European Journal of Combinatorics: PDF for review

Journal	European Journal of Combinatorics
Article ID	YEUJC_171
Title	Self-complementary two-graphs and almost self-complementary double covers
Version	1
Article type	Full-length article
Submitted	26 Oct 05

Files submitted

Name	Fig No	Format	Use	Description
Potocnik_Sajna.tex		Manuscript (Ascii text)		

Self-complementary two-graphs and almost self-complementary double covers

Primož Potočnik
Department of Mathematics and Mechanics, University of Ljubljana
Jadranska 19, 1000 Ljubljana, Slovenia

Mateja Šajna*
Department of Mathematics and Statistics, University of Ottawa
585 King Edward Avenue, Ottawa, ON, K1N 6N5, Canada

October 31, 2005

Abstract

A graph X is called almost self-complementary with respect to a perfect matching $\mathcal{I}$ if it is isomorphic to the graph obtained from its complement X^{c} by removing the edges of $\mathcal{I}$. A two-graph on a vertex set Ω is a collection $\mathcal{T}$ of 3-subsets of Ω such that each 4-subset of Ω contains an even number of elements of $\mathcal{T}$. In this paper we investigate the relationship between self-complementary two-graphs and double covers over complete graphs that are almost self-complementary with respect to a set of fibres. In particular, we classify all doubly transitive self-complementary two-graphs, and thus all almost self-complementary graphs with an automorphism group acting 2-transitively on the corresponding perfect matching.

Keywords: Almost self-complementary graph, homogeneously almost self-complementary graph, self-complementary two-graph, double cover, covering projection, lifting automorphisms.

1 Introduction

A graph X is called *almost self-complementary* if it is isomorphic to a graph obtained from its complement X^{c} by removing the edges of a perfect matching $\mathcal{I}$ of X^{c}. In this case we say that X is almost self-complementary with respect to $\mathcal{I}$, and use the symbol $\mathrm{Aut}_{\mathcal{I}}(X)$ to denote the largest subgroup of $\mathrm{Aut}(X)$ that preserves $\mathcal{I}$ setwise. Almost self-complementary graphs were introduced by Alspach in the 1990s, first studied by Dobson and the second author in [7], and systematically investigated by the present authors in [13, 14]. The initial goal of the research leading to this article was to provide examples and analyze the structure

*This author gratefully acknowledges support by the Natural Sciences and Engineering Research Council of Canada.

of almost self-complementary graphs that occur as 2-fold covers over the complete graphs. As it turns out, such graphs are closely related to a special class of combinatorial objects called two-graphs. A *two-graph* on a vertex set Ω is a collection $\mathcal{T}$ of 3-subsets of Ω with the property that every 4-subset of Ω contains an even number of elements of $\mathcal{T}$ as subsets. Hence a two-graph is not a graph, but rather can be thought of as a special kind of a 3-uniform hypergraph. Two-graphs were introduced by Higman, and later studied by Taylor [15, 16]. Taylor's classification of doubly transitive two-graphs [16] allowed us to determine all graphs X that are almost self-complementary with respect to some perfect matching $\mathcal{I}$ for which $\mathrm{Aut}_{\mathcal{I}}(X)$ acts doubly transitively on $\mathcal{I}$. These graphs are constructed as follows.

Construction 1 Let q be an odd prime power congruent to 1 modulo 4, $\mathbb{F}$ a finite field of cardinality q, and $\Omega = \mathbb{F} \cup \{\infty\}$. We define K_{q+1}^2 to be the graph with vertex set $\Omega \times \mathbb{Z}_2$ and edge set consisting of all edges of the form $\{(\infty, i), (x, i)\}$ for all $i \in \mathbb{Z}_2$ and $x \in \mathbb{F}$; $\{(x, i), (y, i)\}$ for all $i \in \mathbb{Z}_2$ and $x, y \in \mathbb{F}$ such that $y - x$ is a square in $\mathbb{F}^*$; and $\{(x, i), (y, i + 1)\}$ for all $i \in \mathbb{Z}_2$ and $x, y \in \mathbb{F}$ such that $y - x$ is a non-square in $\mathbb{F}^*$.

With this construction in mind, the main result of the paper can be stated as follows.

Theorem 2 *A graph X on $2n$ vertices is almost self-complementary with respect to a perfect matching $\mathcal{I}$ in X^{C} such that $\mathrm{Aut}_{\mathcal{I}}(X)$ acts 2-transitively on $\mathcal{I}$ if and only if $n = q+1$ for some prime power q congruent to 1 modulo 4 and X is isomorphic to the graph K_{q+1}^2 defined in Construction 1.*

This result was one of the crucial steps in the classification of all homogeneusly almost self-complementary graphs on $4p$ vertices, where p is a prime, presented in [14].

This article is organized as follows. In Section 2 we shall present the necessary background on almost self-complementary graphs, graph covers, and two-graphs, and in Section 3 some preliminary observations on almost self-complementary double covers. In Section 4 we shall discuss two-graphs and their relationship with double covers over complete graphs, and in particular, self-complementary two-graphs and their relationship with almost self-complementary double covers. The paper concludes with a classification of doubly transitive self-complementary two-graphs and a proof of the main result, Theorem 2.

2 Preliminaries

In this section we shall review the necessary background and terminology on almost self-complementary graphs, graph covers, and two-graphs.

2.1 Almost self-complementary graphs

Throughout this paper, let $V^{(k)}$ denote the set of all k-subsets of a finite set V. By a *graph* we shall mean a finite undirected simple graph; that is, an ordered pair (V, E), where the vertex set V is finite and the edge set E is a subset of $V^{(2)}$. The *dart set* of a graph (V, E) is the set of ordered pairs $(u, v) \in V^2$ such that $\{u, v\} \in E$. For a graph X, the vertex set, the edge set, and the dart set of X will be denoted by V_X, E_X, and D_X. The adjacency relation in a graph X will be denoted by $\sim_X$, or simply by $\sim$ if the graph X is clear from the context. The set of all vertices adjacent to a vertex v in X (that is, the *neighbourhood* of v) is denoted by $X(v)$. We use the symbol K_V to denote the *complete graph* with vertex set V.

A partition of a set V into subsets of size 2 is called a *perfect matching on* V. If X is a graph and $\mathcal{I}$ a perfect matching on V_X disjoint from E_X, then the *almost complement* $\mathrm{AC}_{\mathcal{I}}(X)$ *of* X *with respect to* $\mathcal{I}$ is defined as the graph $(V_X, {V_X}^{(2)} \setminus (E_X \cup \mathcal{I}))$. If X is isomorphic to $\mathrm{AC}_{\mathcal{I}}(X)$, then we say that X is *almost self-complementary with respect to* $\mathcal{I}$. An isomorphism from X to $\mathrm{AC}_{\mathcal{I}}(X)$ is called an $\mathcal{I}$*-antimorphism* of X. An $\mathcal{I}$*-fair antimorphism* of X is an $\mathcal{I}$-antimorphism of X that preserves the perfect matching $\mathcal{I}$ setwise; the set of all $\mathcal{I}$-fair antimorphisms of X is denoted by $\mathrm{Ant}_{\mathcal{I}}(X)$. Similarly, an automorphism of X preserving a perfect matching $\mathcal{I}$ setwise is called $\mathcal{I}$*-fair*, and the group of all $\mathcal{I}$-fair automorphisms of X is denoted by $\mathrm{Aut}_{\mathcal{I}}(X)$.

A graph X with at least four vertices that is almost self-complementary with respect to $\mathcal{I}$ is said to be *homogeneously almost self-complementary with respect to* $\mathcal{I}$ if it admits an $\mathcal{I}$-fair antimorphism and $\mathrm{Aut}_{\mathcal{I}}(X)$ acts transitively on V_X. Homogeneously almost self-complementary graphs are equivalent to index-2 homogeneous factorizations of the complete multipartite graph with all parts of size two. The reader is referred to [8] for more information on homogeneous factorizations.

2.2 Graph covers

Our main tool for the study of the relationship between almost self-complementary graphs and two-graphs is the theory of lifting automorphisms along regular covering projections, which was developed in [10] and upgraded for a special case of elementary abelian graph covers in [11]. In this section, we shall summarize and adapt for our purposes the relevant results from [10, 11].

In what follows, let X and $\tilde{X}$ be graphs. If $\wp\colon \tilde{X} \to X$ is a graph homomorphism and $v \in V_X$, then the preimage $\wp^{-1}(v)$ is called a $\wp$*-fibre*, and the set of all $\wp$-fibres is denoted by $\mathcal{F}_{\wp}$. A surjective homomorphism $\wp\colon \tilde{X} \to X$ is called a *covering projection* if the restriction of $\wp$ to $\tilde{X}(\tilde{v})$ is a bijection from $\tilde{X}(\tilde{v})$ to $X(\wp(\tilde{v}))$ for every $\tilde{v} \in V_{\tilde{X}}$. A covering projection $\wp$ with all $\wp$-fibres of size r is called r*-fold*.

