Symmetries of Hilbert space effect algebras *

Peter Šemrl
Faculty of Mathematics and Physics
University of Ljubljana
Jadranska 19
SI-1000 Ljubljana
Slovenia
peter.semrl@fmf.uni-lj.si

Abstract

Let \( H \) be a Hilbert space and \( E(H) \) the effect algebra on \( H \), that is, \( E(H) \) is the set of all self-adjoint operators \( A : H \to H \) satisfying \( 0 \leq A \leq I \). The effect algebra can be equipped with several operations and relations that are important in mathematical foundations of quantum mechanics. Automorphisms with respect to these operations or relations are called symmetries. We present a new method that can be used to describe the general form of such maps. The main idea is to reduce this kind of problems to the study of adjacency preserving maps. The efficiency of this approach is illustrated by reproofing some known results as well as by obtaining some new theorems.


1 Introduction

Quantum effects play an important role in mathematical foundations of quantum mechanics. They correspond to yes-no measurements that can be unsharp. A detailed explanation can be found in [2] and [9]. In the Hilbert space formalism they are represented by linear bounded self-adjoint operators \( A \) acting on a Hilbert space \( H \) satisfying \( 0 \leq A \leq I \). Such operators are usually called (Hilbert space) effects. The Hilbert space effect algebra \( E(H) \) is the set of all effects. Symmetries of the effect algebra \( E(H) \) are bijective maps \( \phi : E(H) \to E(H) \) which preserve certain operations and/or relations defined on \( E(H) \) that are important in various aspects of quantum theory. The study of symmetries of

*This work was partially supported by a grant from the Ministry of Science of Slovenia
E(H) was initiated by Ludwig [11, Section V.5]. He proved that every ortho-order automorphism of E(H), dim H ≥ 3, is a unitary or antiunitary similarity transformation. The proof was later clarified in [4] and the two-dimensional case was resolved in [21].

The partial order ≤ on E(H) is defined by A ≤ B if and only if ⟨Ax,x⟩ ≤ ⟨Bx,x⟩ for every x ∈ H. By ⊥ we denote the orthocomplementation on E(H) given by A ↦→ A⊥ = I − A, A ∈ E(H). Then the precise formulation of Ludwig’s theorem reads as follows: If dim H ≥ 2 and φ : E(H) → E(H) is a bijective map such that for any A, B ∈ E(H) we have

\[ A \leq B \iff \phi(A) \leq \phi(B) \quad (1) \]

and \( \phi(A^\perp) = \phi(A)^\perp \), then there exists a unitary or antiunitary operator U : H → H such that \( \phi(A) = UAU^* \) for every \( A \in E(H) \).

Besides the partial order and the orthocomplementation there are other operations and relations on E(H) that are important in quantum theory. Let us mention just few of them: sequential product, Jordan triple product, mixture, and coexistency. The results on the structure of corresponding symmetries can be found in [12, 13, 15, 16, 17, 20, 22, 25] and the references therein.

We denote by S(H) the real space of all bounded linear self-adjoint operators acting on H. In mathematical foundations of quantum mechanics these operators represent bounded observables. Symmetries on bounded observables were studied a lot as well. Let us mention here the main result from [14]: if φ : S(H) → S(H) is a bijective map satisfying A ≤ B ⇐⇒ φ(A) ≤ φ(B), A, B ∈ S(H), then there exist an operator C ∈ S(H), a constant c ∈ \{−1, 1\}, and an invertible bounded linear or conjugate-linear operator T : H → H such that

\[ \phi(A) = cTAT^* + C \quad (2) \]

for every \( A \in S(H) \). Neither the assumptions, nor the conclusion are affected if we compose the map φ with a translation. Hence, there is no loss of generality in assuming that φ(0) = 0. Then C = 0. So, after this harmless normalization we conclude that symmetries on S(H) with respect to the usual partial order are real linear. It turns out that many symmetries on S(H) behave like this. Various results of this type have been proved by ad hoc methods. We have recently discovered that most of them can be proved simultaneously using a unified approach based on adjacency preserving maps [26].

In this paper we will develop a unified approach to symmetries on effect algebras. As in the case of bounded observables the main idea is to reduce various problems on symmetries to the problem of characterizing adjacency preserving maps. However, the case of the effect algebra is far more difficult than the case of S(H). To illustrate this note that all bijective maps on S(H) preserving order are of the nice form given in (2). It was observed in [20] that
for any fixed invertible operator $T \in E(H)$, the transformation

$$A \mapsto \left( \frac{T^2}{2I-T^2} \right)^{-1/2} \left( (I-T^2 + T(I+A)^{-1}T)^{-1} - I \right) \left( \frac{T^2}{2I-T^2} \right)^{-1/2} \quad (3)$$

is a bijective order preserving map of $E(H)$ onto itself. Thus, the behaviour of symmetries with respect to $\leq$ on $E(H)$ can be much wilder than on $S(H)$.

Nevertheless, in [25] we succeeded to describe the general form of bijective order preserving maps on $E(H)$. The formulation of the result is not as nice as in the case of $S(H)$, but still quite simple if we have in mind that the description of the general form of such maps has to cover examples like (3). Needless to say, the proof was also more difficult. We will reprove this result using a different approach. We will first show that bijective order preserving maps preserve adjacency. Thus, the next step is to describe the general form of bijective adjacency preserving maps on $E(H)$. But once we solve this problem we have a general tool that can be used to prove rather easily various characterizations of symmetries. We will show this by reprovings most of known results. The proofs are very short. The efficiency of our method will be demonstrated also by proving several new results.

## 2 Adjacency preserving maps

Recall that $A, B \in E(H)$ are said to be adjacent if $B - A$ is an operator of rank one. We will say that a bijective map $\phi : E(H) \to E(H)$ preserves adjacency if for every pair $A, B \in E(H)$ the operators $\phi(A)$ and $\phi(B)$ are adjacent if and only if $A$ and $B$ are adjacent (note that the terminology in the literature is not unique: in many papers such maps are said to preserve adjacency strongly or in both directions to distinguish them from the maps satisfying the weaker assumption that the adjacency of $A$ and $B$ implies the adjacency of $\phi(A)$ and $\phi(B)$; there is no possibility for confusion in our paper as all properties will be always assumed to be preserved in both directions).

When studying adjacency preserving maps on $S(H)$ [26] the first step was to reduce the general problem to the two-dimensional case. A rather straightforward modification works for the effect algebra as well. Self-adjoint operators on a two-dimensional Hilbert space can be identified with $2 \times 2$ hermitian matrices. It has been known for a long time (see [7, Chapter 5.3]) that the space of $2 \times 2$ hermitian matrices can be identified with four-dimensional Lorentz-Minkowski space in such a way that the adjacency of matrices corresponds to the coherency of the corresponding space-time events. Hence, adjacency preserving maps on the effect algebra on a two-dimensional Hilbert space can be considered as coherency preserving maps on the four-dimensional Lorentz-Minkowski space. Coherency preserving maps have been extensively studied because of their importance in the relativity theory. For some results concerning such maps
defined not on the whole Lorentz-Minkowski space, but only on certain subsets, we refer to papers of Lester and Popovici and Radulescu [10, 23]. We could use these results to describe the general form of adjacency preserving bijective maps on \( E(H) \) where \( \dim H = 2 \). This approach is not optimal as the results of Lester and Popovici and Radulescu characterize the semigroup of coherency preserving maps on a certain subset (we are interested in the case when this subset corresponds to the set of all effects) by describing the set of its generators. If we start with a bijective adjacency preserving map \( \phi : E(H) \to E(H) \) and if we use this approach, that is, we have a description of the general form of the restrictions of \( \phi \) to invariant two-dimensional pieces, then we would like to recover the behaviour of \( \phi \) on the whole effect algebra \( E(H) \) from the behaviour of \( \phi \) on these small pieces. But this seems to be rather difficult as on these small pieces we do not have explicit formulae, but just the list of generators.

Hence, when characterizing adjacency preserving maps on effect algebras we will not use the above reduction idea that has been applied in the case of maps defined on the set of all bounded observables. Instead, we will first gain a better understanding of adjacency preserving maps on the effect algebra by recalling some examples from [25]. The crucial observation will be that in all these examples an invertible effect \( A \) with the property that \( A^\perp \) is invertible as well is mapped into an effect of the same type. We will prove that this is true for all bijective adjacency preserving maps \( \phi \) on \( E(H) \). In particular, \( \phi((1/2)I) \) and \( \phi((1/2)I)^\perp \) are both invertible. In the next step we will show that \( \phi((1/2)I) \) can be transformed back to \((1/2)I\) with a bijective adjacency preserving map of a quite simple form. This reduces the problem of describing the general form of bijective adjacency preserving maps \( \phi \) on \( E(H) \) to the special case when \( \phi((1/2)I) = (1/2)I \). It is then easy to show that this problem is equivalent to the problem of characterizing unital bijective adjacency preserving maps on the set of all positive invertible self-adjoint operators that has been solved in [26].

