Preddoktorski izpit (Predoctoral exam)

20. februarja 2019 (20 February 2019)

Čas reševanja je 150 minut. Vsaka naloga se oceni z 0, 1, 2 ali 3 točkami. Za pozitivno oceno morate doseči vsaj 15 točk. Če ni izrecno povedano drugače, morate vse odgovore utemeljiti. Veliko uspeha!

The exam duration is 150 minutes. Each question will be awarded 0, 1, 2 or 3 marks. To pass the exam, you must achieve a total of at least 15 marks. All answers must be justified, unless explicitly instructed otherwise. Good luck!

Ime in priimek

Vpisna številka

1	
2	
3	
4	
5	
6	
7	
8	
9	
10	
Σ	

1. naloga (3 točke/marks)

Naj bosta p in q praštevili, za kateri je $pq + 1$ kvadrat naravnega števila. Izračunajte $|p - q|$.

Let p and q be prime numbers such that $pq + 1$ is a square number. Calculate $|p - q|$.

2. naloga (3 točke/marks)

Ali obstaja dvakrat odvedljiva omejena funkcija $f\colon \mathbb{R} \to \mathbb{R}$, za katero velja $f''(x) > 0$ za vse x?

Does there exist a twice differentiable bounded function $f\colon \mathbb{R} \to \mathbb{R}$, such that $f''(x) > 0$ for all x?

3. naloga (3 točke/marks)

Poiščite vse realne 2×2 matrike, ki komutirajo z vsemi 2×2 realnimi matrikami.

Find all 2×2 real matrices that commute with all 2×2 real matrices.

4. naloga (3 točke/marks)

Označimo z G topološki prostor vseh obrnljivih 2×2 realnih matrik, ki ga gledamo kot podprostor evklidskega prostora $\mathbb{R}^{2\times2}$.

1. Ali je G odprt podprostor v $\mathbb{R}^{2\times2}$?
2. Ali je prostor G povezan?
3. Kaj je zaprtje G v $\mathbb{R}^{2\times2}$?

Let G denote the topological space of all invertible 2×2 real matrices, viewed as a subspace of the Euclidean space $\mathbb{R}^{2\times2}$.

1. Is G an open subspace of $\mathbb{R}^{2\times2}$?
2. Is the space G connected?
3. What is the closure of G in $\mathbb{R}^{2\times2}$?

5. naloga (3 točke/marks)

Naj bo $f\colon [0,1] \to \mathbb{R}$ dvakrat odvedjiva funkcija, za katero velja $f(0) = 0$, $f(1) = 1$ in $f''(x) \leq 0$ za vse x. Poiščite minimalno vrednost integrala $\int_0^1 f(x)\,\mathrm{d}x$.

Let $f\colon [0,1] \to \mathbb{R}$ be a twice differentiable function such that $f(0) = 0$, $f(1) = 1$ and $f''(x) \leq 0$ for all x. What is the minimum possible value of $\int_0^1 f(x)\,\mathrm{d}x$?

6. naloga (3 točke/marks)

Naj bo K kolobar z enico in $a, b \in K$. Dokažite, da velja: če je element $1 - ab$ obrnljiv, je tudi $1 - ba$ obrnljiv.

Let K be a ring with identity. Prove that if the element $1 - ab$ is invertible then so is $1 - ba$.

7. naloga (3 točke/marks)

Rešite *natanko eno* od naslednjih nalog.
Answer *exactly one* of the following questions.

1. Naj bo X kompakten in naj bo $f\colon X \to \mathbb{R}$ funkcija, ki ni nujno zvezna, vendar je za vsak $a \in \mathbb{R}$ množica $Z_a := \{x \in X \mid f(x) \geq a\}$ zaprta v X (včasih pravimo, da je f *navzgor polzvezna* funkcija).

   (a) Pokažite, da je funkcija f navzgor omejena.
   (Namig: opazujte zaporedje množic $Z_1 \supseteq Z_2 \supseteq Z_3 \supseteq \ldots$.)

   (b) Pokažite, da funkcija f zavzame maksimum na X.

   Let X be compact, and let $f\colon X \to \mathbb{R}$ be a (not necessarily continuous) function, such that the sets $Z_a := \{x \in X \mid f(x) \geq a\}$ are closed in X for every $a \in \mathbb{R}$ (such an f is called *upper semi-comtinuous*).