Let $\wp_1\colon \tilde{X}_1 \to X_1$ and $\wp_2\colon \tilde{X}_2 \to X_2$ be covering projections, and let $\alpha\colon X_1 \to X_2$ be a graph isomorphism. If there exists an isomorphism $\tilde{\alpha}\colon \tilde{X}_1 \to \tilde{X}_2$ such that $\wp_2 \circ \tilde{\alpha} = \alpha \circ \wp_1$, then we say that α *lifts along* $(\wp_1, \wp_2)$, that $\tilde{\alpha}$ is a *lift of* α *along* $(\wp_1, \wp_2)$, and that $(\alpha, \tilde{\alpha})\colon \wp_1 \to \wp_2$ is an *isomorphism of covering projections.* In particular, if $\wp_1 = \wp_2 = \wp$ (and hence α is an automorphism of $\tilde{X}_1 = \tilde{X}_2$), then we say that α *lifts along* $\wp$ and that $\tilde{\alpha}$ is a *lift of* α *along* $\wp$. If G is a subgroup of $\mathrm{Aut}(X)$, then the set of all lifts along $\wp$ of the elements in G is a group called the *lift of* G *along* $\wp$. The lift of the trivial group $\{\mathrm{id}_X\}$ is denoted by $\mathrm{CT}(\wp)$ and called the *group of covering transformations* of $\wp$. If $\wp_1\colon \tilde{X}_1 \to X$ and $\wp_2\colon \tilde{X}_2 \to X$ are covering projections onto the same graph, then isomorphisms of the form $(\mathrm{id}_X, \tilde{\alpha})\colon \wp_1 \to \wp_2$ are called *equivalences of covering projections.* Two covering projections are called *isomorphic* (*equivalent*) if there exists an isomorphism (equivalence) between them. It is not difficult to see that isomorphism and equivalence of covering projections are equivalence relations. The following lemma follows directly from the above definitions, hence the proof is omitted.

Lemma 3 *For each $i \in \{1,2\}$, let $\wp_i\colon \tilde{X}_i \to X_i$ and $\wp'_i\colon \tilde{X}'_i \to X_i$ be equivalent covering projections. Then a graph isomorphism $\alpha\colon X_1 \to X_2$ lifts along $(\wp_1, \wp_2)$ if and only if it lifts along $(\wp'_1, \wp'_2)$. In other words, $\wp_1$ and $\wp_2$ are isomorphic if and only if $\wp'_1$ and $\wp'_2$ are isomorphic.*

A $\mathbb{Z}_2$-*voltage assignment* on X is a mapping $\zeta\colon D_X \to \mathbb{Z}_2$ satisfying the condition $\zeta(v,u) = \zeta(u,v)$ for every dart $(u,v) \in D_X$. The *derived cover* $\tilde{X}_\zeta = \mathrm{Cov}(X;\zeta)$ is the graph with vertex set $V_X \times \mathbb{Z}_2$, where $(u,i) \sim_{\tilde{X}_\zeta} (v,j)$ if and only if $u \sim_X v$ and $j = i + \zeta(u,v)$. The *derived covering projection associated with* ζ is the covering projection $\wp_\zeta\colon \tilde{X}_\zeta \to X$ defined by $\wp_\zeta(u,i) = u$ for every $(u,i) \in V_X \times \mathbb{Z}_2$. We say that voltage assignments $\zeta_1, \zeta_2\colon D_X \to \mathbb{Z}_2$ are *isomorphic* (*equivalent*) if the derived covering projections $\wp_{\zeta_1}$ and $\wp_{\zeta_2}$ are isomorphic (equivalent).

In the remainder of this subsection, we shall assume that X is a connected graph.The following is easy to verify (or see [10]).

Proposition 4 *If T is a spanning tree of a graph X and $\wp\colon \tilde{X} \to X$ is a 2-fold covering projection, then there exists a $\mathbb{Z}_2$-voltage assignment ζ on X such that $\zeta(u,v) = 0$ for all darts (u,v) of T and the derived covering projection $\wp_\zeta$ is equivalent to $\wp$.*

One of the most important tools used repeatedly in this paper is Corollary 5 below — a simple criterion for an automorphism $\alpha \in \mathrm{Aut}(X)$ to have a lift along a 2-fold covering projection $\wp\colon \tilde{X} \to X$. We encourage the interested reader to look up this result in [11], where it is developed in the more general context of elementary abelian covering projections.

For any set S, let $\mathbb{Z}_2 S$ denote the set of all formal sums $\sum_{x \in S} c_x x$ with coefficients c_x from $\mathbb{Z}_2$. Thus $\mathbb{Z}_2 S$ is a $\mathbb{Z}_2$-module. Let $\partial\colon \mathbb{Z}_2 D_X \to \mathbb{Z}_2 V_X$ denote the $\mathbb{Z}_2$-linear transformation induced by the rule $\partial(u,v) = v - u$ for every dart $(u,v) \in D_X$. The kernel of ∂, denoted by $H_1(X;\mathbb{Z}_2)$, is called the *first homology group of X with coefficients in* $\mathbb{Z}_2$, and the elements of $H_1(X;\mathbb{Z}_2)$ are called *cycles* of X. Every $\mathbb{Z}_2$-voltage assignment ζ extends uniquely to a $\mathbb{Z}_2$-linear mapping from $\mathbb{Z}_2 D_X$ to $\mathbb{Z}_2$, and therefore also to its restriction $\zeta^*\colon H_1(X;\mathbb{Z}_2) \to \mathbb{Z}_2$ on $\mathrm{Ker}(\partial) = H_1(X;\mathbb{Z}_2)$. Similarly, every automorphism $\alpha \in \mathrm{Aut}(X)$ induces a permutation on D_X, which extends uniquely to a $\mathbb{Z}_2$-linear mapping $\alpha^*\colon H_1(X;\mathbb{Z}_2) \to \mathbb{Z}_2 D_X$. Note that the image of α^* is $H_1(X;\mathbb{Z}_2)$, whence α^* can be viewed as an automorphism of $H_1(X;\mathbb{Z}_2)$.

The following can be deduced from [11, Theorem 3.2] with the aid of the discussion preceding [11, Proposition 5.1].

Corollary 5 *Let X be a connected graph. For $i = 1,2$, let $\zeta_i\colon D_X \to \mathbb{Z}_2$ be a voltage assignment on X with the derived covering projection $\wp_{\zeta_i}$ and the unique $\mathbb{Z}_2$-linear extension ζ_i^* to $H_1(X;\mathbb{Z}_2)$. For an automorphism $\alpha \in \mathrm{Aut}(X)$, let α^* denote the unique $\mathbb{Z}_2$-linear extension of α (as a permutation on D_X) to $H_1(X;\mathbb{Z}_2)$. Then α lifts along $(\wp_{\zeta_1}, \wp_{\zeta_2})$ if and only if $\zeta_1^* = \zeta_2^* \circ \alpha^*$.*

2.3 Two-graphs

In [15] Taylor writes that "regular 2-graphs were introduced by Professor G. Higman in his Oxford lectures as a means of studying Conway's sporadic simple group ·3 in its doubly transitive representation of degree 276." Subsequently, two-graphs were extensively studied by Taylor in [15, 16] and, in particular, all two-graphs with a 2-transitive group of automorphisms were classified. Later, two-graphs were considered mostly in the context of antipodal distance-regular graphs (see, for example, [1, 9, 17]).

To convey the charming flavour of algebraically influenced combinatorics in the 1970s we shall adopt the terminology of [15]. A *two-graph* on a vertex set Ω is a function $\Phi\colon \Omega^{(3)} \to \mathbb{Z}_2$ that satisfies the rule

$$\Phi(\{v,w,z\}) + \Phi(\{u,w,z\}) + \Phi(\{u,v,z\}) + \Phi(\{u,v,w\}) = 0 \qquad (1)$$

for every quadruple of pairwise distinct vertices $u, v, w, z \in \Omega$. The size of the vertex set Ω is called the *order* of Φ. Note that Condition (1) for a function $\Phi\colon \Omega^{(3)} \to \mathbb{Z}_2$ is equivalent to the requirement that each element of $\Omega^{(4)}$ contain an even number of elements $C \in \Omega^{(3)}$ with $\Phi(C) = 1$, called *coherent triples* of Φ. The set of coherent triples of Φ will be denoted by $\mathcal{T}_\Phi$.

A bijection α between vertex sets of two-graphs Φ_1 and Φ_2 is an *isomorphism of two-graphs* if it induces a bijection of the sets of coherent triples $\mathcal{T}_{\Phi_1}$ and $\mathcal{T}_{\Phi_2}$; that is, if $\Phi_2 \circ \alpha = \Phi_1$. The group of automorphisms of a two-graph Φ (that is, isomorphisms from Φ to Φ) will be denoted by the usual symbol $\mathrm{Aut}(\Phi)$. A two-graph Φ is called *vertex-transitive* and *doubly transitive* if $\mathrm{Aut}(\Phi)$ acts transitively and 2-transitively, respectively, on the vertex set Ω. A two-graph Φ is called *regular* if the number of coherent triples of Φ containing two distinct vertices $\omega, \omega' \in \Omega$ is independent of the choice of ω and ω'. It is easy to see that every doubly transitive two-graph is regular.

