Regular self-complementary uniform hypergraphs

Primož Potočnik
Institute of Mathematics, Physics, and Mechanics
and Faculty of Mathematics and Physics, University of Ljubljana
Jadranska 19, 1000 Ljubljana, Slovenia

Mateja Šajna*
Department of Mathematics and Statistics, University of Ottawa
585 King Edward Avenue, Ottawa, ON K1N 6N5, Canada

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Abstract

A k-uniform hypergraph with vertex set V and edge set E is called t-subset-regular if every t-element subset of V lies in the same number of elements of E. In this paper we establish a necessary condition on n for there to exist a t-subset-regular self-complementary k-uniform hypergraph with n vertices. In addition, we show that this necessary condition is also sufficient in the case $k = 3$ and $t = 1$; that is, we show that a 1-subset-regular self-complementary 3-uniform hypergraph with n vertices exists if and only if $n \geq 5$ and n is congruent to 1 or 2 modulo 4.

Keywords: Self-complementary hypergraph, uniform hypergraph, regular hypergraph.

1 Introduction

It is well known that there exists a regular self-complementary graph with n vertices if and only if n is congruent to 1 modulo 4. The goal of this paper is to extend the above result to regular self-complementary uniform hypergraphs. In particular, in Theorem 2 below, we obtain a necessary condition on the order of a t-subset-regular self-complementary k-uniform hypergraph (that is, a self-complementary k-uniform hypergraph with the property that every t-subset of vertices lies in the same number of edges). Before stating the theorem, however, we need to introduce the following notation, used throughout the paper.

Notation 1 For a prime p and positive integers r and m, let $r_{[m]}$ denote the unique integer in $\{0, 1, \ldots, m-1\}$ such that $r \equiv r_{[m]} \pmod{m}$, and let $r_{(p)}$ denote the largest integer i such that p^i that divides r.

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Theorem 2 *Let X be a t-subset-regular self-complementary k-uniform hypergraph with n vertices. Then $n_{[2^q]} \in \{t, t+1, \ldots, k_{[2^q]} - 1\}$ for some q with $k_{(2)} < q \leq \min\{i : 2^i > k\}$.*

Observe that, as in the case of regular self-complementary graphs, the condition on the order n given in Theorem 2 involves the congruence class of n modulo the smallest power of 2 that exceeds the rank k. This result naturally leads to the following problem.

Question 3 *Is it true that the necessary condition on the order of a t-subset-regular self-complementary k-uniform hypergraph from Theorem 2 is also sufficient?*

As mentioned above, it is well known that the answer to Question 3 is affirmative for graphs, that is, for $k = 2$. In the second part of this paper we show that the answer is affirmative also in the case $k = 3$ and $t = 1$; that is, we prove the following theorem.

Theorem 4 *There exists a 1-subset-regular self-complementary 3-uniform hypergraph of order n if and only if $n \geq 5$ and n is congruent to 1 or 2 modulo 4 .*

Question 3 remains open for all other values of k and t, in particular, for $k = 3$ and $t = 2$. However, some partial results were obtained in [2], where members of a family of Taylor's two-graphs were shown to be self-complementary, and are therefore 2-subset-regular (in fact, 2-transitive) self-complementary 3-uniform hypergraphs. These 3-hypergraphs exist for all orders of the form $1+q$, where q is a prime power congruent to 1 modulo 4. Some additional existence results for $t = 1$ (more precisely, for vertex-transitive self-complementary uniform hypergraphs) can be found in [3].

Theorems 2 and 4 will be proved in Sections 2 and 3, respectively, while Section 4 is reserved for more technical lemmas that will be needed in the proof of Theorem 2. In the rest of this section we introduce the terminology and notation used in the paper.

For a finite set V and a positive integer k, let $V^{(k)}$ denote the set of all k-subsets of V. A *hypergraph* with vertex set V and edge set E is a pair (V, E), where V is a finite set and E is a collection of subsets of V. The *order* of a hypergraph X is the number of vertices of X. A hypergraph (V, E) is called *k-uniform* (shortly, a *k-hypergraph*) if E a subset of $V^{(k)}$. All hypergraphs in this paper will be uniform, and so by a *hypergraph* we shall mean a k-hypergraph for some k. Observe that a 2-hypergraph is simply a *graph*. The vertex set and edge set of a hypergraph X will be denoted by V_X and E_X, respectively.

