Preddoktorski izpit (Predoctoral exam)

1. marca 2018 (1st March 2018)

Čas reševanja je 150 minut. Vsaka naloga se oceni z 0, 1, 2 ali 3 točkami. Za pozitivno oceno morate doseči vsaj 15 točk. Če ni izrecno povedano drugače, morate vse odgovore utemeljiti. Veliko uspeha!

The exam duration is 150 minutes. Each question will be awarded 0, 1, 2 or 3 marks. To pass the exam, you must achieve a total of at least 15 marks. All answers must be justified, unless explicitly instructed otherwise. Good luck!

1	2	3	4	5	6	7	8	9	10	Σ

Ime in priimek ____________________

Vpisna številka [| | | | | | |]

1. naloga (3 točke/marks)

Izračunaj

$$\sum_{k=0}^{n-1} \cos\left(\frac{2\pi k}{n}\right) \quad ,$$

če je $n > 1$ naravno število.

Calculate

$$\sum_{k=0}^{n-1} \cos\left(\frac{2\pi k}{n}\right) \quad ,$$

where $n > 1$ is a natural number.

2. naloga (3 točke/marks)

Zapiši matriko

$$\begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

kot produkt matrik oblike $I \pm E_{ij}$, $j > i$, kjer je E_{ij} matrika, ki ima na (i, j)-tem mestu 1, povsod drugje pa 0.

Write the matrix

$$\begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

as a product of matrices of the form $I \pm E_{ij}$, where $j > i$ and E_{ij} is the matrix with 1 in the (i, j)-th position and 0 everywhere else.

3. naloga (3 točke/marks)

Naj bo $\{a_n\}_n$ tako padajoče zaporedje pozitivnih realnih števil, da je $\sum_n a_n < \infty$. Dokažite, da $\lim_{n\to\infty} na_n = 0$.

Let $\{a_n\}_n$ be a decreasing sequence of positive real numbers such that $\sum_n a_n < \infty$. Prove that $\lim_{n\to\infty} na_n = 0$.

4. naloga (3 točke/marks)

Naj bosta A in B kompaktni podmnožici v $\mathbb{R}^n$ z običajno (evklidsko) metriko. Pokaži, da je tudi množica

$$A + B = \{a + b \mid a \in A,\, b \in B\}$$

kompaktna.

Let A and B be compact subsets of $\mathbb{R}^n$ with the usual (Euclidean) metric. Prove that the subset

$$A + B = \{a + b \mid a \in A,\, b \in B\}$$

is also compact.

5. naloga (3 točke/marks)

Katero je najmanjše tako število n, da v simetrični grupi S_n obstaja element reda 6?

What is the smallest n such that the symmetric group S_n contains an element of order 6?

6. naloga (3 točke/marks)

Naj bo $f\colon [0,1] \to \mathbb{R}$, $f(0) = 1$, $f(1) = 3$ zvezno odvedljiva naraščajoča funkcija. Dokažite, da je dolžina njenega grafa kvečjemu 3.

Let $f\colon [0,1] \to \mathbb{R}$, with $f(0) = 1$ and $f(1) = 3$, be a continuously differentiable monotone function. Prove that the length of the curve of the graph of f is not greater than 3.

7. naloga (3 točke/marks)

Rešite *natanko eno* od naslednjih nalog.
Answer *exactly one* of the following questions.

1. Naj bo $I \subset \mathbb{R}$ zaprt interval in (f_n) zaporedje realnih zveznih funkcij na I, ki po točkah konvergirajo k zvezni funkciji f na I. Denimo, da je zaporedje (f_n) padajoče.

   (a) Dokaži, da (f_n) konvergira proti f enakomerno na I.

   (b) Ali trditev velja za odprt interval I?

   Let $I \subset \mathbb{R}$ be a closed interval, and (f_n) a sequence of continuous real-valued functions on I that converges pointwise to a continuous function f on I. Suppose that the sequence (f_n) is decreasing.

   (a) Prove that (f_n) converges uniformly to f on I.