The *complement* of a two-graph $\Phi\colon \Omega^{(3)} \to \mathbb{Z}_2$ is the two-graph $\Phi^{\mathrm{C}}\colon \Omega^{(3)} \to \mathbb{Z}_2$ defined by

$$\Phi^{\mathrm{C}}(C) = 1 + \Phi(C) \qquad \text{for every } C \in \Omega^{(3)}, \qquad (2)$$

where the addition is performed modulo 2. A two-graph Φ is said to be *self-complementary* if there exists an isomorphism $\varphi\colon \Phi \to \Phi^{\mathrm{C}}$. Lemmas 6 and 7 below will be used in the proof of Theorem 19 to determine which doubly transitive two-graphs are self-complementary.

Lemma 6 *If Φ is a regular self-complementary two-graph of order n, then $n \equiv 2 \pmod 4$.*

Proof. Let $r_\Phi(v)$ and $\lambda_\Phi(e)$ denote the number of coherent triples of Φ containing $v \in \Omega$ and $e \in \Omega^{(2)}$, respectively. For any two-graph Φ of order n we have $r_\Phi(v) + r_{\Phi^{\mathrm{C}}}(v) = \frac{1}{2}(n-1)(n-2)$ and $\lambda_\Phi(e) + \lambda_{\Phi^{\mathrm{C}}}(e) = n-2$. If Φ is regular, then by definition λ_Φ is a constant, and, as it is not difficult to see, so is r_Φ.

Since Φ is self-complementary two-graph, we have that $|\mathcal{T}_\Phi| = |\mathcal{T}_{\Phi^{\mathrm{C}}}|$ and that $|\Omega^{(3)}| = \frac{1}{6}n(n-1)(n-2)$ is an even number, whence n is congruent to 0, 1, or 2 modulo 4. Since, in addition, Φ is regular, we also have $\lambda_\Phi = \lambda_{\Phi^{\mathrm{C}}}$ and $r_\Phi = r_{\Phi^{\mathrm{C}}}$, whence by the above, $\lambda_\Phi = \frac{1}{2}(n-2)$ and $r_\Phi = \frac{1}{4}(n-1)(n-2)$. The conclusion follows easily. ■

Lemma 7 *If Φ is a self-complementary two-graph on a set Ω, then the index of $\mathrm{Aut}(\Phi)$ in its normalizer in $\mathrm{Sym}(\Omega)$ is even.*

Proof. If Φ is a self-complementary two-graph on a set Ω, then clearly $\varphi^2 \in \mathrm{Aut}(\Phi)$ and $\varphi \mathrm{Aut}(\Phi)\varphi^{-1} = \mathrm{Aut}(\Phi)$ for every isomorphism φ from Φ to Φ^{C}. Hence $\mathrm{Aut}(\Phi)$ is a normal subgroup of index 2 in $\langle \mathrm{Aut}(\Phi), \varphi\rangle$, and the statement of the proposition follows. ■

3 Almost self-complementary double covers

A *double cover over a complete graph* is a graph Y isomorphic to $\mathrm{Cov}(K_n;\zeta)$ for some $\mathbb{Z}_2$-voltage assignment ζ and complete graph K_n. A *set of fibres* of the double cover Y is a perfect matching $\mathcal{F}$ of Y^{C} that is mapped to the set of $\wp_\zeta$-fibres by some isomorphism $Y \to \mathrm{Cov}(K_n;\zeta)$. Many double covers over complete graphs are almost self-complementary graphs (see [14]), however, they need not be almost self-complementary with respect to a set of fibres; an example of such a graph is the 6-cycle. To distinguish between the two cases, we shall reserve the term *almost self-complementary double cover* for double covers over complete graphs that are almost self-complementary with respect to a set of fibres. (Note that if a graph is almost self-complementary and a double cover over some graph, then this graph is necessarily complete.) Similarly, we define a *homogeneously almost self-complementary double cover* as a graph Y that is a double cover over a complete graph with a set of fibres $\mathcal{F}$, and is almost self-complementary with respect to $\mathcal{F}$ so that $\mathrm{Aut}_{\mathcal{F}}(Y)$ acts transitively on the vertex set. As we shall see in Corollary 9, any double cover that is almost self-complementary with respect to a set of fibres $\mathcal{F}$ admits a fair $\mathcal{F}$-antimorphism, whence a homogeneously almost self-complementary double cover is necessarily a homogeneously almost self-complementary graph.

The following lemma will be crucial for establishing a correspondence between double covers over complete graphs and two-graphs. It shows that an isomorphism of covering graphs $\mathrm{Cov}(K_\Omega;\zeta_1)$ and $\mathrm{Cov}(K_\Omega;\zeta_2)$ implies the existence of an isomorphism of these graphs that induces an isomorphism of the two covering projections. But first observe that if ζ_1, ζ_2 are $\mathbb{Z}_2$-voltage assignments on a graph X, then $\wp_{\zeta_1}$ and $\wp_{\zeta_2}$, viewed as functions from $V_X \times \mathbb{Z}_2$ to V_X, coincide. In particular, they have the same set of fibres $\mathcal{F} = \mathcal{F}_{\wp_{\zeta_1}} = \mathcal{F}_{\wp_{\zeta_2}}$.

Lemma 8 *Let Ω be a finite set, $\zeta_1, \zeta_2 \colon D_{K_\Omega} \to \mathbb{Z}_2$ voltage assignments on K_Ω, and $\mathcal{F}$ the set of fibres of $\wp_{\zeta_1}$ and therefore also of $\wp_{\zeta_2}$. Then the following hold.*

(i) *An isomorphism $\tilde{\alpha} : \mathrm{Cov}(K_\Omega;\zeta_1) \to \mathrm{Cov}(K_\Omega;\zeta_2)$ preserves $\mathcal{F}$ if and only if it is a lift along $(\wp_{\zeta_1}, \wp_{\zeta_2})$ of some permutation α on Ω.*

(ii) *If $\zeta_1 = \zeta_2 = \zeta$, then the group $\tilde{G} = \mathrm{Aut}_{\mathcal{F}}(\mathrm{Cov}(K_\Omega;\zeta))$ of $\mathcal{F}$-preserving automorphisms of $\mathrm{Cov}(K_\Omega;\zeta)$ is the lift of the largest subgroup $\mathcal{L}(\zeta)$ of $\mathrm{Sym}(\Omega)$ that lifts along $\wp_\zeta$, and its permutation representation $\tilde{G}^{\mathcal{F}}$ (in its action on $\mathcal{F}$) is permutation-isomorphic to $\mathcal{L}(\zeta)$.*

(iii) *Covering graphs $\mathrm{Cov}(K_\Omega;\zeta_1)$ and $\mathrm{Cov}(K_\Omega;\zeta_2)$ are isomorphic if and only if they are isomorphic via an $\mathcal{F}$-preserving isomorphism, that is, if and only if p_{ζ_1} and p_{ζ_2} are isomorphic covering projections.*

Proof. Statement (i) is easy to see: a permutation $\tilde{\alpha} \in \mathrm{Sym}(\Omega\times\mathbb{Z}_2)$ preserves $\mathcal{F}$ if and only if there exists a permutation $\alpha \in \mathrm{Sym}(\Omega)$ such that $\wp_{\zeta_2} \circ \tilde{\alpha} = \alpha \circ \wp_{\zeta_1}$. Moreover, such a permutation α (if it exists) is uniquely determined by $\tilde{\alpha}$, and the mapping $\wp^*\colon \tilde{\alpha} \mapsto \alpha$ is a group homomorphism from the largest subgroup of $\mathrm{Sym}(\Omega \times \mathbb{Z}_2)$ preserving $\mathcal{F}$ to $\mathrm{Sym}(\Omega)$.

Assuming $\zeta_1 = \zeta_2 = \zeta$, it then easily follows that $\tilde{G}$ is the lift of $\mathcal{L}(\zeta)$. Moreover, the image and the kernel of the restriction $\wp^*|_{\tilde{G}}$ of the group homomorphism $\wp^*$ to $\tilde{G}$ are $\mathcal{L}(\zeta)$ and $\mathrm{CT}(\wp_\zeta)$, respectively. Hence, $\wp^*|_{\tilde{G}}$ induces an isomorphism of permutation groups $\tilde{G}^{\mathcal{F}}$ and $\mathcal{L}(\zeta)$, and (ii) follows.