Let \( A, B \in S(H) \) be positive invertible operators. Then \( A \) and \( B \) are adjacent if and only if \( A^{-1} \) and \( B^{-1} \) are adjacent. Indeed,

\[
B^{-1} - A^{-1} = B^{-1}(A - B)A^{-1}
\]

is of rank one if and only if \( A - B \) is a rank one operator. For two self-adjoint operators \( A, B \) such that \( B - A \) is a positive invertible operator, we set \( [A, B] = \{C \in S(H) : A \leq C \leq B\} \). In particular, \( E(H) = [0, I] \). We further denote by \( (A, B) \) the set of all \( C \in [A, B] \) such that both \( C - A \) and \( B - C \) are invertible. In particular, \( C \in [0, I] \) belongs to \( (0, I) \) if and only if both \( C \) and \( C^\perp \) are invertible. Let \( A, B \in S(H) \) be such that \( B - A \) is a positive invertible operator and let \( T \in S(H) \) be any self-adjoint operator. Then the translation map \( X \mapsto X + T \) is a bijective adjacency preserving map of \( [A, B] \) onto \( [A + T, B + T] \). Further, if \( T : H \to H \) is any invertible bounded linear or conjugate-linear map, then the transformation \( X \mapsto TXT^* \) is a bijective adjacency preserving map of \( [A, B] \) onto \( [TAT^*, TBT^*] \). And finally, if \( A, B \in S(H) \) with both \( A \) and \( B - A \) being
positive invertible operators, then the bijective map \( X \mapsto X^{-1} \) of \([A, B]\) onto \([B^{-1}, A^{-1}]\) preserves adjacency as well (the fact that the map \( X \mapsto X^{-1} \) is a bijection of \([A, B]\) onto \([B^{-1}, A^{-1}]\) is well-known although not entirely trivial). The map (3) is a product of such maps. Indeed, the map \( \xi \) defined as a product of maps:

\[
\begin{align*}
A & \mapsto I + A \mapsto (I + A)^{-1} \mapsto T(I + A)^{-1}T \\
I - T^2 + T(I + A)^{-1}T & \mapsto (I - T^2 + T(I + A)^{-1}T)^{-1} \\
(I - T^2 + T(I + A)^{-1}T)^{-1} & \mapsto (I - T^2 + T(I + A)^{-1}T)^{-1} - I,
\end{align*}
\]

is a bijective adjacency preserving map of \( E(H) \) onto \([\xi(0), \xi(I)]\). Clearly, \( \xi(0) = 0 \) and \( \xi(I) = T^2(2I - T^2)^{-1} \). Composing \( \xi \) with the suitable congruence transformation we obtain the map (3).

We have given several examples of bijective adjacency preserving maps \( \phi \) between operator intervals \([A, B]\) and \([C, D]\). Note that in all these examples we have \( \phi((0, I)) = (C, D) \). This motivates the following result.

**Proposition 2.1** Let \( \phi : E(H) \rightarrow E(H) \) be a bijective adjacency preserving map. Suppose that \( \dim H \geq 3 \). Then \( \phi((0, I)) = (0, I) \).

Several lemmas are needed for the proof of this statement. Let \( p \) be any real number satisfying \( p < 1 \). Define a real function \( f_p : [0, 1] \rightarrow [0, 1] \) by

\[
f_p(x) = \frac{x}{px + (1 - p)}, \quad x \in [0, 1].
\]

(4)

**Lemma 2.2** Let \( p < 1 \). Then the map \( \phi : E(H) \rightarrow E(H) \) defined by

\[
\phi(A) = f_p(A), \quad A \in E(H),
\]

where \( f_p(A) \) denotes the image of the function \( f_p \) under the continuous functional calculus of a self-adjoint operator \( A \), is a bijective adjacency preserving map.

**Proof.** Note that for a positive real number \( r \) the map

\[
\delta(A) = r(r + 1) \left[ (rI + A)^{-1} - \frac{1}{r + 1}I \right]
\]

is a product of translations, the map \( A \mapsto A^{-1} \) and the map \( A \mapsto r(r+1)A \), and is therefore a bijection of \( E(H) \) onto \([\delta(I), \delta(0)] = [0, I]\) preserving adjacency. Clearly,

\[
\delta(A) = g_r(A),
\]

where

\[
g_r(x) = r \frac{1 - x}{r + x}, \quad x \in [0, 1].
\]
It is straightforward to verify that
\[ g_s(g_r(x)) = \frac{x}{px + (1 - p)}, \quad x \in [0, 1], \]
with
\[ p = \frac{s - r}{s(1 + r)}. \]
Observe that \( p = \frac{s - r}{s(1 + r)} \in (-\infty, 1) \), \( s, r > 0 \), and for every \( p \in (-\infty, 1) \) we can find positive real numbers \( r, s \) such that \( p = \frac{s - r}{s(1 + r)} \). This completes the proof.

Later on we will need the fact that the map \( A \mapsto f_p(A) \), \( A \in E(H) \), preserves order, that is, for every pair \( A, B \in E(H) \) we have \( A \leq B \iff f_p(A) \leq f_p(B) \). This follows easily from the above proof and the well-known fact that if \( A, B \) are invertible positive operators and \( A \leq B \), then \( B^{-1} \leq A^{-1} \). Moreover, for every pair \( A, B \in E(H) \) the operator \( B - A \) is invertible if and only if \( f_p(B) - f_p(A) \) is invertible.

Let \( S \in (0, I) \). Then \( A \mapsto SAS \) is a bijective adjacency preserving map of \( E(H) \) onto \([0, S^2] \). It follows that \( A \mapsto (f_p(S^2))^{-1/2} f_p(SAS)(f_p(S^2))^{-1/2} \) is a bijective adjacency preserving map of \( E(H) \) onto itself. For the proof of the following lemma see [25, Lemma 2.5].

**Lemma 2.3** Let \( R \in (0, I) \). Then there exist real numbers \( p, q < 1 \) and an operator \( S \in (0, I) \) such that

\[ \xi \left( \frac{1}{2} I \right) = R, \]

where \( \xi : E(H) \to E(H) \) is a bijective adjacency preserving map defined by

\[ \xi(A) = f_q \left( (f_p(S^2))^{-1/2} f_p(SAS)(f_p(S^2))^{-1/2} \right), \quad A \in E(H). \]

A subset \( \mathcal{M} \subset E(H) \) is called an adjacent set if every pair of elements \( A, B \in \mathcal{M} \), \( A \neq B \), is adjacent. It is called a maximal adjacent set if it is maximal among such sets. If two effects \( A \) and \( B \) are adjacent, then \( B - A \) is a self-adjoint operator of rank one. Any such operator is a scalar multiple of a projection of rank one (a rank one self-adjoint idempotent operator). Thus, \( B = A + tP \) for some projection \( P \) of rank one and some nonzero real number \( t \). Let further \( C \) be any effect adjacent to both \( A \) and \( B \). Then the self-adjoint operator \( C - A \) is adjacent to both \( B - A = tP \) and \( 0 = A - A \). It is an elementary linear algebra exercise to show that then \( C - A = sP \) for some scalar \( s \). It follows easily that if \( \mathcal{M} \subset E(H) \) is a maximal adjacent set, then there exist \( A \in E(H) \) and a projection \( P \) of rank one such that \( \mathcal{M} \) is the intersection of the effect algebra \( E(H) \) and the line through \( A \) with a direction vector \( P \), that is,

\[ \mathcal{M} = E(H) \cap \{ A + tP : t \in \mathbb{R} \}. \]
One needs to be careful here. The converse is not true. Namely, it can happen that for some \( A \in E(H) \) and some projection \( P \) of rank one the set \( E(H) \cap \{ A + tP : t \in \mathbb{R} \} \) is not a maximal adjacent set. Clearly, this happens if and only if \( E(H) \cap \{ A + tP : t \in \mathbb{R} \} = \{ A \} \). To give an example write \( H \) as an orthogonal direct sum \( H = H_1 \oplus H_2 \) with \( \dim H_1 = 2 \), and let

\[
A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \oplus 0 \quad \text{and} \quad P = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \oplus 0.
\]

Then for \( t \in \mathbb{R} \) we have \( A + tP \in E(H) \) if and only if \( t = 0 \).

Obviously, the set \( \mathcal{M} \) defined by (5) is a line segment \( \{ A + tP : t \in [t_1, t_2] \} \). Here, \( t_1 = \min \{ t \in \mathbb{R} : A + tP \geq 0 \} \) and \( t_2 = \max \{ t \in \mathbb{R} : A + tP \leq 1 \} \).

Let \( A \) be any effect and \( P \) any projection of rank one. We write \( \mathcal{M}(A, P) = \{ A + tP : t \in \mathbb{R} \quad \text{and} \quad 0 \leq A + tP \leq 1 \} \). Denote by \( \mathcal{L}(A) \) the set of all maximal adjacent sets \( \mathcal{M} \subset E(H) \) satisfying \( A \in \mathcal{M} \), that is, \( \mathcal{L}(A) \) is the collection of all sets \( \mathcal{M}(A, P) \), where \( P \) runs over all projections of rank one with the property that \( \mathcal{M}(A, P) \) is not a singleton.

Let \( R \in E(H) \) be any effect. Denote by \( \mathcal{R} \) the set of all effects \( A \in E(H) \) with the property that there exists a finite sequence of effects \( A = A_0, A_1, \ldots, A_n = R \) such that \( A_j \) and \( A_{j+1} \) are adjacent for all \( j = 0, \ldots, n-1 \). Set \( d(R, R) = 0 \).

For any \( A \in \mathcal{R} \), \( A \neq R \), we define \( d(R, A) \) to be the least positive integer \( n \) for which there is a sequence of effects \( A = A_0, A_1, \ldots, A_n = R \) such that \( A_j \) and \( A_{j+1} \) are adjacent for all \( j = 0, \ldots, n-1 \). If \( d(R, A) = 2 \) for some \( A \in E(H) \), then there exists a sequence of effects \( B \in E(H) \) such that \( A \) and \( B \) are adjacent and \( B \) and \( R \) are adjacent, that is, \( \text{rank} (B - A) = 1 \). As rank is subadditive, we have \( \text{rank} (R - A) \leq 2 \). The other hand, \( R \neq A \) and \( R \) and \( A \) are not adjacent. Hence, \( \text{rank} (R - A) = 2 \).

Let \( \phi : E(H) \to E(H) \) be a bijective adjacency preserving map. Then \( \mathcal{M} \subset E(H) \) is a maximal adjacent set if and only if \( \phi(\mathcal{M}) \) is a maximal adjacent set. And if \( \mathcal{M} \subset E(H) \) is a maximal adjacent set, then for every \( A \in E(H) \) we have \( \mathcal{M} \in \mathcal{L}(A) \) if and only if \( \phi(\mathcal{M}) \in \mathcal{L}(\phi(A)) \). Further, for every pair \( A, B \in E(H) \) we have \( d(A, B) = 2 \) if and only if \( d(\phi(A), \phi(B)) = 2 \).

