complementary graph, while we observe that non-cyclically almost self-complementary circulants can be either homogeneously almost self-complementary or not. Since $2K_2$ is both non-cyclically and homogeneously almost self-complementary, the circulants of order $4n$ from Corollary 6.2 are all non-cyclically and homogeneously almost self-complementary by Theorem 5.5. On the other hand, since $\mathrm{Aut}(X \wr C_6) = \mathrm{Aut}(X) \wr \mathrm{Aut}(C_6)$ for infinitely many self-complementary circulants $X$ (for example, self-complementary circulants of prime order) and since $C_6$ is non-cyclically but not homogeneously almost self-complementary, by Theorem 5.5 and Corollary 6.2, the cycle $C_6$ gives rise to an infinite family of non-cyclically almost self-complementary circulants that are not homogeneously almost self-complementary.

## 7 Conclusion and open problems

As an introduction to general almost self-complementary graphs, this paper naturally raises more questions than it answers. We list some of the open questions below.

Some of the almost self-complementary graphs $X$ from Construction 4.2 (the case $n \equiv 3 \pmod 4$) have the property that for some perfect matching $\mathcal{I}$ in $X^{\mathrm{C}}$, $\mathrm{Aut}_{\mathcal{I}}(X)$ is trivial and $\mathrm{Ant}_{\mathcal{I}}(X) = \emptyset$; that is, the automorphism group of the isomorphic factorization $\{X, \mathrm{AC}_{\mathcal{I}}(X)\}$ is trivial. However, these graphs could be almost self-complementary with respect to another perfect matching $\mathcal{I}'$ in $X^{\mathrm{C}}$ such that $\mathrm{Aut}_{\mathcal{I}'}(X) \cup \mathrm{Ant}_{\mathcal{I}'}(X) \neq \{\mathrm{id}\}$. Hence the following problem.

**Problem 7.1** Does there exist an almost self-complementary graph such that the automorphism groups of all corresponding isomorphic factorizations act trivially on $V(X)$?

Non-cyclically almost-complementary circulants seem to be particularly difficult to find. The only "generic" examples that we have at the moment are $2K_2$ and $C_6$; all other known examples arise from these two by a wreath product construction.

**Problem 7.2** Does there exist a non-cyclically almost self-complementary circulant graph that is not obtained by the construction of Corollary 6.2? In particular, does there exist a non-cyclically almost self-complementary circulant graph of order twice a prime distinct from $2K_2$ and $C_6$?

No known non-cyclically almost self-complementary circulant (see Theorem 6.1) is at the same time cyclically almost self-complementary. Conversely, it is clear from the definition that a circulant $X$ which is cyclically almost self-complementary with respect to a (necessarily unique) perfect matching $\mathcal{I}$ in $X^{\mathrm{C}}$ can not be at the same time non-cyclically almost self-complementary with respect to $\mathcal{I}$. But could it be non-cyclically almost self-complementary with respect to a perfect matching $\mathcal{I}' \neq \mathcal{I}$?

**Problem 7.3** Does there exist a circulant that is both cyclically and non-cyclically almost self-complementary?

In [10], the following problem was addressed for orders $4p$ and $2p^k$, where $p$ is a prime. Partial results proved there indicate that most likely a complete solution will not be obtained in an elementary way (as is the case for vertex-transitive self-complementary graphs – see [9]) and without extensive use of the Classification of Finite Simple Groups.

**Problem 7.4** Determine all possible orders of homogeneously almost self-complementary graphs.

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