To prove the "only if" part of (iii), assume that $Y_1 = \mathrm{Cov}(K_\Omega; \zeta_1)$ and $Y_2 = \mathrm{Cov}(K_\Omega; \zeta_2)$ are isomorphic graphs. Let $\tilde{\alpha} : Y_1 \to Y_2$ be an isomorphism with the smallest number of fibres in $\mathcal{F}$ not mapped to fibres in $\mathcal{F}$. Suppose this number is positive. Then there exist pairwise distinct vertices $v, u, w \in \Omega\times\mathbb{Z}_2$ such that $\{u, v\} \in \mathcal{F}$ and $\{\tilde{\alpha}(u), \tilde{\alpha}(w)\} \in \mathcal{F}$. Since $\{u, v\}$ is a fibre in a double cover over a complete graph, every vertex in $\Omega \times \mathbb{Z}_2 \setminus \{u, v\}$ is adjacent to either u or v in Y_1. Similarly, every vertex in $\Omega \times \mathbb{Z}_2 \setminus \{\tilde{\alpha}(u), \tilde{\alpha}(w)\}$ is adjacent to either $\tilde{\alpha}(u)$ or $\tilde{\alpha}(w)$ in Y_2, whence every vertex in $\Omega \times \mathbb{Z}_2 \setminus \{u, w\}$ is adjacent to either u or w in Y_1. It follows that v and w have exactly the same neighbours in Y_1 that lie in the set $\Omega \times \mathbb{Z}_2 \setminus \{v, w\}$, and hence the permutation $\tilde{\beta}$ on $\Omega \times \mathbb{Z}_2$ that swaps v with w and leaves all other vertices fixed is an automorphism of Y_1. From this it follows that $\tilde{\alpha}\tilde{\beta}$ is an isomorphism from Y_1 to Y_2 that contradicts the assumption on $\tilde{\alpha}$. Hence $\tilde{\alpha}$ preserves $\mathcal{F}$. From (i) it then follows that $\tilde{\alpha}$ is a lift of some permutation α on Ω and therefore that p_{ζ_1} and p_{ζ_2} are isomorphic covering projections. The "if" part of (iii) is obvious. ■

Corollary 9 *Let X be an almost self-complementary double cover, that is, a double cover over a complete graph that is almost self-complementary with respect to a set of fibres $\mathcal{F}$. Then X admits a fair $\mathcal{F}$-antimorphism.*

Proof. It is easy to see that the almost complement of $X = \mathrm{Cov}(K_\Omega; \zeta)$ with respect to the set $\mathcal{F}$ of $\wp_\zeta$-fibres is $\mathrm{Cov}(K_\Omega; 1 + \zeta)$. The rest follows immediately from Part (iii) of Lemma 8. ■

The next lemma shows that the observations of Lemma 8 can be simplified further in the case of double covers over complete graphs with the property that the group of automorphisms preserving the set $\mathcal{F}$ of fibres acts 2-transitively on $\mathcal{F}$.

Lemma 10 *Let X be a connected double cover over a complete graph and $\mathcal{F}$ a set of fibres of X. If $\mathrm{Aut}_{\mathcal{F}}(X)$ acts 2-transitively on $\mathcal{F}$, then $\mathcal{F}$ is the unique set of fibres of X and hence $\mathrm{Aut}(X) = \mathrm{Aut}_{\mathcal{F}}(X)$.*

Proof. Suppose, on the contrary, that $\mathcal{F}'$ is a set of fibres of the double cover X distinct from $\mathcal{F}$. Then there exist vertices $u, v, v' \in V_X$ such that $\{u, v\} \in \mathcal{F} \setminus \mathcal{F}'$ and $\{u, v'\} \in \mathcal{F}' \setminus \mathcal{F}$. Hence both v and v' are at distance at least 3 from u in the graph X. Let $u' \in V_X$ be such that $\{u', v'\} \in \mathcal{F}$. Then $u \sim_X u'$, and since $\mathrm{Aut}_{\mathcal{F}}(X)$ acts 2-transitively on $\mathcal{F}$, every fibre in $\mathcal{F}$ other than $\{u, v\}$ contains a vertex at distance 1 and a vertex at distance at least 3 from u. But then X contains no vertex at distance 2 from u, a contradiction. Hence $\mathcal{F}$ is the unique set of fibres of X and clearly $\mathrm{Aut}(X) = \mathrm{Aut}_{\mathcal{F}}(X)$. ■

4 Two-graphs and double covers over complete graphs

Let **Volt**, **DCov**, and **TGrph** denote the sets of all $\mathbb{Z}_2$-voltage assignments over complete graphs, double covers over complete graphs, and two-graphs, respectively. Let $[\zeta]$, $[X]$, and $[\varphi]$ denote the isomorphism classes of $\zeta \in \mathbf{Volt}$, $X \in \mathbf{DCov}$, and $\Phi \in \mathbf{TGrph}$, respectively. Furthermore, let $[\mathbf{Volt}] = \{[\zeta] : \zeta \in \mathbf{Volt}\}$, $[\mathbf{DCov}] = \{[X] : X \in \mathbf{DCov}\}$, and $[\mathbf{TGrph}] = \{[\Phi] : \Phi \in \mathbf{TGrph}\}$ be the corresponding sets of isomorphism classes on **Volt**, **DCov**, and **TGrph**, respectively. Part (iii) of Lemma 8 shows that there is a bijection between the set [**DCov**] of isomorphism classes of double covers over complete graphs and the set [**Volt**] of isomorphism classes of voltage assignments over complete graphs. In this section we shall establish a bijection between [**Volt**] and [**TGrph**], and therefore between the set [**DCov**] of isomorphism classes of double covers over complete graphs and the set [**TGrph**] of isomorphism classes of two-graphs.

We begin by defining a mapping $T : \mathbf{Volt} \to \mathbf{TGrph}$ that will give rise to a bijection $[\mathbf{Volt}] \to [\mathbf{TGrph}]$. For a voltage assignment $\zeta \colon D_{K_\Omega} \to \mathbb{Z}_2$, let $\Phi_\zeta \colon \Omega^{(3)} \to \mathbb{Z}_2$ be the mapping defined by the rule

$$\Phi_\zeta(\{u, v, w\}) = \zeta(u, v) + \zeta(v, w) + \zeta(w, u) \tag{3}$$

for every triple of pairwise distinct elements $u, v, w \in \Omega$. Observe that Φ_ζ is indeed a well-defined function. A straightforward calculation shows that Φ_ζ satisfies (1), whence it is a two-graph. Let $T : \mathbf{Volt} \to \mathbf{TGrph}$ be defined by $T(\zeta) = \Phi_\zeta$. First we show that T is surjective.

Let Φ be a two-graph on a set Ω. For any vertex $\omega \in \Omega$ we define a voltage assignment $\zeta_{\omega,\Phi} : D_{K_\Omega} \to \mathbb{Z}_2$ by

$$\zeta_{\omega,\Phi}(u, v) = \begin{cases} \Phi(\{u, v, \omega\}) & \text{if } \omega \notin \{u, v\} \\ 0 & \text{if } \omega \in \{u, v\}. \end{cases} \tag{4}$$

An easy calculation shows the following.

Lemma 11 *For any two-graph Φ on a set Ω and any $\omega \in \Omega$ we have that $\Phi_{\zeta_{\omega,\Phi}} = \Phi$. Consequently, the mapping $T : \mathbf{Volt} \to \mathbf{TGrph}$ defined by $T(\zeta) = \Phi_\zeta$ is surjective.*

Next we show that T maps equivalent voltage assignments to the same two-graph, and isomorphic voltage assignments to isomorphic two-graphs.

Lemma 12 *For a set Ω, let $\zeta_1, \zeta_2 \colon D_{K_\Omega} \to \mathbb{Z}_2$ be voltage assignments, and let α be an automorphism of K_Ω. Then α is an isomorphism from Φ_{ζ_1} to Φ_{ζ_2} if and only if it lifts along the pair of covering projections $(\wp_{\zeta_1}, \wp_{\zeta_2})$.*

Proof. For an arbitrary element $\{u, v, w\}$ of $\Omega^{(3)}$ consider the cycle $C = (u, v) + (v, w) + (w, u)$ in the first homology group $H_1(K_\Omega; \mathbb{Z}_2)$. Recall that ζ_i^* is the unique $\mathbb{Z}_2$-linear extension of ζ_i to $H_1(K_\Omega; \mathbb{Z}_2)$. By the definition (3) of Φ_{ζ_1} and Φ_{ζ_2} we have that $\zeta_1^*(C) = \Phi_{\zeta_1}(\{u, v, w\})$ and $(\zeta_2^* \circ \alpha^*)(C) = \Phi_{\zeta_2}(\alpha(\{u, v, w\}))$. Hence ζ_1^* and $\zeta_2^* \circ \alpha^*$ coincide on the set $\mathcal{C}_3$ of 3-cycles in $H_1(K_\Omega; \mathbb{Z}_2)$ if and only if α is an isomorphism from Φ_{ζ_1} to Φ_{ζ_2}.