Assume now that \( A, B \in E(H) \) with \( d(A, B) = 2 \). Denote by \( \mathcal{K}_A(B) \) the set of all maximal adjacent sets \( \mathcal{M} \in \mathcal{L}(A) \) such that there exists \( C \in \mathcal{M} \) which is adjacent to \( B \). In other words, \( \mathcal{K}_A(B) \) is the collection of all sets \( \mathcal{M}(A, P) \) with \( P \) being a projection of rank one such that there exists \( A + tP \in \mathcal{M}(A,P) \) satisfying \( d(A + tP, B) = 1 \). Clearly, we have \( \phi(\mathcal{K}_A(B)) = \mathcal{K}_{\phi(A)}(\phi(B)) \).

Let \( A \in E(H) \). We define \( \mathcal{S}(A) \) to be the set of all effects \( B \in E(H) \) with \( d(A, B) = 2 \) and the property that for every \( C \in E(H) \) satisfying \( d(A, C) = 2 \) and \( \mathcal{K}_A(B) \subset \mathcal{K}_A(C) \) we have \( \mathcal{K}_A(B) = \mathcal{K}_A(C) \). Once again it is clear that for every \( A \in E(H) \) we have \( \phi(\mathcal{S}(A)) = \mathcal{S}(\phi(A)) \).

For effects \( B, C \in \mathcal{S}(A) \) we write \( B \sim C \) if and only if \( \mathcal{K}_A(B) = \mathcal{K}_A(C) \) and there exists \( D \in E(H) \) such that \( D \) and \( A \) are adjacent, \( D \) and \( B \) are adjacent, and \( D \) and \( C \) are adjacent. We further write \( B \equiv C \) if and only
if we can find in \( S(A) \) a finite sequence \( B = B_0, B_1, \ldots, B_n = C \) such that 
\( B_0 \sim B_1, B_1 \sim B_2, \ldots, B_{n-1} \sim B_n \). It is trivial to check that for \( B, C \in S(A) \) we have \( B \equiv C \) if and only if \( \phi(B) \equiv \phi(C) \).

We continue with some simple observations.

**Lemma 2.4** Let \( A, B \in E(H) \) and assume that \( d(A, B) = 2 \). Then
\[
K_A(B) \subset \{ \mathcal{M}(A, P) \in \mathcal{L}(A) : \text{Im } P \subset \text{Im } (B - A) \}.
\]
Here, \( \text{Im } P \) stands for the image of \( P \).

**Proof.** Let \( \mathcal{M}(A, P) \in K_A(B) \) for some rank one projection \( P \). Then there exist a scalar \( t \) and a rank one self-adjoint operator \( R \) such that \( B - (A + tP) = R \), that is, \( B - A = tP + R \). It follows that \( \text{Im } (B - A) \subset \text{Im } P + \text{Im } R \). As \( \text{Im } (B - A) \) is two-dimensional and both images of \( P \) and \( R \) are one-dimensional, we have
\[
\text{Im } (B - A) = \text{Im } P \oplus \text{Im } R.
\]
In particular, \( \text{Im } P \subset \text{Im } (B - A) \), as desired.

\( \square \)

**Lemma 2.5** Let \( A, B \in E(H) \) with \( d(A, B) = 2 \). If \( A \leq B \) or \( B \leq A \), then
\[
K_A(B) = \{ \mathcal{M}(A, P) \in \mathcal{L}(A) : \text{Im } P \subset \text{Im } (B - A) \}
\]
\[
= \{ \mathcal{M}(A, P) : \text{Im } P \subset \text{Im } (B - A) \}.
\]

Note that the last equality tells that for every projection \( P \) of rank one satisfying \( \text{Im } P \subset \text{Im } (B - A) \) the set \( \mathcal{M}(A, P) \) is not a singleton.

**Proof.** We will consider only one of the two possibilities, say \( A \leq B \). We choose a projection \( P \) of rank one with \( \text{Im } P \subset \text{Im } (B - A) \). As \( B - A \) is a finite rank operator, its restriction to \( \text{Im } (B - A) \) is an invertible operator of \( \text{Im } (B - A) \) onto itself. Let \( T : H \to H \) be a bounded self-adjoint linear operator defined by
\[
Tx = \left( (B - A)_{|\text{Im } (B - A)} \right)^{-1/2} x
\]
for every \( x \in \text{Im } (B - A) \) and \( Tx = 0 \), \( x \in \text{Ker } (B - A) \). Then \( TPT \) is a positive rank one operator whose image is contained in the image of \( Q \), the projection of rank two onto \( \text{Im } (B - A) \). Hence, \( TPT = sR \) for some positive real number \( s \) and some projection \( R \) of rank one satisfying \( R \leq Q \). It follows that
\[
B - \left( A + \frac{1}{s} P \right) = (B - A)^{1/2} Q (B - A)^{1/2} - (B - A)^{1/2} R (B - A)^{1/2}
\]
\[
= (B - A)^{1/2} (Q - R) (B - A)^{1/2} \geq 0.
\]
Hence, \( A + \frac{1}{s} P \leq B \), and therefore, \( A + \frac{1}{s} P \in E(H) \), and \( A + \frac{1}{s} P \) and \( B \) are adjacent. Consequently, \( \mathcal{M}(A, P) \in K_A(B) \). This completes the proof.
Lemma 2.6 Let $A, B \in E(H)$ and assume that $d(A, B) = 2$. If $A \leq B$, then $B \in S(A)$. Similarly, if $B \leq A$, then $B \in S(A)$.

Proof. We will prove just one of the two statements. So, assume that $A \leq B$ and let $C \in E(H)$ with $d(A, C) = 2$ and $K_A(B) \subset K_A(C)$. We need to show that $K_A(B) = K_A(C)$. We already know that $K_A(C) \subset \{M(A, P) \in \mathcal{L}(A) : \text{Im } P \subset \text{Im } (C - A)\}$ and $\{M(A, P) : \text{Im } P \subset \text{Im } (B - A)\} \subset K_A(B)$. Then $\{M(A, P) : \text{Im } P \subset \text{Im } (B - A)\} \subset \{M(A, P) \in \mathcal{L}(A) : \text{Im } P \subset \text{Im } (C - A)\}$, and since both $B - A$ and $C - A$ are of rank two, we actually have $\{M(A, P) \in \mathcal{L}(A) : \text{Im } P \subset \text{Im } (B - A)\} = \{M(A, P) \in \mathcal{L}(A) : \text{Im } P \subset \text{Im } (C - A)\}$ yielding that $K_A(B) = K_A(C)$.

Lemma 2.7 Let $\dim H \geq 3$ and let $A \in E(H)$ and $A \notin (0, I)$. Then there exist projections $P$ of rank one and $Q$ of rank two, and a positive real number $t$, such that $P \leq Q$, and either

- $A + tQ \in E(H)$ and $A - sP \notin E(H)$ for every positive real number $s$; or
- $A - tQ \in E(H)$ and $A + sP \notin E(H)$ for every positive real number $s$.

Proof. Assume first that the spectrum of $A$ is contained in $\{0, 1\}$. Then $A$ is a projection and the conclusion follows easily from the assumption that $\dim H \geq 3$. It remains to consider the case that there is a real number $q_0 \in (0, 1)$ such that both $q_0$ and 0 belong to the spectrum of $A$ or both $q_0$ and 1 belong to the spectrum of $A$. We will consider just one of these two possibilities, say the first one. Let $q$ be a real number, $q_0 < q < 1$. Denote by $R$ the spectral projection of $A$ corresponding to the interval $[0, q]$. Then $\dim \text{Im } R \geq 2$. If $A$ has a nontrivial kernel, then for every rank one projection $P$ whose image is contained in the kernel of $A$, every rank two projection $Q$ satisfying $P \leq Q \leq R$, and every positive real number $t$, $t \leq 1 - q$, we have $A + tQ \in E(H)$ and $A - sP \notin E(H)$ for every positive real number $s$.

Thus, it remains to consider the case when $A$ is injective. By the spectral theorem for self-adjoint operators there exists a sequence of orthonormal vectors $(x_n) \subset \text{Im } R$ such that

$$\langle Ax_n, x_n \rangle \leq \frac{1}{2^n}, \quad n = 1, 2, \ldots$$

Let $P$ be a rank one projection whose image is spanned by the unit vector

$$u = \sum_{n=1}^{\infty} 2^{-\frac{n}{2}} x_n.$$
As before we take any rank two projection $Q$ satisfying $P \leq Q \leq R$ and any positive real number $t$, $t \leq 1 - q$. Obviously, $A + tQ \in E(H)$. Let $s$ be any positive real number. We need to show that $A - sP \not\in E(H)$. Assume on the contrary that $A - sP \in E(H)$. Then $sP \leq A$ which yields
\[
\frac{s}{2^n} = s(x_n, u)^2 = \langle sPx_n, x_n \rangle \leq \langle Ax_n, x_n \rangle \leq \frac{1}{2^n}
\]
for every positive integer $n$, contradicting the fact that $s$ is positive.

\[\square\]

Now we are ready to prove the crucial lemma of this section.

**Lemma 2.8** Let $\dim H \geq 3$. For $A \in E(H)$ the following two statements are equivalent:

- $A \in (0, I)$;
- for every $B \in S(A)$ there exists $B_1 \in S(A)$ such that $K_A(B) = K_A(B_1)$ and $B \not\equiv B_1$.

**Proof.** Assume first that $A \in (0, I)$. Then the spectrum of $A$ is contained in some closed interval $[a, b]$ with $0 < a$ and $b < 1$. It follows that there exist positive real numbers $s, t$ such that $0 \leq A - sI \leq A \leq A + tI \leq I$.

