Line graph operator and small world

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Large networks
Examples

They are all around us and we are part of them.

Figure: Facebook as a social network

Figure: Airplane network as a transportation system
Large networks

Examples

Figure: WWW around Wikipedia

Figure: Protein Interaction Network
**Large networks**

**Definitions**

*Network = Graph + Data*

**Definition 1**

A simple undirected graph $G = (V(G), E(G))$ consists of a set $V(G)$ of elements called vertices and a set $E(G)$ of unordered pairs of vertices called edges.

**Small networks** (some tens of vertices) – can be represented by a picture and analyzed by many algorithms.

**Large networks** are reality - several thousands or millions of vertices; can be stored in computer memory.
How to capture the global properties of large networks?

Develop new models to mimic the growth of a network and to reproduce the structural properties observed in real topologies.

Properties of real networks:

- sparse networks;
- relatively short paths between any two vertices;
- presence of a large number of short cycles or specific motifs;
- degree distributions that approximately follow a power law.
Small world networks

Properties

**Definition 2**
The *diameter* of a graph $G$: $\text{diam}(G) = \max_{u,v \in V(G)} d(u, v)$.

**Definition 3**
The *clustering coefficient*: $\text{CC}(G) = \frac{1}{|V(G)|} \sum_{v \in V(G)} \text{CC}_G(v)$.

**(A1) Graph $G$ is sparse:** $|E(G)|/|V(G)| \in O(\lg |V(G)|)$.

**(A2) The diameter of $G$ is small:** $\text{diam}(G) \in O(\lg |V(G)|)$.

**(A3) The clustering coefficient $\text{CC}(G)$ is large:** $\text{CC}(G) \geq c$, for a positive constant $c$. 
**Deterministic model**

**Regular ring lattice**

A **regular ring lattice model.** A deterministic regular network where each vertex is clustered (connected) to \( k \) nearest neighbors.

- large clustering coefficient
- large diameter
Random network model
ER model

The Erdős-Rényi model, $G(n, p)$. A graph is constructed by connecting nodes randomly, with probability $p$.

- $p = 0.1$
- $p = 0.25$
- $p = 0.5$

- small clustering coefficient
- small diameter
The Watts and Strogatz model. A random graph model that produces graphs with small world properties, using random rewiring.

- Regular network: $p = 0$
- Small world network: $p = 0.0001$
- Random network: $p = 1$

- Large clustering coefficient
- Small diameter
**Definition 4**

Given a graph $G$, its line graph $L(G)$ is a graph such that:

- a vertex of $L(G)$ is assigned to each edge of $G$;
- two vertices of $L(G)$ are adjacent iff their corresponding edges are adjacent in $G$.

**Example 1**

![Image showing graphs $G$ and $L(G)$]
**Theorem 5 (Niepel, Knor, Šoltés, 1996)**

Let $G$ be a connected graph with at least one edge. Then,

$$diam(G) - 1 \leq diam(L(G)) \leq diam(G) + 1.$$
Main results
Implication of properties

Proposition 6 (·, Knor, Škrekovski, 2011)

If $G$ satisfies $(A2)$, then also $L(G)$ satisfies $(A2)$.

Proposition 7 (·, Knor, Škrekovski, 2011)

Let $G$ be a connected graph satisfying $(A3)$. Then, $L(G)$ satisfies $(A3)$ as well.
Proposition 8 (Knor, Škrekovski, 2011)

There exist graphs satisfying (A1) and (A2), but their line graphs do not satisfy (A3).

Counterexample 1
**Proposition 9 (Knor, Škrekovski, 2011)**

*There exist graphs satisfying (A1) and (A3), but their line graphs do not satisfy (A2).*

**Counterexample 2**

\[ G = P_n^2 \]

\[ L(G) \]
Proposition 10 (·, Knor, Škrekovski, 2011)

There exist small worlds, which line graphs are not small worlds. In particular, these line graphs do not satisfy (A1).
Main results
Sufficient conditions

Theorem 11 (·, Knor, Škrekovski, 2011)

Let $G$ be a graph whose all vertices are of degree $O(\lg m)$. Suppose that $(\text{diam})(G) \in O(\lg m)$ and $G$ has at most $c m$ bad edges, where $c < 1$ is some prescribed constant. Then, $L(G)$ is a small world with clustering coefficient at least $\frac{(1-c)}{3}$.

Corollary 12 (·, Knor, Škrekovski, 2011)

If a regular graph satisfies (A1) and (A2), then its line graph is a small world.
Network graphs
Complete $t$-ary trees, $T_{t,d}$

Example 3

- $t$ – every non-leaf vertex has precisely $t$ sons,
- $d$ – depth of these trees.
Network graphs
Toroidal graphs, $C^d_a$

$C^d_a = C_a \square C_a \square \cdots \square C_a$

Example 4

$C^2_5 = C_5 \square C_5$
Network graphs

Grid powers, $P_a^d$

$$P_a^d = P_a \square P_a \square \cdots \square P_a$$

Example 5

$$P_a^2 = P_a \square P_a$$
Questions?