Since $\mathcal{C}_3$ spans $H_1(K_\Omega;\mathbb{Z}_2)$, α is an isomorphism from Φ_{ζ_1} to Φ_{ζ_2} if and only if $\zeta_1^* = \zeta_2^* \circ \alpha^*$ (on $H_1(K_\Omega;\mathbb{Z}_2)$), and by Corollary 5, the latter statement holds if and only if α lifts along the pair of covering projections $(\wp_{\zeta_1}, \wp_{\zeta_2})$. ■

Corollary 13 *If $\zeta_1, \zeta_2 \colon D_{K_\Omega} \to \mathbb{Z}_2$ are voltage assignments, then Φ_{ζ_1} and Φ_{ζ_2} are isomorphic two-graphs if and only if ζ_1 and ζ_2 are isomorphic voltage assignments, and $\Phi_{\zeta_1} = \Phi_{\zeta_2}$ if and only if ζ_1 and ζ_2 are equivalent voltage assignments.*

Proof. The first statement of the corollary follows directly from Lemma 12. If $\Phi_{\zeta_1} = \Phi_{\zeta_2}$, then id_Ω is an isomorphism between Φ_{ζ_1} and Φ_{ζ_2}. By Lemma 12, id_Ω lifts along $(\wp_{\zeta_1}, \wp_{\zeta_2})$, whence ζ_1 and ζ_2 are equivalent. Conversely, if ζ_1 and ζ_2 are equivalent, then id_Ω lifts along $(\wp_{\zeta_1}, \wp_{\zeta_2})$, and by By Lemma 12, id_Ω is an isomorphism between Φ_{ζ_1} and Φ_{ζ_2}. Therefore, $\Phi_{\zeta_1} = \Phi_{\zeta_2}$. ■

We mention that the last statement of Corollary 13 has been observed previously in a different form; see for example [3, 12].

It is now clear that T : **Volt** $\to$ **TGrph** induces a bijection from the set of equivalence classes of $\mathbb{Z}_2$-voltage assignments over complete graphs to the set of all two-graphs, and also a bijection from the set [**Volt**] of isomorphism classes of $\mathbb{Z}_2$-voltage assignments over complete graphs to the set [**TGrph**] of isomorphism classes of two-graphs.

Moreover, since any double cover over a complete graph is isomorphic to a derived cover $\mathrm{Cov}(K_\Omega;\zeta)$ for some voltage assignment $\zeta \colon D_{K_\Omega} \to \mathbb{Z}_2$, and since by Lemma 8, derived covers $\mathrm{Cov}(K_\Omega;\zeta_1)$ and $\mathrm{Cov}(K_\Omega;\zeta_2)$ are isomorphic if and only if the voltage assignments ζ_1 and ζ_2 are isomorphic, the mapping T also implies a bijection from [**DCov**] to [**TGrph**], and therefore from [**TGrph**] to [**DCov**]. The next theorem explains this more precisely.

Theorem 14 *For any two-graph Φ on a set Ω choose a vertex $\omega \in \Omega$, and let* $\mathrm{F}(\Omega) = \mathrm{Cov}(K_\Omega;\zeta_{\omega,\Phi})$. *Then* F : **TGrph** $\to$ **DCov** *induces a bijection* $\underline{\mathrm{F}}$: [**TGrph**] $\to$ [**DCov**]. *Moreover, if Φ is a two-graph on a set Ω, ω a fixed vertex in Ω, and $\mathcal{F}$ the set of $\wp_{\zeta_{\omega,\Phi}}$-fibres of the double cover* $\mathrm{F}(\Phi) = \mathrm{Cov}(K_\Omega;\zeta_{\omega,\Phi})$*, then the permutation representation* $\mathrm{Aut}_{\mathcal{F}}(\mathrm{F}(\Phi))^{\mathcal{F}}$ *of the group* $\mathrm{Aut}_{\mathcal{F}}(\mathrm{F}(\Phi))$ *of $\mathcal{F}$-preserving automorphisms of* $\mathrm{F}(\Phi)$ *(in its action on $\mathcal{F}$) is permutation-isomorphic to* $\mathrm{Aut}(\Phi)$.

Proof. Let Φ_1 and Φ_2 be two-graphs on a set Ω, and ω_1, ω_2 fixed vertices in Ω. By Lemma 11, $\Phi_i = \Phi_{\zeta_{\omega_i,\Phi_i}}$, and hence Corollary 13 tells us that two-graphs Φ_1 and Φ_2 are isomorphic if and only if voltage assignments ζ_{ω_1,Φ_1} and ζ_{ω_2,Φ_2} are isomorphic. However, by Part (iii) of Lemma 8, voltage assignments ζ_{ω_1,Φ_1} and ζ_{ω_2,Φ_2} are isomorphic if and only if the derived covering graphs $\mathrm{Cov}(K_\Omega;\zeta_{\omega_1,\Phi_1})$ and $\mathrm{Cov}(K_\Omega;\zeta_{\omega_2,\Phi_2})$ are isomorphic. Thus two-graphs Φ_1 and Φ_2 are isomorphic if and only if the derived covering graphs $\mathrm{Cov}(K_\Omega;\zeta_{\omega_1,\Phi_1})$ and $\mathrm{Cov}(K_\Omega;\zeta_{\omega_2,\Phi_2})$ are isomorphic, implying both that $\underline{\mathrm{F}}$ is a well-defined mapping from [**TGrph**] to [**DCov**] and that it is injective.

Let $X \in$ **DCov** be any double cover over a complete graph. Then there exist a set Ω and a voltage assignment $\zeta : D_{K_\Omega} \to \mathbb{Z}_2$ such that $X \cong \mathrm{Cov}(K_\Omega;\zeta)$. Let $\Phi = \Phi_\zeta$ and fix some vertex $\omega \in \Omega$. Lemma 11 shows that $\Phi_\zeta = \Phi = \Phi_{\zeta_{\omega,\Phi}}$, and so voltage assignments

ζ and $\zeta_{\omega,\Phi}$ are equivalent by Corollary 13. But then X and $\mathrm{Cov}(K_\Omega;\zeta_{\omega,\Phi})$ are isomorphic double covers and $\underline{\mathrm{F}}([\Phi]) = [X]$, implying that $\underline{\mathrm{F}} : [\mathbf{TGrph}] \to [\mathbf{DCov}]$ is also surjective.

Let Φ be a two-graph on a set Ω, ω a fixed vertex in Ω, and $\zeta = \zeta_{\omega,\Phi}$. Note that $\Phi = \Phi_\zeta$ by Lemma 11, and hence by Lemma 12, a permutation $\alpha \in \mathrm{Sym}(\Omega)$ is an automorphism of Φ if and only if it lifts along $\wp_\zeta$. Thus $\mathrm{Aut}(\Phi)$ is the largest subgroup $\mathcal{L}(\zeta)$ of $\mathrm{Sym}(\Omega)$ that lifts along $\wp_\zeta$. Let $\mathrm{F}(\Phi) = \mathrm{Cov}(K_\Omega;\zeta)$ and let $\mathcal{F}$ be the set of $\wp_\zeta$-fibres. Now by Lemma 8, the group $\mathrm{Aut}_{\mathcal{F}}(\mathrm{F}(\Phi))$ of $\mathcal{F}$-preserving automorphisms of $\mathrm{F}(\Phi)$ is the lift of $\mathcal{L}(\zeta)$ and its permutation representation $\mathrm{Aut}_{\mathcal{F}}(\mathrm{F}(\Phi))^{\mathcal{F}}$ (acting on the set of fibres $\mathcal{F}$) is permutation isomorphic to $\mathcal{L}(\zeta) = \mathrm{Aut}(\Phi)$. ■

The next lemma will be crucial in showing that the mapping F from Theorem 14 induces a bijection between the isomorphism classes of self-complementary two-graphs and isomorphism classes of almost self-complementary double covers (Theorem 16). But first observe that for a two-graph Φ, a set Ω, and any $\omega \in \Omega$, definition (4) of the voltage assignment $\zeta_{\omega,\Phi} : D_{K_\Omega} \to \mathbb{Z}_2$ implies that

$$\zeta_{\omega,\Phi^{\mathrm{C}}}(u,v) = \begin{cases} 1 + \zeta_{\omega,\Phi}(u,v) & \text{if } \omega \notin \{u,v\} \\ \zeta_{\omega,\Phi}(u,v) & \text{if } \omega \in \{u,v\}. \end{cases} \tag{5}$$

Lemma 15 *Let Φ be a two-graph on a set Ω and $\omega \in \Omega$ a fixed vertex. Furthermore, let ζ and ζ^{C} be the voltage assignments $D_{K_\Omega} \to \mathbb{Z}_2$ induced by Φ and Φ^{C}, respectively; that is, $\zeta = \zeta_{\omega,\Phi}$ and $\zeta^{\mathrm{C}} = \zeta_{\omega,\Phi^{\mathrm{C}}}$. Finally, let $\mathcal{F}$ be the set of fibres of the covering projection $\wp_\zeta$. Then the following statements are equivalent:*