In the next step we will show that if $B \in S(A)$, then either $A \leq B$, or $B \leq A$. So, assume that $B \in S(A)$. Then $B = A + R$ where $R$ is an operator of rank two. Let $Q$ be the projection of rank two such that $\text{Im} Q = \text{Im} R$ and denote $C = A + tQ \in E(H)$. By Lemma 2.5, $K_A(C) = \{M(A, P) : P \leq Q\}$. It follows then from the definition of $S(A)$ and Lemma 2.4 that
\[
K_A(B) = \{M(A, P) : P \leq Q\}.
\]

In particular, for every projection $P$ of rank one satisfying $P \leq Q$ we can find a real number $r$ such that $A + rP$ and $A + R$ are adjacent, or equivalently, $rP$ and $R$ are adjacent. As $P \leq Q$ and $\text{Im} Q = \text{Im} R$ we can consider only the restrictions of $P$, $Q$, and $R$ to the two-dimensional subspace $\text{Im} Q$. These restrictions can be identified with $2 \times 2$ hermitian matrices. Hence, we have a $2 \times 2$ invertible hermitian matrix $R$ with the property that for every $2 \times 2$ projection $P$ of rank one there exists a real number $r$ such that $R - rP$ is of rank one. In other words, $\det(R - rP) = 0$, which yields that $\det(I - rR^{-1}P) = 0$. It follows that the trace of a rank one matrix $R^{-1}P$ is nonzero for each projection $P$ of rank one. Let $x$ be a unit vector whose span is the image of $P$. Then $\text{tr}(R^{-1}P) = \langle R^{-1}x, x \rangle$. In other words, the numerical range of the hermitian matrix $R^{-1}$ does not contain 0. Hence, $R^{-1}$ has either both eigenvalues positive, or both eigenvalues negative, which yields that $R$ is either positive, or negative. Consequently, $A \leq B$ or $B \leq A$, as desired.

10
We have proved that each member of \( S(A) \) is comparable with \( A \). In order to complete the proof of one direction of our lemma it is enough to verify that if \( B, C \in S(A) \) and \( B \sim C \) then either \( B, C \geq A \), or \( B, C \leq A \). Namely, it follows immediately that if \( B, C \in S(A) \) and \( B \equiv C \) then either \( B, C \geq A \), or \( B, C \leq A \). This further yields that if \( B \) is any member of \( S(A) \), and \( R \) and \( Q \) are as above, and \( B \geq A \), then \( B_1 = A - sQ \) has all the desired properties, that is, \( B_1 \in S(A), \, K_A(B) = K_A(B_1), \) and \( B \not\equiv B_1 \). Of course, in the case when \( A \geq B \) we take \( B_1 = A + tQ \) to see that the second condition of our lemma is satisfied.

Thus, we need to show that if \( B, C \in S(A) \) and \( B \sim C \) then either \( B, C \geq A \), or \( B, C \leq A \). We know that both \( B \) and \( C \) are as above, and \( B, C \leq A \). Of course, in the case when \( B \geq A \) we take \( B_1 = A - sQ \) has all the desired properties, that is, \( B_1 \in S(A), \, K_A(B) = K_A(B_1), \) and \( B \not\equiv B_1 \). Of course, in the case when \( A \geq B \) we take \( B_1 = A + tQ \) to see that the second condition of our lemma is satisfied.

To prove the other direction assume that \( A \not\in (0, I) \). Then by Lemma 2.7 we have two possibilities. We will treat just one of them, say \( B \geq A \). From \( B \sim C \) we get the existence of \( D \in E(H) \) such that \( D \) is adjacent with each of the operators \( A, B, C \). In particular, \( D = A + qQ \) for some nonzero real number \( q \) and a projection \( Q \) of rank one. Because \( B - A \geq 0 \) and \( qQ \) are adjacent, we have necessarily \( q > 0 \), since otherwise \( B - D = (B - A) - qQ \geq B - A \) would be of rank at least two. But \( C - A \) and \( qQ \) are adjacent as well, and consequently, \( C - A \) is positive, as desired.

To prove the other direction assume that \( A \not\in (0, I) \). Then by Lemma 2.7 we have two possibilities. We will treat just one of them, say the first one, that is, we assume that there exist projections \( R \) of rank one and \( Q \) of rank two, and a positive real number \( t \) such that \( R \leq Q, \, A + tQ \in E(H), \) and \( A - sR \not\in E(H) \) for every positive real number \( s \). Set \( B = A + tQ \). It follows from Lemma 2.5 that \( K_A(B) = \{ M(A, P) : P \leq Q \} \). By Lemma 2.6, \( B \in S(A) \). As above we see that for every \( B_1 \in S(A) \) such that \( K_A(B) = K_A(B_1) \) we have either \( B_1 \leq A \) or \( B_1 \geq A \). We want to show that the first possibility cannot occur. Indeed, if \( B_1 = A - S \) with \( S \) positive operator of rank two and \( K_A(B) = K_A(B_1) \), then \( \text{Im } S = \text{Im } Q \), and because \( S \) is positive we have \( S \geq pQ \) for some positive real number \( p \). But then

\[
0 \leq B_1 = A - S \leq A - pQ \leq A - pR \leq A \leq I
\]

yielding that \( A - pR \in E(H) \), a contradiction.

Hence, for every \( B_1 \in S(A) \) such that \( K_A(B) = K_A(B_1) \) we have \( B_1 \geq A \). We then have \( B_1 = A + r_1R_1 + r_2R_2 \) where \( R_1, R_2 \) are orthogonal rank one projections with \( R_1 + R_2 = Q \) and \( 0 < r_1 \leq r_2 \). It follows that \( A + r_1R_1 \) is adjacent to each of the following three operators: \( A, \, A + r_1Q \) and \( B_1 \). Thus, \( B_1 \sim A + r_1Q \). It is trivial to check that \( A + r_1Q \equiv A + tQ = B \). Thus, \( B_1 \equiv B \). We have proved that if \( A \not\in (0, I) \), then \( A \) does not satisfy the second condition. This completes the proof.

\[ \square \]

Note that Proposition 2.1 is a straightforward consequence of the above lemma.
The main results of this section characterize bijective adjacency preserving maps on $E(H)$. Denote by $F(H)$ the set of all effects that are finite rank perturbations of $\frac{1}{2}I$,

$$F(H) = \left\{ \frac{1}{2}I + C \in E(H) : C \text{ is of finite rank} \right\}.$$ Choose and fix a unitary operator $U : H \to H$ and define $\phi : E(H) \to E(H)$ by $\phi(A) = UAU^*$ whenever $A \in F(H)$ and $\phi(A) = A$ otherwise. Since two adjacent effects $A, B \in E(H)$ either both belong to $F(H)$, or both belong to the complement of $F(H)$, the bijective map $\phi$ preserves adjacency. It is now clear that the behaviour of a bijective adjacency preserving map $\phi : E(H) \to E(H)$ on $F(H)$ is completely unrelated to the behaviour of $\phi$ on the complement of $F(H)$. Thus, when studying such maps we can expect to have a nice description only if we consider the restrictions of such maps to $F(H)$.

Let $\phi : E(H) \to E(H)$ be an adjacency preserving bijective map satisfying $\phi(\frac{1}{2}I) = \frac{1}{2}I$. For any $B = \frac{1}{2}I + A \in F(H)$ there exist a nonnegative integer $r$, real numbers $t_1, \ldots, t_r \in [-\frac{1}{2}, \frac{1}{2}]$, and pairwise orthogonal rank one projections $P_1, \ldots, P_r$ such that $B = \frac{1}{2}I + \sum_{j=1}^r t_jP_j$. Then $B$ is adjacent to $\frac{1}{2}I + \sum_{j=1}^{r-1} t_jP_j$, $\frac{1}{2}I + \sum_{j=1}^{r-1} t_jP_j$ is adjacent to $\frac{1}{2}I + \sum_{j=1}^{r-2} t_jP_j$, ..., and $\frac{1}{2}I + r_1P_1$ is adjacent to $\frac{1}{2}I$. Therefore, $\phi(B)$ is adjacent to $\phi(\frac{1}{2}I + \sum_{j=1}^{r-1} t_jP_j)$, $\phi(\frac{1}{2}I + \sum_{j=1}^{r-2} t_jP_j)$ is adjacent to $\phi(\frac{1}{2}I + \sum_{j=1}^{r-3} t_jP_j)$, ..., and $\phi(\frac{1}{2}I + r_1P_1)$ is adjacent to $\phi(\frac{1}{2}I) = \frac{1}{2}I$. It follows that $\phi(B) - \frac{1}{2}I$ is a finite rank operator. We have shown that $\phi(F(H)) \subset F(H)$. But the inverse of $\phi$ has the same properties as $\phi$, and therefore we have $\phi(F(H)) = F(H)$. Set $G(H) = F(H) \cap (0, I)$.

**Theorem 2.9** Let $\dim H \geq 3$. Assume that $\phi : E(H) \to E(H)$ is a bijective adjacency preserving map. Then there exist real numbers $p, q \in (-\infty, 1)$ and a bijective linear or conjugate-linear bounded operator $T : H \to H$ with $\|T\| \leq 1$ such that either

$$\phi(A) = f_q \left( (f_p(T^*T))^{-1/2} f_p(TAT^*) (f_p(T^*T))^{-1/2} \right), \quad A \in G(H),$$

or

$$\phi(A) = f_q \left( (f_p(T^*T))^{-1/2} f_p(T(I - A)T^*) (f_p(T^*T))^{-1/2} \right), \quad A \in G(H).$$

**Remark.** We conjecture that the theorem holds also when $\dim H = 2$. Unfortunately, the assumption that $\dim H \geq 3$ is essential when using our approach that depends heavily on Lemma 2.8. Let us mention that another essential ingredient of the proof, that is, [26, Theorem 2.3], holds also in the case when $\dim H = 2$. Thus, all that one needs to do to extend the theorem to this low-dimensional case is to verify Proposition 2.1 in the case when $\dim H = 2$. This should not be too difficult as in this special case the set $E(H) \setminus (0, I)$ is rather
simple. Namely, it consists of the zero operator, the identity operator, and all matrices of the form $tP$ or $I - tP$, where $0 < t \leq 1$ and $P$ is a projection of rank one. For our purposes the case when $\dim H = 2$ is not that important and thus, we leave it as an open question.