The *complement* X^{C} of a k-hypergraph X is the k-hypergraph with vertex set V_X and edge set $V_X^{(k)} \setminus E_X$. For a positive integer t, $t \leq k$, and a t-subset $f \in V^{(t)}$, we define the *t-valency of f in X* to be the number of edges of E_X that contain f, and denote it by $\mathrm{val}_X^t(f)$, or shortly by $\mathrm{val}^t(f)$ if X is understood. A k-hypergraph X is called *t-subset-regular* if $\mathrm{val}^t(f)$ is independent of the choice of $f \in V_X^{(t)}$, and *regular* if it is t-subset-regular for some positive integer t.

An isomorphism between k-hypergraphs X and X' is a bijection $\varphi \colon V_X \to V_{X'}$ that induces a bijection between E_X and $E_{X'}$. A k-hypergraph X is called *self-complementary* if it is isomorphic to its complement X^{C}. An *antimorphism* of a self-complementary k-hypergraph X is an isomorphism between X and its complement; that is, a permutation on V_X that induces a bijection from E_X to $V_X^{(k)} \setminus E_X$.

2 The orders of regular self-complementary hypergraphs

In the lemma below, part (iii), we establish a necessary condition on n for there to exist a t-subset-regular self-complementary k-hypergraph of order n.

Lemma 5 *Let X be a t-subset-regular self-complementary k-hypergraph with n vertices. Then the following hold.*

(i) $\mathrm{val}^t(f) = \frac{1}{2}\binom{n-t}{k-t}$ *for every* $f \in V_X^{(t)}$.

(ii) X *is* t'*-subset-regular for every positive integer* $t' \le t$.

(iii) $\binom{n-i}{k}$ *is an even number for all* $i = 0, 1, \dots, t$.

Proof. To prove (i), let $r = \mathrm{val}^t(f)$ for some t-subset f of V_X. Then every t-subset of V_X is contained in exactly r elements of E_X. On the other hand, every element of E_X contains $\binom{k}{t}$ t-subsets of V_X. Therefore, $\binom{n}{t}r = |E_X|\binom{k}{t}$. From $|V^{(k)}| = \binom{n}{k}$ and $|E_X| = |E_{X^{\mathrm{C}}}|$ it follows that $|E_X| = \frac{1}{2}\binom{n}{k}$. Hence,

$$r = \frac{\binom{n}{k}\binom{k}{t}}{2\binom{n}{t}} = \frac{1}{2}\binom{n-t}{k-t}.$$

To show (ii), let f be a fixed t'-subset of V_X for $t' \le t$. We shall count the number of ordered pairs (g, e) in $V^{(t)} \times E_X$ such that $f \subseteq g \subseteq e$. Now f lies in $\binom{n-t'}{t-t'}$ t-subsets of V_X, each contained in r edges by (i). On the other hand, f lies in $\mathrm{val}^{t'}(f)$ edges, each containing $\binom{k-t'}{t-t'}$ t-subsets that contain f. Hence $\mathrm{val}^{t'}(f) = r\binom{n-t'}{t-t'}/\binom{k-t'}{t-t'}$, which is independent of the choice of f, and X is t'-subset-regular.

We shall now prove statement (iii). Since X is self-complementary, the number $\binom{n}{k}$ of all k-subsets of V must be even. By (ii), X is i-subset-regular for all $i = 1, \dots, t$, and so every i-subset of V_X must lie in the same number of edges of X as of X^{C}. Hence the total number $\binom{n-i}{k-i}$ of k-subsets containing a fixed i-subset of V must be even. We thus have that $\binom{n-i}{k-i}$ is even for all $i = 0, 1, \dots, t$. From Lemma 11 it now follows that $\binom{n-i}{k}$ is even for all $i = 0, 1, \dots, t$. ■

As we shall prove in Section 4, condition (iii) above is equivalent to the necessary condition in Theorem 2, however, the latter is computationally much easier to verify and is analogous to the observation that the order of a regular self-complementary graph is congruent to 1 modulo 4. It is therefore well worth the additional effort (in the form of the technical lemmas in Section 4) required to prove.