   (b) Does the same property hold if I is an open interval?

2. Dokaži, da med poljubnimi devetimi celimi števili vedno lahko najdemo dve, katerih razlika ali pa vsota je deljiva tako s 3 kot s 5.

   Show that, from any collection of nine integers, we can always find two elements such that either their sum or their difference is divisible by both 3 and 5.

3. Igralca A in B igrata tekmo, sestavljeno iz zaporednih iger, v katerih zmaga A z verjetnostjo p in B z verjetnostjo $1 - p$. Tekmo dobi igralec, ki prvi zmaga v dveh igrah zapored.

   (a) Kolikšna je verjetnost, da tekmo dobi A?

   (b) Svoj rezultat smiselno preizkusi.

   Two players A and B play a match consisting of consecutive games, each of which A wins with probability p and B with probability $1 - p$. The match is won by the player who first wins two consecutive games.

   (a) What is the probability that A wins the match?

   (b) Test your answer in a sensible way.

8. naloga (3 točke/marks)

Rešite *natanko eno* od naslednjih nalog.
Answer *exactly one* of the following questions.

1. Naj bosta V in W neskončno razsežna realna vektorska prostora. Naj bo $\mathrm{Lin}(V, W)$ vektorski prostor linearnih preslikav iz V v W.

   (a) Ali je $X = \{f \in \mathrm{Lin}(V, W); f \text{ ima končen rang}\}$ podprostor prostora $\mathrm{Lin}(V, W)$?

   (b) Ali je $Y = \{f \in \mathrm{Lin}(V, W); \mathrm{Ker}(f) \text{ je končno razsežen}\}$ podprostor prostora $\mathrm{Lin}(V, W)$?

   (c) Kaj je presek $X \cap Y$?

   Let V and W be infinite dimensional real vector spaces. Let $\mathrm{Lin}(V, W)$ be the vector space of linear functions from V to W.

   (a) Is $X = \{f \in \mathrm{Lin}(V, W); f \text{ has finite rank}\}$ a vector subspace of $\mathrm{Lin}(V, W)$?

   (b) Is $Y = \{f \in \mathrm{Lin}(V, W); \mathrm{Ker}(f) \text{ is finite dimensional}\}$ a vector subspace of $\mathrm{Lin}(V, W)$?

   (c) What is the intersection $X \cap Y$?

2. Kaj je *universalni* Turingov stroj? (Podajte splošen odgovor, ki pojasnjuje značilnosti univerzalnega Turingovega stroja. Ni vam potrebno definirati, kaj je Turingov stroj.)

   What is meant by a *universal* Turing machine? (Give an informal answer that explains the main properties of a universal Turing machine. You do not need to define what a Turing machine is.)

3. Navedi primer absolutno zvezne in diskretne slučajne spremenljivke, za kateri: a) ne velja krepki zakon velikih števil, b) krepki zakon velikih števil velja, ne velja pa centralni limitni izrek.

   Give examples of an absolutely continuous and a discrete random variable for which: a) the strong law of large numbers does not apply, b) the strong law of large numbers does apply but not the central limit theorem.

9. naloga (3 točke/marks)

Rešite *natanko eno* od naslednjih nalog.
Answer *exactly one* of the following questions.

1. Naj bo K kolobar (ne nujno komutativen), v katerem iz pogoja $xy = 0$ vedno sledi $x = 0$ ali $y = 0$. Naj bosta $a, b \in K$ in m ter n tuji naravni števili, za kateri velja $a^n = b^n$ in $a^m = b^m$. Pokaži, da je $a = b$.

   Let K be a (not necessarily commutative) ring, in which $xy = 0$ implies $x = 0$ or $y = 0$. Suppose $a, b \in K$, and suppose m and n are coprime natural numbers, for which $a^n = b^n$ and $a^m = b^m$. Show that $a = b$.

2. Koliko nizov dolžine n, sestavljenih iz simbolov $0, 1, 2$, je takšnih, da simbol 2 nikoli neposredno ne sledi simbolu 0?