Proof. By Proposition 2.1 we have $\phi((0, I)) = (0, I)$. In particular, $\phi((1/2)I) = R$ for some $R \in (0, I)$. Let $\xi$ be as in Lemma 2.3. Set

$$\psi = \xi^{-1} \circ \phi : E(H) \to E(H).$$

Then $\psi$ is a bijective map of $E(H)$ onto itself preserving adjacency and satisfying $\psi((1/2)I) = (1/2)I$. We further know that the restriction of $\psi$ to $G(H)$ is a bijective map of $G(H)$ onto itself.

We denote by $S_F(H)^{> - I}$ the set of all bounded self-adjoint linear finite rank operators $A$ on $H$ satisfying $A > -I$ ($T > S$ means that $T - S$ is a bounded positive invertible linear operator). We already know that $X \mapsto X^{-1}$ is a bijective adjacency preserving map of $(0, I)$ onto the set of all positive operators $A$ satisfying $A > I$. It is then easy to conclude that the map $X \mapsto X^{-1} - 2I$ is a bijective adjacency preserving map of $G(H)$ onto $S_F(H)^{> - I}$. Its inverse is given by the formula $X \mapsto (2I + X)^{-1}, X \in S_F(H)^{> - I}$. Hence, the map $\tau$ defined by

$$\tau(A) = \psi \left( ((2I + A)^{-1})^{-1} - 2I \right)$$

is a bijective adjacency preserving map of $S_F(H)^{> - I}$ onto itself. A straightforward calculation shows that $\tau(0) = 0$. By [26, Theorem 2.3], there exists a unitary or antiunitary operator $U$ such that either

$$\tau(A) = UAU^*$$

for every $A \in S_F(H)^{> - I}$, or

$$\tau(A) = U(I + A)^{-1}U^* - I$$

for every $A \in S_F(H)^{> - I}$. It follows that either

$$\psi(A) = UAU^*, \quad A \in G(H),$$

or

$$\psi(A) = U(I - A)U^*, \quad A \in G(H).$$

We will consider just one of the above two possibilities, say the second one. Then it follows from $\psi = \xi^{-1} \circ \phi$ that

$$\phi(A) = \xi(U(I - A)U^*)$$

$$= f_q \left( f_p(S^2)^{-1/2} f_p(SU(I - A)U^* S)(f_p(S^2))^{-1/2} \right), \quad A \in G(H).$$
Set $SU = T$ and note that $S^2 = TT^*$ to conclude that

$$\phi(A) = f_q \left( (f_p(TT^*))^{-1/2} f_p(T(I - A)T^*) (f_p(TT^*))^{-1/2} \right)$$

for all $A \in G(H)$, as desired.

\[\square\]

**Corollary 2.10** Let $\dim H \geq 3$. Assume that $\phi : E(H) \to E(H)$ is a bijective adjacency preserving map and suppose that

$$\phi \left( \frac{1}{2} I \right) = \frac{1}{2} I.$$

Then there exists a unitary or an antiunitary operator $U : H \to H$ such that either

$$\phi(A) = UAU^*, \quad A \in G(H),$$

or

$$\phi(A) = U(I - A)U^*, \quad A \in G(H).$$

**Proof.** We have here the additional assumption that $\phi \left( \frac{1}{2} I \right) = \frac{1}{2} I$. Thus, exactly the same proof works with $\xi$ being the identity map, or equivalently, $\phi(A) = \psi(A)$, $A \in E(H)$.

\[\square\]

**Corollary 2.11** Let $\dim H \geq 3$. Assume that $\phi : E(H) \to E(H)$ is a bijective adjacency preserving map and suppose that there exists $r \in (0, 1)$ such that

$$\phi \left( \frac{1}{2} I \right) = rI.$$

Then there exist a real number $p < 1$ and a unitary or an antiunitary operator $U : H \to H$ such that either

$$\phi(A) = f_p(UAU^*), \quad A \in G(H),$$

or

$$\phi(A) = f_p(U(I - A)U^*), \quad A \in G(H).$$

**Proof.** We have here the additional assumption that $\phi \left( \frac{1}{2} I \right) = rI$. Thus, exactly the same proof works with $\xi$ being the map $A \mapsto f_p(A)$ for some $p < 1$.

\[\square\]
Corollary 2.12 Let $\dim H \geq 3$. Assume that $\phi : (0, I) \to (0, I)$ is a bijective adjacency preserving map. Then there exist real numbers $p, q \in (-\infty, 1)$ and a bijective linear or conjugate-linear bounded operator $T : H \to H$ with $\|T\| \leq 1$ such that either

$$
\phi(A) = f_q \left( (f_p(TT^*))^{-1/2} f_p(TAT^*) (f_p(TT^*))^{-1/2} \right), \quad A \in G(H),
$$
or

$$
\phi(A) = f_q \left( (f_p(TT^*))^{-1/2} f_p(T(I - A)T^*) (f_p(TT^*))^{-1/2} \right), \quad A \in G(H).
$$

Proof. In the first step of the proof of the main theorem we have shown that every bijective adjacency preserving map on $E(H)$ maps $(0, I)$ onto itself. The rest of the proof was devoted to the verification of the above statement.

3 Automorphisms of effect algebras

The aim of this section is to apply the main results of the previous section to describe the general form of various symmetries of $E(H)$.

3.1 Order automorphisms

In the first subsection we will reprove the following result from [25]. Recall that a map $\phi : E(H) \to E(H)$ is called an order automorphism if it is bijective and for every pair $A, B \in E(H)$ we have $A \leq B \iff \phi(A) \leq \phi(B)$.

Theorem 3.1 Let $\dim H \geq 3$. Assume that $\phi : E(H) \to E(H)$ is an order automorphism. Then there exist real numbers $p, q \in (-\infty, 1)$ and a bijective linear or conjugate-linear bounded operator $T : H \to H$ with $\|T\| \leq 1$ such that

$$
\phi(A) = f_q \left( (f_p(TT^*))^{-1/2} f_p(TAT^*) (f_p(TT^*))^{-1/2} \right), \quad A \in E(H).
$$

Once we have Theorem 2.9, the proof of the above statement is short and simple. Moreover, as we will see in the other subsections, many other characterizations of automorphisms of effect algebras can be deduced rather quickly from this theorem and its corollaries.

We say that two effects $A$ and $B$ are comparable if $A \leq B$ or $B \leq A$. In both cases we can define an operator interval between $A$ and $B$: in the first case we have $[A, B] = \{ C \in E(H) : A \leq C \leq B \}$, while in the second case we have $[B, A] = \{ C \in E(H) : B \leq C \leq A \}$ (note that we have already defined operator intervals in the previous section with an essential difference that there we have assumed that the difference of the endpoints is invertible, while here we allow $B - A$ to be singular).
It is clear that every order automorphism \( \phi \) of \( E(H) \) preserves comparability, that is, for every pair \( A, B \in E(H) \) the effects \( A \) and \( B \) are comparable if and only if \( \phi(A) \) and \( \phi(B) \) are comparable. The crucial lemma of this subsection characterizes adjacency in terms of comparability.

**Lemma 3.2** For \( A, B \in E(H), A \neq B \), the following are equivalent:

- \( A \) and \( B \) are adjacent,
- \( A \) and \( B \) are comparable and any two elements \( C, D \) from the operator interval between \( A \) and \( B \) are comparable as well.

**Proof.** Assume first that \( A \) and \( B \) are adjacent. Then \( B = A + tP \) for some rank one projection \( P \) and some nonzero real number \( t \). We may assume that \( t > 0 \), since otherwise we just interchange \( A \) and \( B \). Then \( A \leq B \). We have \( C \in [A, B] \) if and only if \( 0 \leq C - A \leq tP \), and it is rather easy to verify that this is true if and only if \( C - A = sP \) for some real number \( s \), \( 0 \leq s \leq t \). Hence, if \( C \) and \( D \) belong to the interval \( [A, B] \), then they are of the form \( C = A + sP \) and \( D = A + tP \) for some real numbers \( s \) and \( r \), and are therefore comparable.

Assume now that \( A \) and \( B \) are not adjacent. If they are not comparable, we are done. Thus, we assume from now on that they are comparable. Then we have two possibilities: either \( A \leq B \), or \( B \leq A \). Once again we will consider just one of the two possibilities, say \( A \leq B \). As \( B - A \geq 0 \) is neither the zero operator nor a rank one operator we can find a projection \( P \) of rank two and a positive real number \( c \) such that \( cP \leq B - A \). Let \( Q \) be a rank one projection satisfying \( Q \leq P \). Then both \( A + cQ \) and \( A + c(P - Q) \) belong to \( [A, B] \) but are not comparable.

\[ \square \]

**Proof of Theorem 3.1.** It follows from Lemma 3.2 that \( \phi \) preserves adjacency. By Theorem 2.9, there exist real numbers \( p, q \in (-\infty, 1) \) and a bijective linear or conjugate-linear bounded operator \( T : H \to H \) with \( \|T\| \leq 1 \) such that either

\[ \phi(A) = f_q \left( (f_p(TT^*))^{-1/2} f_p(TAT^*) (f_p(TT^*))^{-1/2} \right), \quad A \in G(H), \]

or

\[ \phi(A) = f_q \left( (f_p(TT^*))^{-1/2} f_p(T(I - A)T^*) (f_p(TT^*))^{-1/2} \right), \quad A \in G(H). \]

Because of order preserving property we have the first possibility. Composing the map \( \phi \) with the inverse of the order automorphism of \( E(H) \) given by \( A \mapsto f_q \left( (f_p(TT^*))^{-1/2} f_p(TAT^*) (f_p(TT^*))^{-1/2} \right), \quad A \in E(H) \), we may assume that

\[ \phi(A) = A \]

16
for every $A \in G(H)$ and we need to prove that $\phi(A) = A$ for every $A \in E(H)$.