Proof of Theorem 2. Let X be a t-subset-regular self-complementary k-hypergraph with n vertices. By Lemma 5, $\binom{n-i}{k}$ is even for all $i = 0, 1, \dots, t$, and hence by Lemma 15, we can see that $n_{[2^q]} \in \{t, t+1, \dots, k_{[2^q]} - 1\}$ for some q with $k_{(2)} < q \le \min\{i : 2^i > k\}$. ■

The corollary below gives more transparent necessary conditions for $t = 1$ and particular values of k.

Corollary 6 *Let X be a 1-subset-regular self-complementary k-hypergraph with n vertices, where $k = 2^\ell$ or $k = 2^\ell + 1$ for some positive integer ℓ. Then $n \equiv n_{[2^{\ell+1}]} \pmod{2^{\ell+1}}$ where $n_{[2^{\ell+1}]} \in \{1, \ldots, k-1\}$. In particular,*

- *if $k = 2$, then $n \equiv 1 \pmod 4$;*
- *if $k = 3$, then $n \equiv 1$ or $2 \pmod 4$;*
- *if $k = 4$, then $n \equiv 1$, 2, or $3 \pmod 8$;*
- *if $k = 5$, then $n \equiv 1$, 2, 3, or $4 \pmod 8$.*

PROOF. Let a positive integer q be as in the statement of Theorem 2, and note that $\min\{i : 2^i > k\} = \ell+1$. If $k = 2^\ell$, then $k_{(2)} = \ell$ and so we must have $q = \ell+1$. If $k = 2^\ell+1$, then $k_{(2)} = 0$, whence $q \in \{1, \ldots, \ell+1\}$. However, $k_{[2^i]} = 1$ for all $i \leq \ell$, which would give $n_{[2^i]} \in \emptyset$. Hence we must also have $q = \ell + 1$. Since in both cases $k_{[2^{\ell+1}]} = k$, we conclude from Theorem 2 that $n_{[2^{\ell+1}]} \in \{1, 2, \ldots, k-1\}$. ∎

3 Constructing 1-subset-regular self-complementary 3-uniform hypergraphs

In this section we prove Theorem 4 by constructing 1-subset-regular self-complementary 3-hypergraphs of all possible orders. In other words, we prove that the necessary conditions established in Theorem 2 are also sufficient in the case $t = 1$ and $k = 3$. Construction 7 below produces 1-subset-regular self-complementary 3-uniform hypergraphs of all admissible orders congruent to 1 modulo 4, while Construction 9 produces those of orders congruent to 2 modulo 4.

Construction 7 *Let m be a positive integer and $V = \{u\} \cup V_0 \cup V_1 \cup V_2 \cup V_3$, where $V_i = \{v^i_j : j \in \mathbb{Z}_m\}$ for all $i \in \mathbb{Z}_4$. For pairwise distinct $i, i', i'' \in \mathbb{Z}_4$ define the following subsets of $V^{(3)}$.*

$$\begin{aligned}
E_i &= V_i^{(3)}, \\
E_{(i,i')} &= \{\{v^i_{j_1}, v^i_{j_2}, v^{i'}_{j'}\} : j_1, j_2, j' \in \mathbb{Z}_m, j_1 \neq j_2\}, \\
E_{i,i',i''} &= \{\{v^i_j, v^{i'}_{j'}, v^{i''}_{j''}\} : j, j', j'' \in \mathbb{Z}_m\}, \\
E^u_i &= \{\{u, v^i_{j_1}, v^i_{j_2}\} : j_1, j_2 \in \mathbb{Z}_m, j_1 \neq j_2\}, \\
E^u_{i,i'} &= \{\{u, v^i_j, v^{i'}_{j'}\} : j, j' \in \mathbb{Z}_m\}.
\end{aligned}$$

Let

$$E = \bigcup_{i=0,1} (E_i \cup E_{(2,i)} \cup E_{(3,i)} \cup E_{i,2,3} \cup E^u_i) \cup E_{(0,1)} \cup E_{(1,0)} \cup \bigcup_{i=0,1,3} E^u_{i,i+1},$$

where the additions in the subscripts are carried out in $\mathbb{Z}_4$. Let X^3_{4m+1} be the 3-hypergraph with vertex set V and edge set E as defined above.