   How many sequences of length n, comprising of symbols $0, 1, 2$, are such that the symbol 2 never appears immediately after the symbol 0?

3. Iz žare, ki vsebuje dve beli in $b > 0$ črnih krogel žrebamo brez vračanja. Naj bo X število žrebanj, da prvič izvlečem belo, in Y število žrebanj, da prvič izvlečemo črno kroglo. Izračunaj $\mathrm{Cov}(X, Y)$.

   We draw without replacement from an urn containing two white and $b > 0$ black balls. Let X be the number of draws until we first draw a white ball and Y the number of draws until we first draw a black one. Calculate $\mathrm{Cov}(X, Y)$

10. naloga (3 točke/marks)

Rešite *natanko eno* od naslednjih nalog.
Answer *exactly one* of the following questions.

1. Dokaži ali ovrzi: za vsako odvedljivo, padajočo, nenegativno funkcijo $f\colon [0,\infty) \to \mathbb{R}$ velja $\lim_{x\to\infty} f'(x) = 0$.

   Prove or disprove: for every differentiable, monotonically decreasing, nonnegative function $f\colon [0,\infty) \to \mathbb{R}$ it holds that $\lim_{x\to\infty} f'(x) = 0$.

2. Miller-Rabinov test praštevil je randomizirani algoritem z naslednjimi lastnostmi. Vnos je zaporedje l bitov, ki predstavlja naravno število $n < 2^l$. Algoritem da enega od dveh možnih izhodov: sestavljeno število ali možno praštevilo. Če je n praštevilo, algoritem vedno vrne možno praštevilo. Če pa je n sestavljeno število, algoritem naključno vrne sestavljeno število z verjetnostjo $\geq \frac{1}{2}$ in možno praštevilo z verjetnostjo $\leq \frac{1}{2}$. Čas poteka algoritma je $\mathcal{O}(l^3)$.

   (a) Opišite algoritem z vnosom naravnih števil l, k, ki vrne kot izhod tako naključno celo število p, da $2^{l-1} < p < 2^l$ in je verjetnost, da je p praštevilo, najmanj $1 - \frac{1}{2^k}$. Vaš algoritem bi moral uporabiti Miller-Rabinov test praštevil.

   (b) Napišite neformalni argument, ki potrjuje, da je pričakovani čas poteka algoritma $\mathcal{O}(kl^4)$?

   The Miller-Rabin primality test is a randomised algorithm with the following properties. The input is a sequence of l bits, representing a non-negative integer $n < 2^l$. The algorithm returns one of two possible outputs: composite number or possible prime. If n is a prime number then the algorithm always returns possible prime. If n is a composite number then the algorithm randomly returns composite number with probability $\geq \frac{1}{2}$, and possible prime with probability $\leq \frac{1}{2}$. The running time of the algorithm is $\mathcal{O}(l^3)$.

   (a) Describe an algorithm that takes, as input, positive integers l, k and returns, as output, a random integer p with $2^{l-1} < p < 2^l$ whose probability of being prime is at least $1 - \frac{1}{2^k}$. Your algorithm should make use of the Miller-Rabin primality test.

   (b) Give an informal argument as to why the expected running time of your algorithm is $\mathcal{O}(kl^4)$.

3. Naj bo slučajni vektor (X, Y) porazdeljen enakomerno po trikotniku z oglišči $(0,0)$, $(0,1)$ in $(1,0)$. Naj bo R dolžina tega slučajnega vektorja.

   (a) Izračunaj $\mathrm{Cov}(X,Y)$.

   (b) Izračunaj $\mathbb{E}(R^2|X)$ in $\mathbb{E}(R^2)$.

   The random vector (X, Y) is distributed uniformly on the triangle with vertices $(0,0)$, $(0,1)$ and $(1,0)$. Let R be its length.

   (a) Calculate $\mathrm{Cov}(X,Y)$.

   (b) Calculate $\mathbb{E}(R^2|X)$ and $\mathbb{E}(R^2)$.