Let us first show that this is true for every $A \in E(H)$ of the form $A = sP$ for some projection $P$ of rank one and some positive real number $s < 1$. Indeed, for every $\varepsilon$, $0 < \varepsilon < 2$, and every finite rank projection $Q$ orthogonal to $P$ we have

$$A \leq sP + \frac{1}{2}(I - P - (1 - \varepsilon)Q),$$

and consequently,

$$\phi(A) \leq \phi \left( sP + \frac{1}{2}(I - P - (1 - \varepsilon)Q) \right) = sP + \frac{1}{2}(I - P - (1 - \varepsilon)Q).$$

It follows easily that $\phi(A) \leq A$. Thus, $\phi(A) = rP$ for some positive real number $r \leq s$ (note that $r$ cannot be zero as it is clear that $\phi(0) = 0$ and $\phi$ is bijective). Because the inverse of $\phi$ has the same properties as $\phi$ we conclude that $\phi(A) = A$.

It is now easy to conclude the proof. Our theorem is a known result and it was included only to show the efficiency of our method. Therefore we will just outline the rest of the proof. For the details see [26]. One can easily see that $\phi(P) = P$ for every projection $P$ of rank one. Thus, we have $\phi(A) = A$ for every $A \in E(H)$ of rank one. If $A \in E(H)$ is invertible, then for every projection $P$ of rank one there is a positive real number $c$ such that $cP \leq A$. It follows that $cP \leq \phi(A)$. Hence, $\phi(A)$ is invertible as well. For every rank one operator $R \in E(H)$ we have $R \leq A$ if and only if $\phi(R) = R \leq A$. It follows easily that $\phi(A) = A$. For an arbitrary $A \in E(H)$ we have $\phi(A) \leq \phi(B) = B$ for every invertible $B \in E(H)$ satisfying $A \leq B$. It follows that $\phi(A) \leq A$ for every $A \in E(H)$ and since the inverse of $\phi$ has the same properties as $\phi$, we finally conclude that $\phi(A) = A$, $A \in E(H)$.

\[ \square \]

### 3.2 Sequential automorphisms

For $A, B \in E(H)$ their sequential product is defined by $A \circ B = A^{1/2}BA^{1/2}$. It is then clear that a map $\phi : E(H) \to E(H)$ is called a sequential automorphism if $\phi$ is bijective and

$$\phi \left( A^{\frac{1}{2}}BA^{\frac{1}{2}} \right) = \phi(A)^{\frac{1}{2}} \phi(B) \phi(A)^{\frac{1}{2}}$$

for all pairs $A, B \in E(H)$. The following characterization of sequential automorphisms of $E(H)$ has been known before [17, Corollary 2.7.7].

**Theorem 3.3** Let $\dim H \geq 3$. Assume that $\phi : E(H) \to E(H)$ is a sequential automorphism. Then there exists a unitary or antiunitary operator $U$ on $H$ such that

$$\phi(A) = UAU^*$$
for all $A \in E(H)$.

In [17] this result was deduced from the characterization of zero-product preserving order automorphisms of $E(H)$. We will reprove it using our characterization of adjacency preserving maps on $E(H)$. Because the result is not new we will give just the outline of the proof.

If $P \in E(H)$ is a projection then putting $A = B = P$ in (6) we conclude that $\phi(P)$ is a projection as well. It is well-known that $A \circ B = B \circ A$ if and only if $AB = BA$ (note that $A \circ B \in E(H)$ for all $A, B \in E(H)$, while $AB$ need not be even self-adjoint). And finally $A \circ B = 0$ if and only if $AB = 0$ which is equivalent to $BA = 0$. It follows that $\phi$ preserves commutativity and zero products. As scalar operators are the only effects commuting with all effects we necessarily have $\phi(tI) = g(t)I$ for some bijective function $g : [0, 1] \to [0, 1]$. Consequently, $\phi(0) = 0$ and $\phi(I) = I$. It follows from (6) that $g(ts) = g(t)g(s)$, $t, s \in [0, 1]$. It is well-known that any multiplicative function $g$ of the unit interval is of the form $g(t) = t^r$ for some positive real number $r$.

Let $\mathcal{P} \subset E(H)$ denote the set of all projections on $H$. We know that $\phi(\mathcal{P}) = \mathcal{P}$. For $P \in \mathcal{P}$ denote $P(\perp) = \{Q \in \mathcal{P} : PQ = QP = 0\}$. Clearly, we have $\phi(P(\perp)) = \phi(P)(\perp)$. If we exclude the zero projection, then rank one projections can be characterized among all projections as those projections $P$ for which the set $P(\perp)$ is maximal. Hence, $\phi$ maps the set of projections of rank one onto itself.

We next observe that every sequential automorphism $\phi$ preserves the Jordan triple product, that is, we have $\phi(ABA) = \phi(A)\phi(B)\phi(A)$ for every pair $A, B \in E(H)$. To verify this it is enough to show that $\phi(A^2) = \phi(A)^2$, $A \in E(H)$. This equation follows from (6) with $A = B$ and the uniqueness of the positive square root of a positive operator.

Let $A, B \in E(H)$. We denote $\mathcal{G}(A, B) = \{P \in \mathcal{P} : PAP = PBP\}$. Clearly, $\phi(\mathcal{G}(A, B)) = \mathcal{G}(\phi(A), \phi(B))$. Assume that $A$ and $B$ are adjacent, that is, $B = A + tR$ for some nonzero real $t$ and some projection $R$ of rank one. Then obviously, $\mathcal{G}(A, B) = R(\perp)$.

Suppose now that for a given pair $A, B \in E(H)$ we have $\mathcal{G}(A, B) = R(\perp)$ for some projection $R$ of rank one. We will prove that then $A$ and $B$ are adjacent. Indeed, we have $(I - R)(B - A)(I - R) = 0$ which implies that either $B - A$ is a nonzero scalar multiple of $R$ and we are done; or $B - A$ is a rank two self-adjoint operator with one positive eigenvalue and one negative eigenvalue. In the second case an elementary linear algebra argument yields the existence of a rank one projection $Q$ which is not orthogonal to $R$ such that $Q(B - A)Q = 0$, a contradiction.

It follows that a pair of effects $A, B$ is adjacent if and only if $\mathcal{G}(A, B)$ is $R(\perp)$ for some projection $R$ of rank one. Consequently, $\phi$ preserves adjacency. By Corollary 2.11, there exist a real number $p < 1$ and a unitary or an antunitary operator $U : H \to H$ such that either $\phi(A) = f_p(UAU^*)$, $A \in G(H)$, or $\phi(A) = f_p(U(I - A)U^*)$, $A \in G(H)$. Replacing $\phi$ by $A \mapsto \phi(U^*AU)$ we may
assume that either $\phi(A) = f_P(A)$ for every $A \in G(H)$, or $\phi(A) = f_P(I - A)$ for every $A \in G(H)$.

If $A, B \in G(H)$, then $(A \circ B)^{\frac{1}{2}} \in G(H)$. We know that $\phi(S^\frac{1}{2})^2 = \phi(S)$ for every $S \in E(H)$, and therefore, either

$$f_P(A)^\frac{1}{2} f_P(B)^{\frac{1}{2}} = \phi(A)^\frac{1}{2} \phi(B)^{\frac{1}{2}} = \phi(A^{\frac{1}{2}} B A^{\frac{1}{2}})$$

$$= \phi \left( \left( A^{\frac{1}{2}} B A^{\frac{1}{2}} \right)^\frac{1}{2} \right)^2 = f_P \left( \left( A^{\frac{1}{2}} B A^{\frac{1}{2}} \right)^\frac{1}{2} \right)^2$$

for all $A, B \in G(H)$, or

$$f_P(I - A)^{\frac{1}{2}} f_P(I - B)^{\frac{1}{2}} = f_P \left( I - (A^{\frac{1}{2}} B A^{\frac{1}{2}})^\frac{1}{2} \right)^2$$

for all $A, B \in G(H)$.

A straightforward verification yields that we necessarily have the first possibility with $p = 0$, that is, $\phi(A) = A$ for every $A \in G(H)$. In particular, $\phi((1/2)I) = (1/2)I$, which further yields that $g(t) = t, t \in [0, 1]$, or equivalently, $\phi(tI) = tI$ for all real $t$ from the unit interval.

We know that for every projection $P$ of rank one there is a projection $Q$ of rank one such that $\phi(P) = Q$. Applying (6) again with $A = tI$ and $B = P$ we arrive at $\phi(tP) = tQ$ for every $t \in [0, 1]$. Using $\phi(A) = A$ for every $A \in G(H)$ and (6) we conclude that $\phi(P) = P$ for every projection $P$ of rank one. It follows that $\phi$ acts like the identity on every rank one operator.

Finally, for an arbitrary $A \in E(H)$ and an arbitrary projection $P$ of rank one we have

$$PAP = \phi(PAP) = \phi(P) \phi(A) \phi(P) = P \phi(A) P,$$

yielding that $\phi(A) = A$ for every $A \in E(H)$. We are done.