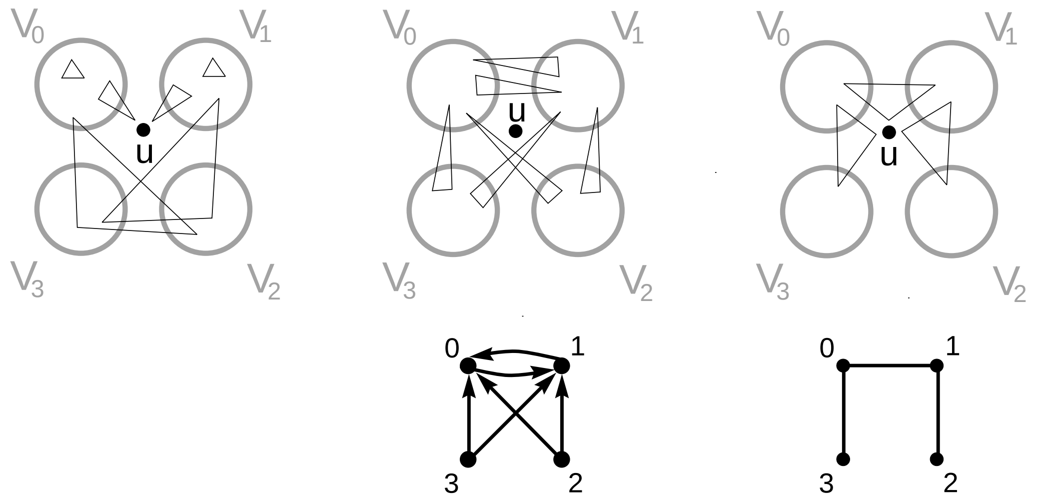

Figure 1: Above: The types of triples making up the edge set of the 3-hypergraph X^3_{4m+1}. Below: The digraph and graph, both admitting the antimorphism $(0,3,1,2)$, that correspond to the subsets of triples illustrated above.

Figure 1 explains the construction of the 3-hypergraph X^3_{4m+1} in another way.

Lemma 8 *The 3-hypergraph X^3_{4m+1} defined in Construction 7 is 1-subset-regular and self-complementary.*

PROOF. First we show that X^3_{4m+1} is 1-subset-regular. Take any vertex v^i_j. Then, for fixed $i', i'' \in \mathbb{Z}_4$ distinct from i, the vertex v^i_j lies in $\binom{m-1}{2}$ triples of E_i, $(m-1)m$ triples of $E_{(i,i')}$, $\binom{m}{2}$ triples of $E_{(i',i)}$, m^2 triples of $E_{i,i',i''}$, $m-1$ triples of E^u_i, and m triples of $E_{i,i'}$. Hence, for every vertex v^i_j with $i \in \{0,1\}$, we have

$$\mathrm{val}^1(v^i_j) \;=\; \binom{m-1}{2} + (m-1)m + 3\binom{m}{2} + m^2 + (m-1) + 2m = 4m^2 - m,$$

and for every vertex v^i_j with $i \in \{2,3\}$, we obtain

$$\mathrm{val}^1(v^i_j) \;=\; 2(m-1)m + 2m^2 + m = 4m^2 - m.$$

Furthermore,

$$\mathrm{val}^1(u) \;=\; 2\binom{m}{2} + 3m^2 = 4m^2 - m.$$

We conclude that X^3_{4m+1} is 1-subset-regular.