(i) Φ *is a self-complementary two-graph;*

(ii) $\wp_\zeta$ *and* $\wp_{\zeta^{\mathrm{C}}}$ *are isomorphic covering projections;*

(iii) $\wp_\zeta$ *and* $\wp_{1+\zeta}$ *are isomorphic covering projections;*

(iv) $\mathrm{Cov}(K_\Omega;\zeta)$ *is an almost self-complementary double cover, that is, almost self-complementary with respect to* $\mathcal{F}$*.*

Proof. (i) ⇔ (ii): Recall that, by Lemma 11, $\Phi = \Phi_\zeta$ and $\Phi^{\mathrm{C}} = \Phi_{\zeta^{\mathrm{C}}}$. If (i) holds, then there exists an isomorphism α between Φ and Φ^{C}. By Lemma 12, α lifts along $(\wp_\zeta, \wp_{\zeta^{\mathrm{C}}})$, whence (ii) follows. Conversely, if (ii) holds, then there exists $\alpha \in \mathrm{Sym}(\Omega)$ that lifts along $(\wp_\zeta, \wp_{\zeta^{\mathrm{C}}})$, and hence by Lemma 12, α is an isomorphism between Φ and Φ^{C}.

(ii) ⇔ (iii): Let $\tilde{\beta}$ be the transposition on $\Omega \times \mathbb{Z}_2$ interchanging $(\omega, 0)$ with $(\omega, 1)$ and fixing all other vertices. Routine calculation using observation (5) shows that $\tilde{\beta}$ is an isomorphism from $\mathrm{Cov}(K_\Omega;\zeta^{\mathrm{C}})$ to $\mathrm{Cov}(K_\Omega; 1+\zeta)$, and clearly $\wp_{1+\zeta} \circ \tilde{\beta} = \wp_{\zeta^{\mathrm{C}}}$, whence $(\mathrm{id}, \tilde{\beta})$ is an equivalence between $\wp_{\zeta^{\mathrm{C}}}$ and $\wp_{1+\zeta}$. Hence, by Lemma 3, $\wp_\zeta$ is isomorphic to $\wp_{\zeta^{\mathrm{C}}}$ if and only if it is isomorphic to $\wp_{1+\zeta}$.

(iii) ⇔ (iv): Observe that the almost complement of $\mathrm{Cov}(\Omega;\zeta)$ with respect to the perfect matching $\mathcal{F}$ is $\mathrm{Cov}(\Omega; 1+\zeta)$. Now, by Lemma 8, $\mathrm{Cov}(\Omega;\zeta)$ and $\mathrm{Cov}(\Omega; 1+\zeta)$ are isomorphic graphs if and only if $\wp_\zeta$ and $\wp_{1+\zeta}$ are isomorphic covering projections. ■

Lemma 15 allows us to extend Theorem 14 to the following result.

Theorem 16 *The bijection* $\underline{\mathrm{F}}\colon [\mathbf{TGrph}] \to [\mathbf{DCov}]$ *defined by* $\underline{\mathrm{F}}([\Phi]) = [\mathrm{Cov}(K_\Omega; \zeta_{\omega,\Phi})]$ *for a fixed* ω *in the vertex set* Ω *of the two graph* Φ *induces a bijection between:*

(i) *the set of isomorphism classes of self-complementary two-graphs and the set of isomorphism classes of almost self-complementary double covers;*

(ii) *the set of isomorphism classes of vertex-transitive self-complementary two-graphs and the set of isomorphism classes of homogeneously almost self-complementary double covers.*

Proof. Equipped with Theorem 14, all we need to prove is the following two statements:

(i') Φ is a self-complementary two-graph if and only if $\mathrm{F}(\Phi)$ is an almost self-complementary double cover.

(ii') Φ is a vertex-transitive self-complementary two-graph if and only if $\mathrm{F}(\Phi)$ is a homogeneously almost self-complementary double cover.

(i') and therefore (i) follows directly from Lemma 15. To see (ii'), recall from Theorem 14 that $\mathrm{Aut}(\Phi)$ is permutation isomorphic to $\mathrm{Aut}_{\mathcal{F}}(\mathrm{F}(\Phi))^{\mathcal{F}}$, where $\mathcal{F}$ is the set of $p_{\zeta_{\omega,\Phi}}$-fibres of $\mathrm{F}(\Phi) = \mathrm{Cov}(K_\Omega; \zeta_{\omega,\Phi})$. Since $\mathrm{F}(\Phi)$ is a double cover, it is not difficult to see that $\mathrm{Aut}_{\mathcal{F}}(\mathrm{F}(\Phi))^{\mathcal{F}}$ is transitive (on $\mathcal{F}$) if and only if $\mathrm{Aut}_{\mathcal{F}}(\mathrm{F}(\Phi))$ is transitive (on the vertex set of $\mathrm{F}(\Phi)$). Hence $\mathrm{Aut}(\Phi)$ is transitive if and only if $\mathrm{Aut}_{\mathcal{F}}(\mathrm{F}(\Phi))$ is transitive, and so by (i'), Φ is a vertex-transitive self-complementary two-graph if and only if $\mathrm{F}(\Phi)$ is a homogeneously almost self-complementary double cover. ∎

5 Doubly transitive self-complementary two-graphs and proof of Theorem 2

In this section we shall prove the main result of this paper, Theorem 2, by showing that there exists, up to isomorphism, a unique doubly transitive two-graph of every given admissible order (Theorem 19). These two-graphs arise from the graphs K^2_{q+1} defined in Construction 1, which are essentially due to Taylor [15, Example 6.2]. In the following lemma we shall prove the essential properties that will show that the graphs K^2_{q+1} are homogeneously almost self-complementary with the group of fair automorphisms acting 2-transitively on a corresponding perfect matching. The lemma also gives an alternative construction of the graphs K^2_{q+1} as derived covers.

Lemma 17 *Let* $q \equiv 1 \pmod 4$ *be a prime power,* $\mathbb{F}$ *a finite field of cardinality* q*, and* $\Omega = \mathbb{F} \cup \{\infty\}$*. Furthermore, let* ζ *be the* $\mathbb{Z}_2$*-voltage assignment on* K_Ω *defined by*

$$\zeta(x,y) = \begin{cases} 0 & \text{if } \infty \in \{x,y\} \\ 0 & \text{if } x,y \in \mathbb{F} \text{ and } x-y \text{ is a square in } \mathbb{F}^* \\ 1 & \text{if } x,y \in \mathbb{F} \text{ and } x-y \text{ is a non-square in } \mathbb{F}^* \end{cases}.$$

Then the following hold.

(i) *The derived cover* $\mathrm{Cov}(K_\Omega, \zeta)$ *is isomorphic to the graph* K^2_{q+1} *defined in Construction 1.*

(ii) *The covering projections* $\wp_\zeta$ *and* $\wp_{1+\zeta}$ *are isomorphic.*

(iii) *The 2-transitive group* $\mathrm{P\Sigma L}(2,q)$ *is the largest subgroup of* $\mathrm{Aut}(K_\Omega)$ *that lifts along* $\wp_\zeta$.

Proof. Statement (i) is easy to see and its proof is left to the reader. Let $q = p^k$, where p is a prime. Furthermore, let $\mathbb{SF}$ and $\mathbb{NF}$ denote the sets of all squares and all non-squares, respectively, in the multiplicative group $\mathbb{F}^*$. To prove (ii) and (iii), we define permutations ι, σ, α_t (for all $t \in \mathbb{F}^*$), and β_c (for all $c \in \mathbb{F}$) on Ω as follows:

- $\iota(0) = \infty$, $\iota(\infty) = 0$, and $\iota(x) = -x^{-1}$ for all $x \in \mathbb{F}^*$;
- $\sigma(\infty) = \infty$ and $\sigma(x) = x^p$ for all $x \in \mathbb{F}$;
- $\alpha_t(\infty) = \infty$ and $\alpha_t(x) = tx$ for all $x \in \mathbb{F}$; and
- $\beta_c(\infty) = \infty$ and $\beta_c(x) = x + c$ for all $x \in \mathbb{F}$.

Let r be a generator of the group $\mathbb{F}^*$. We shall first prove the following:

- Permutations ι, σ, α_{r^2}, and β_c (for every $c \in \mathbb{F}$) lift along $\wp_\zeta$.
- Permutation α_r does not lift along $\wp_\zeta$, however, it lifts along the pair of covering projections $(\wp_{\zeta'}, \wp_\zeta)$, where $\zeta' = 1 + \zeta$.