### 3.3 Geometric and harmonic means

The first two subsections illustrated the efficiency of our method by reproving two known results. In the last two subsections we will use our approach to obtain new results concerning symmetries of the effect algebra. We will first study bijective maps preserving geometric mean or harmonic mean. Such maps have been already studied on the set of all positive operators with different methods [18, 19]. The results in the case of the effect algebra are expected modifications of the known theorems for positive operators. In the last section we will have a more interesting situation where once again the characterization of a certain preserver on the effect algebra is essentially more complicated than in the case of all positive operators or the case of all self-adjoint operators.
Let us first recall the definitions of geometric and harmonic means for positive operators. For positive operators $A,B : H \to H$ the most natural definitions of their geometric mean and harmonic mean were given by Ando [1] by the formulae

$$A \sharp B = \max \left\{ X \geq 0 : \begin{bmatrix} A & X \\ X & B \end{bmatrix} \succeq 0 \right\},$$

and

$$A \odot B = \max \left\{ X \geq 0 : \begin{bmatrix} 2A & 0 \\ 0 & 2B \end{bmatrix} \succeq \begin{bmatrix} X & X \\ X & X \end{bmatrix} \right\},$$

respectively.

Here are some basic properties of these two means. Let $A,B,C,D$ be positive operators. Then $A \sharp B = B \sharp A$ and $A \odot B = B \odot A$. If $A \leq C$ and $B \leq D$, then $A \sharp B \leq C \sharp D$ and $A \odot B \leq C \odot D$. For every invertible bounded linear or conjugate-linear operator $T$ on $H$ we have $T(A \sharp B)T^* = (TAT^*) \sharp (TBT^*)$ and $T(A \odot B)T^* = (TAT^*) \odot (TBT^*)$. For any positive real numbers $t,s$ we have $tA \sharp sB = \sqrt{ts} (A \sharp B)$ and $tA \odot tB = t(A \odot B)$.

If $A$ invertible, then

$$A \sharp B = A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2} \quad (7)$$

and

$$A \odot B = 2A(A + B)^{-1}B. \quad (8)$$

And finally, if $P$ is a rank one projection on $H$ and if we set $\lambda(A,P) = \sup \{ t \in [0,\infty) : tP \leq A \}$, then

$$A \sharp P = \sqrt{\lambda(A,P)}P$$

and

$$A \odot P = \frac{2\lambda(A,P)}{\lambda(A,P) + 1}P.$$  

According to [3, Theorem 4] we have

$$\lambda(A,P) = \begin{cases} \|A^{-1/2}x\|^{-2} & \text{if } x \in \text{Im}(A^{1/2}) \\ 0 & \text{otherwise} \end{cases},$$

where $x$ is a unit vector which spans the image of $P$ and $A^{-1/2}$ denotes the inverse of $A^{1/2}$ on its image.

Assume now that $A,B \in E(H)$. Then $A \leq I$ and $B \leq I$, and therefore, $A \sharp B \leq I \sharp I = I$ and $A \odot B \leq I$. Thus, both $\sharp$ and $\odot$ are binary operations on $E(H)$.

Both theorems in this section can be proved as direct corollaries of the results from the second section. It turns out that it is more efficient to deduce them from our characterization of order automorphisms.
Theorem 3.4 Let $\dim H \geq 3$. Assume that $\phi : E(H) \to E(H)$ is a bijective map such that
\[ \phi(A \sharp B) = \phi(A) \sharp \phi(B), \quad A, B \in E(H). \]
Then there exists a unitary or antiunitary operator $U$ on $H$ such that
\[ \phi(A) = UAU^* \]
for all $A \in E(H)$.

Proof. Let $A \in E(H)$. Set $A^\sharp = \{ C \in E(H) : C = B \sharp A \text{ for some } B \in E(H) \}$.
Then it is easy to check that the following are equivalent:
- $A = I$,
- $A^\sharp = E(H)$.

Indeed, if $A = I$, then for every $B \in E(H)$ we have $B \sharp A = B \sharp I = B^{1/2}$, and since $B \mapsto B^{1/2}$ is a bijection of $E(H)$ onto itself, we have $I^5 = E(H)$. If on
the other hand $A \neq I$, then $A \sharp B \neq I$ for all $B \in E(H)$. Otherwise we would have $I = A \sharp B \leq A \sharp I = A^{1/2} \leq I$ yielding that $A = I$, a contradiction.

Clearly, $\phi(A^\sharp) = \phi(A)^2$. It follows that $\phi(I) = I$.

For every $A \in E(H)$ we have
\[ \phi(A^{1/2}) = \phi(A \sharp I) = \phi(A) \sharp \phi(I) = \phi(A)^{1/2} \]
and consequently,
\[ \phi(A^2) = \phi(A)^2, \quad A \in E(H). \] (9)

In particular, $\phi$ maps projections to projections.

According to [5, Proposition 2] we have $P \sharp Q = P \land Q$ for every pair of projections $P, Q$. Thus, the restriction of $\phi$ to the set of all projections is a lattice automorphism. In particular, $\phi(0) = 0$ and $\phi$ maps the set of projections of rank one onto itself.

Clearly, for every projection $P$ of rank one and any two scalars $t, s \in [0, 1]$ we have
\[ tP \sharp sP = \sqrt{ts}P. \]

Let $P$ be any projection of rank one and $t$ any scalar, $0 \leq t \leq 1$. Since $tP = (t^2P) \sharp P$ we have
\[ \phi(tP) = \phi(t^2P) \sharp \phi(P) = \sqrt{\lambda(\phi(t^2P), \phi(P))} \phi(P). \]

Thus, for every projection $P$ of rank one there exists a function $h_P : [0, 1] \to [0, 1]$ such that
\[ \phi(tP) = h_P(t) \phi(P), \quad t \in [0, 1]. \]

For an arbitrary pair of real numbers $t, s \in [0, 1]$ we calculate
\[ h_P(\sqrt{ts}) \phi(P) = \phi(\sqrt{ts}P) = \phi(tP \sharp sP) = \phi(tP) \sharp \phi(sP) \]
\[ h_P(t) \phi(P) \cdot h_P(s) \phi(P) = \sqrt{h_P(t) h_P(s)} \phi(P). \]

It follows that
\[ h_P(\sqrt{ts}) = \sqrt{h_P(t) h_P(s)}, \quad t, s \in [0, 1]. \]

Using the fact that \( h_P(1) = 1 \) and setting \( s = 1 \) we arrive at \( h_P(\sqrt{t}) = \sqrt{h_P(t)} \)
which further yields that \( h_P \) is multiplicative, that is,
\[ h_P(ts) = h_P(t) h_P(s), \quad t, s \in [0, 1]. \]

Therefore, for every projection \( P \) of rank one there exists a positive constant \( c_P \) such that
\[ \phi(tP) = h_P(t) \phi(P) = t^{c_P} \phi(P), \quad t \in [0, 1]. \quad (10) \]

Let \( A \in E(H) \) be any effect and \( P \) any projection of rank one. Then
\[
\sqrt{\lambda(\phi(A), \phi(P))} \phi(P) = \phi(A) \phi(P) = \phi(t_P P)
= \phi \left( \sqrt{\lambda(A, P) P} \right) = \lambda(A, P)^{c_P} \phi(P),
\]
and therefore,
\[ \lambda(\phi(A), \phi(P)) = \lambda(A, P)^{c_P}. \]

By [3, Theorem 1] we know that for every pair \( A, B \in E(H) \) we have \( A \leq B \)
if and only if \( \lambda(A, P) \leq \lambda(B, P) \) for every projection \( P \) of rank one. It follows
that \( A \leq B \) if and only if \( \phi(A) \leq \phi(B) \).

Hence, Theorem 3.1 tells that there exist real numbers \( p, q \in (-\infty, 1) \) and a
bijective linear or conjugate-linear bounded operator \( T : H \to H \) with \( \|T\| \leq 1 \)
such that
\[ \phi(A) = f_q \left( (f_p(TT^*))^{-1/2} f_p(TAT^*) (f_p(TT^*))^{-1/2} \right), \quad A \in E(H). \quad (11) \]

Clearly, for every projection \( P \) of rank one the operator \( TPT^* \) is of the
form \( rQ \) for some \( r, 0 \leq r \leq 1 \), and some projection \( Q \) of rank one. Further, for
every projection \( P \) of rank one and every real number \( t, 0 \leq t \leq 1, \) we have
\( f_p(tP) = f_p(tP) \). Now, comparing (10) and (11) we conclude after some
straightforward calculations that \( c_P = 1 \). From the fact that \( \phi(tP) = t \phi(P) \)
for every projection \( P \) of rank one and every real \( t, 0 \leq t \leq 1 \), we conclude that
\( \phi(tI) = tI, \quad 0 \leq t \leq 1 \). It follows that \( \phi(A) = UAU^*, \quad A \in E(H) \),
for some unitary or antiunitary \( U \).

\( \square \)

**Theorem 3.5** Let \( \dim H \geq 3 \). Assume that \( \phi : E(H) \to E(H) \) is a bijective
map such that
\[ \phi(A \diamond B) = \phi(A) \diamond \phi(B), \quad A, B \in E(H). \]
Then there exists a unitary or antiunitary operator $U$ on $H$ such that

$$\phi(A) = UAU^*$$

for all $A \in E(H)$.

Proof. In the same way as in the previous theorem we show that $\phi(I) = I$.

It follows from (8) that $A \in E(H)$ is a projection if and only if $I \diamond A = A$. Consequently, $\phi$ maps the set of all projections onto itself.

According to [5, Proposition 2] we have $P \diamond Q = P \land Q$ for every pair of projections $P, Q$. Thus, the restriction of $\phi$ to the set of all projections is a lattice automorphism. In particular, $\phi(0) = 0$ and $\phi$ maps the set of projections of rank one onto itself.

Using exactly the same ideas as in [19, pp. 3063-3064] we prove that $\phi(tP) = t\phi(P)$ for every projection $P$ of rank one and every real $t$, $0 \leq t \leq 1$. It is then easy to complete the proof following the same ideas as in the previous theorem.

\[ \square \]

3.4 Maps preserving the invertibility of the difference of effects

The main results in this paper deal with bijective adjacency preserving maps on $E(H)$. Assume for a moment that $H$ is a finite-dimensional Hilbert space, $\dim H = n$. Note that rank is subadditive, that is, $\text{rank}(A + B) \leq \text{rank} A + \text{rank} B$ for any two matrices $A$ and $B$ of the same size. It follows that $E(H)$ equipped with the distance function $d$ defined by $d(A, B) = \text{rank} (B - A)$, $A, B \in E(H)$, is a metric space. The distance $d$ is called the arithmetic distance. Hence, maps preserving adjacency are maps preserving the minimal nonzero arithmetic distance.