To finish the proof, define a bijection $\varphi : V \to V$ by: $\varphi(u) = u$, $\varphi(v_j^0) = v_j^3$, $\varphi(v_j^1) = v_j^2$, $\varphi(v_j^2) = v_j^0$, and $\varphi(v_j^3) = v_j^1$, for all $j \in \mathbb{Z}_m$. It is not difficult to see that φ acts as an antimorphism of X^3_{4m+1}, and so X^3_{4m+1} is self-complementary. ■

A *tournament* is a directed graph (V, A) with the property that for all pairs of distinct vertices $u, v \in V$, either $(u, v) \in A$ or $(v, u) \in A$. A tournament is said to be *self-converse* if there exists a bijection $\varphi : V \to V$ such that for all distinct $u, v \in V$ we have $(u, v) \in A$ if and only if $(\varphi(u), \varphi(v)) \notin A$. In the following construction of 1-subset-regular self-complementary 3-hypergraphs of even orders we shall use the fact that there exists a regular self-converse tournament with n vertices for every odd integer n [1].

Construction 9 *Let $n \geq 3$ be an odd integer, $n = 2m + 1$, and let $V = V_0 \cup V_1$, where $V_i = \{v_j^i : j \in \mathbb{Z}_n\}$. Furthermore, let $T = (\mathbb{Z}_n, A)$ be a regular self-converse tournament. For $i \in \mathbb{Z}_2$, define the following subsets of $V^{(3)}$.*

$$\begin{aligned} E_i &= V_i^{(3)}, \\ E_{(i,i+1)} &= \{\{v_{j_1}^i, v_{j_2}^i, v_j^{i+1}\} : j_1, j_2, j \in \mathbb{Z}_n, j_1, j_2, j \textit{ pairwise distinct }\}, \\ E_A &= \{\{v_{k_1}^i, v_{k_2}^i, v_{k_1}^{i+1}\} : (k_1, k_2) \in A, i \in \mathbb{Z}_2\}. \end{aligned}$$

Let $E = E_1 \cup E_{(0,1)} \cup E_A$, and let X^3_{4m+2} be the 3-hypergraph with vertex set V and edge set E.

The construction of the 3-hypergraph X^3_{4m+2} is illustrated in Figure 2 below.

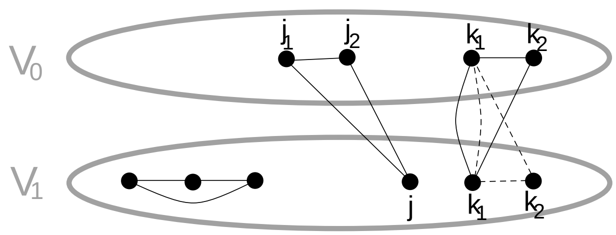

Figure 2: The types of triples making up the edge set of the 3-hypergraph X^3_{4m+2} (where j_1, j_2, j are pairwise distinct and $(k_1, k_2) \in A$).

Lemma 10 *The 3-hypergraph X^3_{4m+2} defined in Construction 9 is 1-subset-regular and self-complementary.*

Proof. First we show X^3_{4m+2} is 1-subset-regular. Take any vertex v_j^i. Then v_j^i lies in $\binom{n-1}{2}$ triples of E_i, $(n-1)(n-2)$ triples of $E_{(i,i+1)}$, $\binom{n-1}{2}$ triples of $E_{(i+1,i)}$, and $\frac{3}{2}(n-1)$ triples of E_A. Hence,

$$\mathrm{val}^1(v_j^0) = (n-1)(n-2) + \frac{3}{2}(n-1) = \frac{1}{2}(n-1)(2n-1)$$

and

$$\mathrm{val}^1(v_j^1) = \binom{n-1}{2} + \binom{n-1}{2} + \frac{3}{2}(n-1) = \frac{1}{2}(n-1)(2n-1).$$

We conclude that X^3_{4m+1} is 1-subset-regular.