   (a) Show that the function f is bounded from above.
   (Hint: consider the sequence $Z_1 \supseteq Z_2 \supseteq Z_3 \supseteq \ldots$.)

   (b) Show that f achieves a maximum value on X.

2. Koliko nizov dolžine 20 nad abecedo a, b, c, d, e vsebuje

   (a) natanko sedem a-jev?

   (b) natanko en b, dva c-ja, tri d-je in štiri e-je?

   (c) vseh pet črk?

   What is the number of sequences of length 20 over the alphabet a, b, c, d, e containing:

   (a) exactly seven a's?

   (b) exactly one b, two c's, three d's and four e's?

   (c) all five symbols?

3. Nepošten kovanec ima verjetnost p, da pokaže cifro. Denimo, da n-krat vržemo kovanec. Navedite splošno formulo za verjetnost, da pade dvakrat toliko cifer kot grbov. Kakšna je verjetnost za primer $n = 6$ in $p = \frac{2}{3}$?

   A biased coin has a probability p of showing heads. Suppose the coin is tossed n times. Give a general formula for the probability that precisely twice as many heads as tails are thrown. What is the probability in the case $n = 6$ and $p = \frac{2}{3}$?

8. naloga (3 točke/marks)

Rešite *natanko eno* od naslednjih nalog.
Answer *exactly one* of the following questions.

1. Naj bo $f_n : [0,1] \to \mathbb{R}$, $n = 1, 2, \dots$, zaporedje enakomerno omejenih zveznih funkcij. Za $x \in [0,1]$ definirajmo

   $$F_n(x) = \frac{1}{n} \int_0^x (x-t)^{n-1} f_n(t)\, dt.$$

   Dokažite, da funkcijska vrsta

   $$\sum_{n=1}^{\infty} F_n(x)$$

   konvergira enakomerno na $[0,1]$.

   Let $f_n : [0,1] \to \mathbb{R}$, $n = 1, 2, \dots$, be a sequence of uniformly bounded continuous functions. For $x \in [0,1]$, define

   $$F_n(x) = \frac{1}{n} \int_0^x (x-t)^{n-1} f_n(t)\, dt.$$

   Show that the function series

   $$\sum_{n=1}^{\infty} F_n(x)$$

   converges uniformly on $[0,1]$.

2. Znano je, da je en od dveh razredov računske zahtevnosti NP (nedetermisitični polinomski čas) in PSPACE (deterministični polinomski prostor) podmnožica drugega. Navedite katera inkluzija velja in orišite njen dokaz.

   One of the two computational complexity classes NP (nondeterministic polynomial-time) and PSPACE (deterministic polynomial-space) is known to be included in the other. State the known inclusion, and give an outline proof of it.

3. Šeststrana kocka ima tri oznake A, B in C na svojih ploskvah; vsaka od oznak je zapisana na natanko dveh ploskvah. Kocko mečemo zaporedoma dokler se vsaka izmed treh oznak A, B in C ne pojavi vsaj enkrat. Kolikšno je pričakovano število metov kocke?

   A six-sided dice has three labels A, B and C on its faces; with each label written on two faces. The dice is thrown repeatedly until each of the three labels A, B and C has come up at least once. What is the expected number of throws of the dice?

9. naloga (3 točke/marks)

Rešite *natanko eno* od naslednjih nalog.
Answer *exactly one* of the following questions.

1. Naj bodo $A \in \mathbb{R}^{n\times n}$ obrnljiva matrika in $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ taka vektorja, da velja $1 + \mathbf{v}^T A^{-1}\mathbf{u} \neq 0$. Preverite, da je

$$(A + \mathbf{u}\mathbf{v}^T)^{-1} = A^{-1} - \frac{A^{-1}\mathbf{u}\mathbf{v}^T A^{-1}}{1 + \mathbf{v}^T A^{-1}\mathbf{u}} \ .$$

Let $A \in \mathbb{R}^{n\times n}$ be an invertible matrix and $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ be vectors such that $1+\mathbf{v}^T A^{-1}\mathbf{u} \neq 0$. Show that

$$(A + \mathbf{u}\mathbf{v}^T)^{-1} = A^{-1} - \frac{A^{-1}\mathbf{u}\mathbf{v}^T A^{-1}}{1 + \mathbf{v}^T A^{-1}\mathbf{u}} \ .$$

2. Koliko lic dolžine 5 ima povezan 3-regularen graf v ravnini, pri katerem so vsa lica dolžine 5 ali 6?

   How many faces of length 5 does a connected 3-regular plane graph possess if all its faces have length 5 or 6?