It is natural to ask what about preserving the maximal possible arithmetic distance? As $d(A, B) \leq n$ for all $A, B \in E(H)$, we are interested in bijective maps $\phi : E(H) \to E(H)$ such that for each pair $A, B \in E(H)$ we have $\text{rank} (B - A) = n$ if and only if $\text{rank} (\phi(B) - \phi(A)) = n$. In other words, we want to characterize bijective maps $\phi$ with the property that $A - B$ is invertible if and only if $\phi(A) - \phi(B)$ is invertible. With this formulation we do not need to restrict ourselves to the finite-dimensional case. Thus, from now on $H$ again denotes any Hilbert space, finite or infinite-dimensional.

Maps preserving invertibility of operator differences defined on the algebra of all bounded linear operators on a Hilbert space $H$ have been already studied in [6], while such maps on self-adjoint operators or positive self-adjoint operators have been characterized in [26]. The background of this kind of problems going back to the famous Kaplansky’s problem on invertibility preserving maps and Kowalski-Słodkowski’s extension of the celebrated Gleason-Kahane-Żelazko theorem is explained in [6].

23
It turns out that under the additional assumption that the endpoints of the operator interval 
\([0, I] = E(H)\) are mapped into themselves, the bijective maps preserving the invertibility of the difference of effects are order automorphisms.

**Theorem 3.6** Let \(\dim H \geq 3\). Assume that \(\phi : E(H) \to E(H)\) is a bijective map such that \(\phi(0) = 0, \phi(I) = I\), and for every pair \(A, B \in E(H)\) the operator \(B - A\) is invertible if and only if \(\phi(B) - \phi(A)\) is invertible. Then there exist real numbers \(p, q \in (-\infty, 1)\) and a bijective linear or conjugate-linear bounded operator \(T : H \to H\) with \(\|T\| \leq 1\) such that

\[
\phi(A) = f_q \left( (f_p(TT^*))^{-1/2} f_p(TAT^*) (f_p(TT^*))^{-1/2} \right), \quad A \in E(H).
\]

Besides our main results on adjacency preserving maps of the effect algebra the main tool for proving this theorem is the following lemma from [26]. We denote by \((0, \infty)\) the set of all positive bounded invertible linear operators on \(H\), and by \((I, \infty)\) the set of all self-adjoint bounded linear operators \(A : H \to H\) such that \(A \geq I\) and \(A - I\) is invertible.

**Lemma 3.7** Assume that \(A, B \in (0, \infty)\) with \(A \neq B\). Then the following are equivalent:

- \(A\) and \(B\) are adjacent,
- there exists \(C \in (0, \infty) \setminus \{A, B\}\) such that for every \(D \in (0, \infty)\) the invertibility of \(D - C\) implies that \(D - A\) is invertible or \(D - B\) is invertible.

For us the following straightforward consequence will be important.

**Corollary 3.8** Assume that \(A, B \in (0, I)\) with \(A \neq B\). Then the following are equivalent:

- \(A\) and \(B\) are adjacent,
- there exists \(C \in (0, I) \setminus \{A, B\}\) such that for every \(D \in (0, I)\) the invertibility of \(D - C\) implies that \(D - A\) is invertible or \(D - B\) is invertible.

**Proof.** Note that the maps \(\varphi_1 : (0, I) \to (I, \infty)\) and \(\varphi_2 : (I, \infty) \to (0, \infty)\) defined by

\[
\varphi_1(A) = A^{-1}, \quad A \in (0, I), \quad \text{and} \quad \varphi_2(A) = A - I, \quad A \in (I, \infty),
\]

are bijective and for each pair of operators \(A, B\) from the domain of \(\varphi_j\) we have

\[
\varphi_j(B) - \varphi_j(A) \text{ is invertible } \iff B - A \text{ is invertible, } j = 1, 2,
\]

and

\[
\varphi_j(A) \text{ and } \varphi_j(B) \text{ are adjacent } \iff A \text{ and } B \text{ are adjacent, } j = 1, 2.
\]

It is now trivial to complete the proof.

24
We are ready to prove the main result of the last subsection.

Proof of Theorem 3.6. Note that for $A \in E(H)$ the following are equivalent:

- $A \in (0, I)$,
- $A = A - 0$ is invertible and $A - I$ is invertible.

Hence, the restriction of $\phi$ to $(0, I)$ is a bijective map of $(0, I)$ onto itself. Moreover, by Corollary 3.8 this restriction preserves adjacency. According to Corollary 2.12 we have either

$$\phi(A) = f_q \left( (f_p(TT^*))^{-1/2} f_p(TAT^*) (f_p(TT^*))^{-1/2} \right), \quad A \in G(H),$$

or

$$\phi(A) = f_q \left( (f_p(TT^*))^{-1/2} f_p(T(I - A)T^*) (f_p(TT^*))^{-1/2} \right), \quad A \in G(H),$$

for some real numbers $p, q \in (-\infty, 1)$ and some bijective linear or conjugate-linear bounded operator $T : H \to H$ with $\|T\| \leq 1$. Composing $\phi$ with the inverse of the bijective preserver of invertibility of operator differences given by

$$A \mapsto f_q \left( (f_p(TT^*))^{-1/2} f_p(TAT^*) (f_p(TT^*))^{-1/2} \right), \quad A \in E(H),$$

we may assume that either $\phi(A) = A$ for every $A \in G(H)$, or $\phi(A) = I - A$ for every $A \in G(H)$. In the first case we need to show that actually $\phi(A) = A$ for all $A \in E(H)$, while in the second case we have to prove that $\phi(A) = I - A$ for every $A \in E(H)$, contradicting the fact that $\phi(0) = 0$.

We will outline the proof only in the second case. Hence, assume that $\phi(A) = I - A$ for every $A \in G(H)$. In particular, for every projection $P$ of rank one and every real number $t \in (-1/2, 1/2)$ we have

$$\phi \left( \frac{1}{2} I + tP \right) = \frac{1}{2} I - tP. \quad (12)$$

Now, let $s \in (0, 1)$ be any real number, $s \neq 1/2$, and $A \in (0, I)$. Then $A = sI$ if and only if $A - (1/2)I$ is invertible and $A - ((1/2)I + (s - (1/2))P)$ is singular for every projection $P$ of rank one. It follows that $\phi(sI) = (1 - s)I$ for every real $s$, $s \in (0, 1)$. Because $\phi$ preserves adjacency every operator from $(0, I)$ of the form $sI$ plus rank one is mapped into an operator of the form $(1 - s)I$ plus rank one. Using (12) and not entirely trivial but elementary linear algebra arguments we can show that $\phi(A) = I - A$ for every $A \in (0, I)$ of the form $A = sI + R$ with $R$ being of rank one. Let now $A$ be any operator in $(0, I)$. We know that then $\phi(A)$ belongs to $(0, I)$ as well. Hence, we can find a positive real number
such that \( sI \leq A, \phi(A) \leq (1-s)I \), and both \( A - sI \) and \( (1-s)I - \phi(A) \) are invertible. We know that for every projection \( P \) of rank one and every positive real number \( t \) satisfying \( s + t < 1 \) the difference \( A - (sI + tP) \) is invertible if and only if \( \phi(A) = ((1-s)I - tP) \) is invertible. From here one can rather quickly (see [26]) conclude that \( \phi(A) = I - A \). A suitable modification of arguments from [26] yields also that \( \phi(A) = I - A \) for every \( A \in E(H) \) such that \( \frac{1}{2} \) is not contained in the spectrum of \( A \).

Let finally \( A \) be any operator from \( E(H) \). We need to show that \( \phi(A) = I - A \). We know that for every \( B \in (0, I) \) the operator \( A - B \) is invertible if and only if \( \phi(A) - (I - B) = -((I - \phi(A)) - B) \) is invertible. Hence, we need to verify that if for given operators \( A, C \in E(H) \) we know that for every \( B \in (0, I) \) the operator \( A - B \) is invertible if and only if \( C - B \) is invertible, then \( A \) must be equal to \( C \). And we know that slightly modified ideas from [26] can be applied to verify this statement when \( \frac{1}{2} \) is not contained in the spectrum of \( A \).

In the remaining case when \( A - \frac{1}{2}I \) is singular we apply the spectral theorem for self-adjoint operators to find a small self-adjoint perturbation \( A - R \) of \( A - \frac{1}{2}I \) which is invertible. If \( A - R \) is a perturbation of \( A - \frac{1}{2}I \) that is small enough, then \( R \) is a perturbation of \( \frac{1}{2}I \) that is small enough and then \( R \in (0, I) \).

By Lemma 2.3 there exists a bijective map \( \xi : E(H) \to E(H) \) preserving the invertibility of the difference of effects such that \( \xi(R) = \frac{1}{2}I \). We know that \( \xi((0, I)) = (0, I) \) and that \( \xi(A) - \frac{1}{2}I = \xi(A) - \xi(R) \) is invertible. Further, for every \( B \in (0, I) \) there exists \( B_1 \in (0, I) \) such that \( \xi(B_1) = B \). Therefore, \( \xi(A) - B = \xi(A) - \xi(B_1) \) is invertible if and only if \( A - B_1 \) is invertible which is equivalent to the invertibility of \( C - B_1 \). The last difference is invertible if and only if \( \xi(C) - \xi(B_1) = \xi(C) - B \) is invertible. By the previous paragraph we have \( \xi(A) = \xi(C) \), and consequently, \( A = C \), as desired.

\[ \square \]

Acknowledgment

I would like to express my gratitude to the referee who found several misprints and whose suggestions help to improve English of the paper and the exposition.

References