Let $\psi : \mathbb{Z}_n \to \mathbb{Z}_n$ be an arc-reversing mapping of the tournament T; that is, ψ is a bijection on $\mathbb{Z}_n$ such that $\psi(a) \notin A$ for all $a \in A$. Define a bijection $\varphi : V \to V$ by $\varphi(v_j^i) = v^{i+1}_{\psi(j)}$. Then φ interchanges the sets E_0 and E_1, and also the sets $E_{(0,1)}$ and $E_{(1,0)}$. Furthermore, for all $(j_1, j_2) \in A$ and $i \in \mathbb{Z}_2$, since ψ is arc-reversing, φ maps the triple $\{v^i_{k_1}, v^i_{k_2}, v^{i+1}_{k_1}\} \in E_A$ to the triple $\{v^{i+1}_{\psi(k_1)}, v^{i+1}_{\psi(k_2)}, v^{i}_{\psi(k_1)}\} \notin E_A$. It follows easily that φ is an antimorphism of X^3_{4m+2}, and X^3_{4m+2} is self-complementary. ■

We are now ready to conclude the proof of our second main result.

Proof of Theorem 4. The necessity of the conditions follows directly from Corollary 6, and the sufficiency is established using Lemmas 8 and 10. ■

4 Technical lemmas

In this section we shall develop the technical lemmas used in the proofs of part (iii) of Lemma 5 and Theorem 2.

Lemma 11 *Let n, k, t be integers with $0 \le t \le k \le n$. If $\binom{n-i}{k-i}$ is even for all $i = 0, 1, \ldots, t$, then $\binom{n-i}{k}$ is even for all $i = 0, 1, \ldots, t$*

Proof. We have that $\binom{n-i}{k-i}$ is even for all $i = 0, 1, \ldots, t$. With this assumption, we shall now prove the following statement:

$$\binom{n-i}{k-i+j} \text{ is even for all } i = 0, 1, \ldots, t \text{ and } j = 0, 1, \ldots, i. \tag{1}$$

We use induction on j nested within induction on i.

Since $\binom{n}{k}$ is even by the assumption, we have that (1) holds for $i = 0$. Suppose that for some i, $0 \le i < t$, Claim (1) holds for all $j = 0, 1, \ldots, i$. By induction on j we shall prove that

$$\binom{n-(i+1)}{k-(i+1)+j} \text{ is even for all } j = 0, 1, \ldots, i+1. \tag{2}$$

By the assumption, $\binom{n-(i+1)}{k-(i+1)}$ is even and so (2) holds for $j = 0$. Suppose $\binom{n-(i+1)}{k-(i+1)+j}$ is even for some j, $0 \le j \le i$. Since

$$\binom{n-(i+1)}{k-(i+1)+(j+1)} = \binom{n-i}{k-i+j} - \binom{n-(i+1)}{k-(i+1)+j}$$

and both terms on the right are even (the first by the induction hypothesis on i and the second by the induction hypothesis on j), we have that $\binom{n-(i+1)}{k-(i+1)+(j+1)}$ is even. Statement (2) then follows by induction, completing the induction step for Statement(1). Finally, we conclude from (1) that $\binom{n-i}{k}$ is even for all $i = 0, 1, \ldots, t$. ■

The following notation will be used in the rest of the section.

Notation 12 For positive integers k, n, and q with $k \leq n$, we let $N_q^k(n)$ denote the number of integers in the set $\{n-(k-1), n-(k-2), \ldots, n\}$ that are divisible by q.

Lemma 13 *Let k, n, and q be positive integers with $k \leq n$. Then the following hold.*

(i) $N_q^k(n) = \lfloor \frac{n}{q} \rfloor - \lfloor \frac{n-k}{q} \rfloor$.

(ii) $N_q^k(n) - N_q^k(k) \in \{0, 1\}$.

(iii) *If $k \equiv 0 \pmod{q}$, then $N_q^k(n) = N_q^k(k)$.*

(iv) *If $k \not\equiv 0 \pmod{q}$, then $N_q^k(n) > N_q^k(k)$ if and only if $n_{[q]} \in \{0, 1, \ldots, k_{[q]} - 1\}$.*

(v) *If p is a prime, then $\binom{n}{k}_{(p)} = \sum_{i=1}^{\infty}(N_{p^i}^k(n) - N_{p^i}^k(k))$.*

Proof.