Let T be the spanning tree (in fact, a star) of K_Ω that contains all edges incident with the vertex ∞. Furthermore, let ϵ_T be an orientation of $X - E_T$, that is, a function $E_X \setminus E_T \to D_X \setminus D_T$ satisfying $\epsilon_T(\{u,v\}) \in \{(u,v),(v,u)\}$ for every edge $\{u,v\} \in E_X \setminus E_T$. As shown in [11], for every $e \in E_X \setminus E_T$ there exists a unique cycle $C_e \in H_1(X;\mathbb{Z}_2)$ of the form $\epsilon_T(e) + \sum_{x \in D_T} a_x x$, and the set $\mathcal{B} = \{C_e : e \in E_X \setminus E_T\}$ is a basis for the $\mathbb{Z}_2$-module $H_1(X;\mathbb{Z}_2)$. By Corollary 5, an automorphism $\varphi \in \mathrm{Aut}(K_\Omega)$ lifts along $\wp_\zeta$ if and only if $\zeta^* \circ \varphi^* = \zeta^*$, where ζ^* and φ^* are the unique $\mathbb{Z}_2$-linear extensions of ζ and φ, respectively, to the $\mathbb{Z}_2$-module $H_1(K_\Omega;\mathbb{Z}_2)$. Clearly, it suffices to check the validity of this equality on the elements of $\mathcal{B}$, which are all possible cycles of the form $(\infty,u) + (u,v) + (v,\infty)$ for $u,v \in \Omega \setminus \{\infty\}$. Therefore, $\varphi \in \mathrm{Aut}(K_\Omega)$ lifts along $\wp_\zeta$ if and only if

$$\zeta(\varphi(\infty),\varphi(u)) + \zeta(\varphi(u),\varphi(v)) + \zeta(\varphi(v),\varphi(\infty)) = \zeta(u,v) \tag{6}$$

for every pair of distinct vertices $u, v \in \Omega \setminus \{\infty\}$.

By (6), the permutation ι lifts along $\wp_\zeta$ if and only if $\zeta(0,-x^{-1}) + \zeta(-x^{-1},-y^{-1}) + \zeta(-y^{-1},0) = \zeta(x,y)$ for every pair of distinct elements $x,y \in \mathbb{F}^*$, and $\zeta(-y^{-1},0) = \zeta(0,y)$ for every $y \in \mathbb{F}^*$. To see that the latter condition holds, observe that $y \in \mathbb{SF}$ if and only if $-y^{-1} \in \mathbb{SF}$. To check the former condition, observe that for $x,y \in \mathbb{F}^*$ we have that $\zeta(-x^{-1},-y^{-1}) = 0$ if and only if $-y^{-1} - (-x^{-1}) = -\frac{x-y}{xy} \in \mathbb{SF}$, which is true if and only if $x - y$ and xy are either both squares or both non-squares. Next, observe that xy is a square if and only if $\zeta(0,x) + \zeta(0,y) = 0$. Therefore, $\zeta(-x^{-1},-y^{-1}) = 0$ if and only if $\zeta(x,y) + \zeta(0,x) + \zeta(0,y) = 0$, and hence $\zeta(-x^{-1},-y^{-1}) = \zeta(x,y) + \zeta(0,x) + \zeta(0,y)$.

But $\zeta(0,x) = -\zeta(0,-x^{-1})$ and $\zeta(0,y) = \zeta(y,0) = -\zeta(-y^{-1},0)$, so indeed, $\zeta(0,-x^{-1}) + \zeta(-x^{-1},-y^{-1}) + \zeta(-y^{-1},0) = \zeta(x,y)$. Therefore, ι lifts along $\wp_\zeta$.

If $\varphi \in \{\sigma, \alpha_r, \alpha_{r^2}\}$ or $\varphi = \beta_c$ for $c \in \mathbb{F}$, then $\infty^\varphi = \infty$, and (6) is equivalent to the condition $\varphi(u) - \varphi(v) \in \mathbb{SF} \Leftrightarrow u - v \in \mathbb{SF}$. This condition is clearly fulfilled for $\varphi = \beta_c$ (where $c \in \mathbb{F}$), $\varphi = \sigma$, and $\varphi = \alpha_{r^2}$. Hence these automorphisms of K_Ω lift along $\wp_\zeta$.

Finally, let $\varphi = \alpha_r$. For $C = (\infty,x) + (x,y) + (y,\infty) \in \mathcal{B}$ we have $\zeta^*(\varphi^*(C)) = \zeta(rx,ry) \neq \zeta(x,y)$ since r is a non-square. On the other hand, $\zeta'^*(C) = (1+\zeta(\infty,x)) + (1+\zeta(x,y)) + (1+\zeta(y,\infty)) = 1+\zeta(x,y)$. Since multiplication by r interchanges the sets $\mathbb{SF}$ and $\mathbb{NF}$, $\zeta(rx,ry) = 1+\zeta(x,y)$ for every pair $\{x,y\} \subseteq \mathbb{F}$. Hence $\zeta^* \circ \varphi^* = \zeta'^*$, and thus by Corollary 5, the permutation α_r lifts along $(\wp_{\zeta'}, \wp_\zeta)$, but it does not lift along $\wp_\zeta$. In particular, we have shown (ii), that is, that $\wp_\zeta$ and $\wp_{\zeta'} = \wp_{1+\zeta}$ are isomorphic covering projections.

To prove Statement (iii) of the lemma, observe first that the group generated by the permutations $\iota, \sigma, \alpha_{r^2}$, and β_c (for all $c \in \mathbb{F}$) is the 2-transitive group $\mathrm{P\Sigma L}(2,p^k)$ acting on the points of the projective line $\Omega = \mathrm{PG}(1,p^k)$. Now, let H be the largest subgroup of $\mathrm{Aut}(K_\Omega) = \mathrm{Sym}(\Omega)$ that lifts along $\wp_\zeta$. Since it contains $\mathrm{P\Sigma L}(2,p^k)$, it acts 2-transitively on Ω. Moreover, since the degree $n = |\Omega|$ of H is congruent to 2 modulo 4 and $n-1$ is a prime power, the classification of 2-transitive permutation groups (see for example [4, Theorem 5.3(S)]) implies that the socle T of H is one of the following groups: the alternating group A_{1+p^k}, the projective special group $\mathrm{PSL}(2,p^k)$, or the projective special unitary group $\mathrm{PSU}(3,q)$ with $q = p^\ell$ and $3\ell = k$. If $T \cong A_{1+p^k}$, then H is 3-transitive, which contradicts the fact that H preserves setwise the set of triples $\{u,v,w\} \subseteq \Omega$ with $\zeta(u,v)+\zeta(v,w)+\zeta(w,u) = 0$. If $T \cong \mathrm{PSU}(3,q)$ with $q = p^\ell$ and $k = 3\ell$, then $H \leq \mathrm{P\Gamma U}(3,q)$, and therefore $|H| \leq \ell p^{3\ell}(p^{2\ell}-1)(p^{3\ell}+1)$ (see [5] or [6] for the orders of groups of Lie type). On the other hand, we have that $|H| \geq |\mathrm{P\Sigma L}(2,p^k)| = \frac{1}{2}3\ell p^{3\ell}(p^{3\ell}+1)(p^{3\ell}-1)$, implying that $3(p^{3\ell}-1) \leq 2(p^{2\ell}-1)$, a contradiction. Finally, if $T \cong \mathrm{PSL}(2,p^k)$, then H is a subgroup of $\mathrm{P\Gamma L}(2,p^k)$, which is the normalizer of $\mathrm{PSL}(2,p^k)$ in $\mathrm{Sym}(\Omega)$. However, the only subgroup of $\mathrm{P\Gamma L}(2,p^k)$ containing $\mathrm{P\Sigma L}(2,p^k)$ but not containing α_r is $\mathrm{P\Sigma L}(2,p^k)$, implying that $H \cong \mathrm{P\Sigma L}(2,p^k)$ as claimed. This concludes the proof of (iii). ■

Corollary 18 *Let $\mathbb{F}$ be a finite field of prime power order q congruent to 1 modulo 4, and let $\Omega = \mathbb{F} \cup \{\infty\}$. Let $\zeta : D_{K_\Omega} \to \mathbb{Z}_2$ and $K^2_{q+1} = \mathrm{Cov}(K_\Omega;\zeta)$ be the voltage assignment and the corresponding derived cover, respectively, as defined in Lemma 17. Then the two-graph Φ_ζ (as defined in Section 4) is a self-complementary doubly transitive two-graph and K^2_{q+1} is a homogeneously almost self-complementary double cover with $\mathrm{Aut}(K^2_{q+1}) = \mathrm{Aut}_{\mathcal{F}}(K^2_{q+1})$ acting 2-transitively on the set $\mathcal{F}$ of $\wp_\zeta$-fibres.*

Proof. A straightforward calculation shows that $\zeta = \zeta_{\infty,\Phi_\zeta}$, so Lemma 15 can be used for Φ_ζ and ζ. Indeed, since the covering projections $\wp_\zeta$ and $\wp_{1+\zeta}$ are isomorphic by Lemma 17, the two-graph Φ_ζ is self-complementary by Lemma 15. Moreover, since by Lemma 17 the 2-transitive group $\mathrm{P\Sigma L}(2,q)$ lifts along $\wp_\zeta$, Lemma 12 implies that $\mathrm{P\Sigma L}(2,q) \leq \mathrm{Aut}(\Phi_\zeta)$, and therefore the two-graph Φ_ζ is doubly transitive.