3. Žarnice so bodisi tipa S, če je njihova življenska doba porazdeljena $Exp(2\lambda)$, ali tipa N, če je njihova življenska doba porazdeljena kot $\min\{Exp(\lambda), d\}$, kjer je d tako število, da sta pričakovani življenski dobi obeh tipov enaki.

   (a) Določite d.

   (b) Denimo, da slučajno izberemo žarnico, pri čemer sta oba tipa enako verjetna. Kolikšna je verjetnost, da bo gorela dlje, kot je njena pričakovana življenska doba?

   Light bulbs are of type S if their life-time is $Exp(2\lambda)$-distributed or type N if their life-time is distributed as $\min\{Exp(\lambda), d\}$, where d is such that the expected life-time of both types of bulb is equal.

   (a) Compute the value of d.

   (b) Suppose we randomly choose a bulb, both types being equally likely. What is the probability, that the life-time of the bulb exceeds its expectation?

10. naloga (3 točke/marks)

Rešite *natanko eno* od naslednjih nalog.
Answer *exactly one* of the following questions.

1. Vektorski prostor P vseh polinomov z realnimi koeficienti v eni spremenljivki opremimo z normo $\|p\| = \max_{0\leq x\leq 1} |p(x)|$. Ali je preslikava $D : P \to P$, $D(p) = p'$ zvezna glede na to normo?

   We endow the vector space P of all single-variable polynomials with real coefficients with the norm $\|p\| = \max_{0\leq x\leq 1} |p(x)|$. Is the function $D : P \to P$, $D(p) = p'$ continuous with respect to this norm?

2. *Dvojiško iskalno drevo* ima dve vrsti vozlišč: *notranje vozlišče* opremljeno s *ključem*, ki je celo število, in pripadajočo *vrednostjo*; in *list*. Sledeči iskalni algoritem išče vrednost pripadajočo ključu `k` v drevesu pod vozliščem `n`.

```
def išči(n, k):
  če je-list(n)    : vrni null          // neuspelo iskanje
     ključ(n) == k : vrni vrednost(n)
     ključ(n) > k  : išči (levi-otrok(n), k)
     ključ(n) < k  : išči (desni-otrok(n), k)
```

   (a) Katere lastnosti mora imeti dvojiško iskalno drevo s korenom `koren`, da bo iskanje `išči(koren,k)` gotovo vrnilo pripadajočo vrednost `v` za vsak par ključ-vrednost `(k,v)` v drevesu?

   (b) Razložite, kako oblika drevesa vpliva na časovno zahtevnost algoritma v najslabšem primeru za drevo z $2^N - 1$ notranjimi vozlišči.

   A *binary search tree* has two kinds of node: *internal nodes* possessing an integer *key* and an associated *value*; and *leaf nodes*. The following search algorithm searches the tree beneath a node `n` for the value associated with the key `k`.

```
def search(n, k):
  if is-leaf(n)   : return null         // failed search
     key(n) == k : return value(n)
     key(n) > k  : search (left-child(n), k)
     key(n) < k  : search (right-child(n), k)
```

   (a) What properties must a binary search tree with root node `root` enjoy for the search `search(root,k)` to be guaranteed to return the associated value `v`, for every key-value pair `(k,v)` in the tree?

   (b) Discuss how the shape of the tree influences the worst-case run time of the algorithm for a tree with $2^N - 1$ internal nodes.

3. Naj bo N porazdeljena geometrijsko s parametrom p (in sicer $P(N = k) = p(1-p)^{k-1},\ k = 1, 2, \dots$) in $(X_k)_{k\geq 1}$ zaporedje neodvisnih enako eksponentno porazdeljenih slučajnih spremenljivk s parametrom λ, neodvisno od N. Naj bo $S = \sum_{i=1}^{N} X_i$. Kako je porazdeljena slučajna spremenljivka S?

   Let N be a geometric random variable with parameter p (i.e., $P(N = k) = p(1-p)^{k-1},\ k = 1, 2, \dots$) and $(X_k)_{k\geq 1}$ a sequence of independent identically exponentially distributed random variables with parameter λ independent of N. Let $S = \sum_{i=1}^{N} X_i$. What is the distribution of the random variable S?