(i) This is easy to see since $\lfloor \frac{n}{q} \rfloor$ is the number of multiples of q among $1, 2, \ldots, n$ and $\lfloor \frac{n-k}{q} \rfloor$ is the number of multiples of q among $1, 2, \ldots, n-k$.

(ii) For any positive real numbers x and y, we have

$$\lfloor x + y \rfloor \geq \lfloor x \rfloor + \lfloor y \rfloor$$

and

$$\lfloor x + y \rfloor - (\lfloor x \rfloor + \lfloor y \rfloor) \leq x - \lfloor x \rfloor + y - \lfloor y \rfloor < 2.$$

Hence

$$0 \leq \lfloor x + y \rfloor - (\lfloor x \rfloor + \lfloor y \rfloor) \leq 1.$$

It follows that

$$N_q^k(n) - N_q^k(k) = \lfloor \frac{n}{q} \rfloor - \lfloor \frac{n-k}{q} \rfloor - \lfloor \frac{k}{q} \rfloor \in \{0, 1\}.$$

(iii) Assume $k \equiv 0 \pmod{q}$. Then

$$\begin{aligned} N_q^k(n) - N_q^k(k) &= \lfloor \frac{n}{q} \rfloor - \lfloor \frac{n-k}{q} \rfloor - \lfloor \frac{k}{q} \rfloor \leq \frac{n-k}{q} - \lfloor \frac{n-k}{q} \rfloor + \frac{k}{q} - \lfloor \frac{k}{q} \rfloor \\ &= \frac{n-k}{q} - \lfloor \frac{n-k}{q} \rfloor < 1. \end{aligned}$$

Hence $N_q^k(n) - N_q^k(k) = 0$.

(iv) We have

$$\begin{aligned} N_q^k(n) - N_q^k(k) &= \lfloor \frac{n}{q} \rfloor - \lfloor \frac{n-k}{q} \rfloor - \lfloor \frac{k}{q} \rfloor \\ &= \frac{1}{q}(n - n_{[q]} - (n-k-(n-k)_{[q]}) - (k - k_{[q]})) \\ &= \frac{1}{q}((n-k)_{[q]} - (n_{[q]} - k_{[q]})). \end{aligned}$$

Now

$$(n-k)_{[q]} = \begin{cases} n_{[q]} - k_{[q]} & \text{if } n_{[q]} \geq k_{[q]}, \\ q + n_{[q]} - k_{[q]} & \text{if } n_{[q]} < k_{[q]}, \end{cases}$$

whence $N_q^k(n) - N_q^k(k) = 1$ if and only if $(n-k)_{[q]} = q + n_{[q]} - k_{[q]}$, that is, if and only if $0 \leq n_{[q]} < k_{[q]}$.

(v) Observe that

$$\begin{aligned} (n(n-1)\dots(n-k+1))_{(p)} &= (n!)_{(p)} - ((n-k)!)_{(p)} = (\sum_{i=1}^{\infty} \lfloor \frac{n}{p^i} \rfloor) - (\sum_{i=1}^{\infty} \lfloor \frac{n-k}{p^i} \rfloor) \\ &= \sum_{i=1}^{\infty} N_{p^i}^k(n). \end{aligned}$$

Hence

$$\binom{n}{k}_{(p)} = (n(n-1)\dots(n-k+1))_{(p)} - (k!)_{(p)} = \sum_{i=1}^{\infty} (N_{p^i}^k(n) - N_{p^i}^k(k)).$$

■

Lemma 14 *Let k and n be positive integers with $k \leq n$, and let $m = \min\{i : 2^i > k\}$. Then $\binom{n}{k}$ is even if and only if $n_{[2^s]} \in \{0, 1, \dots, k_{[2^s]} - 1\}$ for some integer s with $k_{(2)} < s \leq m$.*