Since $K^2_{q+1} = \mathrm{Cov}(K_\Omega;\zeta_{\infty,\Phi_\zeta}) = \mathrm{F}(\Phi_\zeta)$, by Theorem 16, K^2_{q+1} is a homogeneously almost self-complementary double cover, and by Theorem 14, $\mathrm{Aut}_{\mathcal{F}}(K^2_{q+1})^{\mathcal{F}}$ is 2-transitive since $\mathrm{Aut}(\Phi_\zeta)$ is. Finally, since K^2_{q+1} is connected, it follows from Lemma 10 that $\mathrm{Aut}(K^2_{q+1}) = \mathrm{Aut}_{\mathcal{F}}(K^2_{q+1})$, as asserted. ■

Taylor [16] gives a complete list of doubly transitive two-graphs using the classification of 2-transitive permutation groups (based on the Classification of Finite Simple Groups). We shall now determine which two-graphs on this list are self-complementary.

Theorem 19 *A doubly transitive two-graph Φ is self-complementary if and only if it is isomorphic to the two-graph Φ_ζ for the voltage assignment ζ defined in Lemma 17.*

Proof. Since every doubly transitive two-graph is regular, a doubly transitive self-complementary two-graph will have order congruent to 2 modulo 4 by Lemma 6. By [16, Theorem 1], there are only two families of doubly transitive two-graphs of order congruent to 2 modulo 4 — these are the families arising in Subcases (i) and (ii) of Case (A) of [16, Theorem 1]. The family described in Subcase (ii) is associated with the natural 2-transitive action of the group $\mathrm{PSU}(3, q)$, where q is a prime power, on a set of size $q^3 + 1$. By [16, Theorem 2], the full automorphism group of a two-graph associated with $\mathrm{PSU}(3, q)$ for $q > 3$ is $\mathrm{P\Gamma U}(3, q)$, and in our case $q > 3$ since $q \equiv 1 \pmod 4$. However, the normalizer of the permutation group $\mathrm{P\Gamma U}(3, q)$ in the full symmetric group is just $\mathrm{P\Gamma U}(3, q)$ itself, whence by Lemma 7 the corresponding two-graphs can not be self-complementary. Therefore, the only candidates for self-complementary doubly transitive two-graphs are the two-graphs of the family arising from Subcase (i) of [16, Theorem 1] associated with the permutation groups $\mathrm{PSL}(2, q)$, where q is a prime power congruent to 1 modulo 4. By [16, Theorem 2], this family of two-graphs is unique; it is explicitly described in [15, Example 6.2], and is exactly the family of two-graphs Φ_ζ where ζ is the voltage assignment defined in Lemma 17. Lemma 18 then concludes the proof. ■

We are now ready to prove the main result of this paper.

Proof of Theorem 2. The "if" part is immediate: the graph K^2_{q+1} has the required property by Corollary 18.

To prove the "only if" part, let X be a graph on $2n$ vertices that is almost self-complementary with respect to a perfect matching $\mathcal{I}$ in X^{c}, and suppose $\mathrm{Aut}_{\mathcal{I}}(X)^{\mathcal{I}}$ is 2-transitive. By a *brick* we shall mean a subgraph of X induced by two elements of $\mathcal{I}$. Observe that all bricks of X are pairwise isomorphic, and are "symmetric" in the sense that there exists an automorphism of X swapping the two elements of $\mathcal{I}$ that are the bipartition sets of the brick. Moreover, since X has $n(n-1)$ edges and $\binom{n}{2}$ bricks, each brick must contain two edges and therefore be isomorphic to $2K_2$. Hence X is a double cover over the complete graph K_n and thus isomorphic to $\mathrm{Cov}(K_n; \zeta')$ for some voltage assignment $\zeta' : D_{K_n} \to \mathbb{Z}_2$. Moreover, there exists an isomorphism that maps $\mathcal{I}$ to the set $\mathcal{F}$ of $\wp_{\zeta'}$-fibres, and thus $\mathrm{Cov}(K_n; \zeta')$ is an almost self-complementary double cover.

Let ω be a fixed vertex of K_n and consider the spanning tree (in fact, a star) of K_n containing all edges incident with ω. By Proposition 4 there exists a voltage assignment ζ'' equivalent to ζ' such that $\zeta''(\omega, u) = 0$ for all $u \in V(K_n) \setminus \{\omega\}$. Then $\mathrm{Cov}(K_n; \zeta'')$ is also an almost self-complementary double cover isomorphic to X, and $\mathrm{Aut}_{\mathcal{F}}(\mathrm{Cov}(K_n;\zeta''))$ acts 2-transitively on the set $\mathcal{F}$ of $\wp_{\zeta''}$-fibres.

Let Φ be the two-graph $\Phi_{\zeta''}$. A straightforward calculation shows that $\zeta'' = \zeta_{\omega,\Phi}$, whence by Lemma 15 the two-graph Φ is self-complementary since $\mathrm{Cov}(K_n;\zeta'')$ is an almost self-complementary double cover. Moreover, since $\mathrm{Cov}(K_n;\zeta'') = \mathrm{F}(\Phi)$ and since $\mathrm{Aut}_{\mathcal{F}}(\mathrm{Cov}(K_n;\zeta''))^{\mathcal{F}}$ is 2-transitive, Φ is doubly transitive by Theorem 14. But then, by Lemma 18, Φ is isomorphic to Φ_ζ for the voltage assignment $\zeta : D_{K_\Omega} \to \mathbb{Z}_2$ defined in Lemma 17 for some prime power q congruent to 1 modulo 4 and $\Omega = \mathbb{F} \cup \{\infty\}$. But as $\zeta = \zeta_{\infty,\Phi_\zeta}$, the graphs $K^2_{q+1} = \mathrm{Cov}(K_\Omega;\zeta)$ and $\mathrm{Cov}(K_n;\zeta'')$ are isomorphic by Theorem 14. Hence $X \cong K^2_{q+1}$, and necessarily $n = q + 1$. ■

References

[1] A. E. Brouwer, A. M. Cohen, and A. Neumaier, *Distance-regular graphs*, Springer-Verlag, New York, 1998.

[2] C. J. Colbourn and A. Rosa, *Triple Systems*, Oxford Mathematical Monographs, Clarendon Press, Oxford, 1999.

[3] P. J. Cameron, Cohomological aspects of two-graphs, *Math. Zeitsch.* **157** (1977), 101–119.

[4] P. J. Cameron, Finite permutation groups and finite simple groups, *Bull. London Math. Soc.* **13** (1981), 1–22.

[5] P. J. Cameron, *Permutation Groups*, London Mathematical Society Student Texts **45**, Cambridge University Press, Cambridge, 1999.

[6] J. D. Dixon and B. Mortimer, *Permutation Groups*, Graduate Texts in Mathematics **163**, Springer-Verlag, New York, 1996.

[7] E. Dobson and M. Šajna, Almost self-complementary circulant graphs, *Discrete Math.* **278** (2004), 23–44.

[8] M. Giudici, C. H. Li, P. Potočnik, and C. E. Praeger, Homogeneous factorisations of complete multipartite graphs, *Discrete Math.*, to appear.

[9] C. D. Godsil, R. A. Liebler, and C. E. Praeger, Antipodal distance transitive covers of complete graphs, *European J. Combin.* **19** (1998), 455–478.

[10] A. Malnič, R. Nedela, and M. Škoviera, Lifting graph automorphisms by voltage assignments, *Europ. J. Combin.* **21** (2000), 927–947.

[11] A. Malnič, D. Marušič, and P. Potočnik, Elementary abelian covers of graphs, *J. Algebraic Combin.*, to appear.

[12] J. J. Seidel, Graphs and two-graphs, *Congressus Numerantium* **10** (1974), 125–143.

[13] P. Potočnik and M. Šajna, On almost self-complementary graphs, *Discrete Math.*, accepted.

[14] P. Potočnik and M. Šajna, Homogeneously almost self-complementary graphs, submitted.

[15] D. E. Taylor, Regular 2-graphs, *Proc. London Math. Soc.* (3) **35** (1977), 257–274.

[16] D. E. Taylor, Two-graphs and doubly transitive groups, *J. Combin. Theory Ser. A* **61** (1992), 113–122.

[17] D. E. Taylor and R. Levingston, Distance-regular graphs, *Combinatorial Mathematics*, Proc. Internat. Conf. Combinatorial Theory, Australian Nat. Univ., Canberra, 1977, pp. 313–323, Lecture Notes in Math. **686**, Springer, Berlin, 1978.