Proof. By Lemma 13, parts (v) and (ii), we have that $\binom{n}{k}$ is even if and only if $\binom{n}{k}_{(2)} = \sum_{i=1}^{\infty} (N_{2^i}^k(n) - N_{2^i}^k(k)) > 0$, that is, if and only if $N_{2^s}^k(n) > N_{2^s}^k(k)$ for some s. By parts (iii) and (iv) of the same lemma, this happens if and only if $n_{[2^s]} \in \{0, 1, \dots, k_{[2^s]} - 1\}$ for some $s > k_{(2)}$. Now, if $s \geq m$, then $k_{[2^s]} = k_{[2^m]} = k$ since $k < 2^m \leq 2^s$, and $n_{[2^s]} \geq n_{[2^m]}$ since 2^m divides 2^s. It follows that $n_{[2^s]} \in \{0, 1, \dots, k_{[2^s]} - 1\}$ implies $n_{[2^m]} \in \{0, 1, \dots, k_{[2^m]} - 1\}$. ■

Lemma 15 *Let t, k, and n be integers with $0 \leq t \leq k \leq n$, and let $m = \min\{i : 2^i > k\}$. Then $\binom{n}{k}, \binom{n-1}{k}, \dots, \binom{n-t}{k}$ are all even if and only if $n_{[2^q]} \in \{t, t+1, \dots, k_{[2^q]} - 1\}$ for some integer q with $k_{(2)} < q \leq m$.*

Proof. Suppose $n_{[2^q]} \in \{t, t+1, \ldots, k_{[2^q]} - 1\}$ for some q with $k_{(2)} < q \le m$. Then $0 \le t \le k_{[2^q]} - 1$ and so $(n-i)_{[2^q]} \in \{t-i, t-i+1, \ldots, k_{[2^q]} - i - 1\}$ for all $i = 0, 1, \ldots, t$. It now follows by Lemma 14 that $\binom{n}{k}, \binom{n-1}{k}, \ldots, \binom{n-t}{k}$ are all even.

Conversely, suppose $\binom{n}{k}, \binom{n-1}{k}, \ldots, \binom{n-t}{k}$ are all even. The proof is by induction on t. For $t = 0$ the claim follows from Lemma 14. Now suppose that the claim holds for some t, $0 \le t < k$. We need to show that it also holds for $\tau = t + 1$.

Assume $\binom{n}{k}, \binom{n-1}{k}, \ldots, \binom{n-\tau}{k}$ are all even. Then by the induction hypothesis (applied on the first $\tau - 1$ and the last $\tau - 1$ of the above binomial coefficients) we have that

$$n_{[2^r]} \in \{\tau - 1, \tau, \ldots, k_{[2^r]} - 1\} \quad \text{for some } r \text{ with } k_{(2)} < r \le m, \quad \text{and} \tag{3}$$

$$(n-1)_{[2^s]} \in \{\tau - 1, \tau, \ldots, k_{[2^s]} - 1\} \quad \text{for some } s \text{ with } k_{(2)} < s \le m. \tag{4}$$

In particular, $\tau \le \min\{k_{[2^r]}, k_{[2^s]}\} < \min\{2^r, 2^s\}$.

Now (4) is clearly equivalent to $n_{[2^s]} \in \{\tau, \tau + 1, \ldots, k_{[2^s]}\}$. Hence the induction step is proved (with either r or s in place of q) unless

$$n_{[2^r]} = \tau - 1 \quad \text{and } n_{[2^s]} = k_{[2^s]}.$$

Suppose now that this is the case. If $s \le r$, then $\tau - 1 \equiv n_{[2^r]} \equiv n_{[2^s]} \equiv k_{[2^s]} \pmod{2^s}$, and since $\tau - 1 < 2^s$, it follows that $\tau - 1 = k_{[2^s]}$, which contradicts the fact that $\tau \le k_{[2^s]}$.

It follows that $s > r$. In this case, similarly to the previous case, we see that $\tau - 1 \equiv n_{[2^r]} \equiv n_{[2^s]} \equiv k_{[2^s]} \equiv k_{[2^r]} \pmod{2^r}$. Since $\tau - 1 < 2^r$, it follows that $\tau - 1 = k_{[2^r]}$, which contradicts the fact that $\tau \le k_{[2^r]}$. This completes the proof of the induction step. ■

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