Preddoktorski izpit (Predoctoral exam)

12. februarja 2020 (12th February 2020)

Čas reševanja je 150 minut. Vsaka naloga se oceni z 0, 1, 2 ali 3 točkami. Za pozitivno oceno morate doseči vsaj 15 točk. Če ni izrecno povedano drugače, morate vse odgovore utemeljiti. Veliko uspeha!

The exam duration is 150 minutes. Each question will be awarded 0, 1, 2 or 3 marks. To pass the exam, you must achieve a total of at least 15 marks. All answers must be justified, unless explicitly instructed otherwise. Good luck!

1	
2	
3	
4	
5	
6	
7	
8	
9	
10	
Σ	

Ime in priimek

Vpisna številka

1. naloga (3 točke/marks)

Poiščite vse pare praštevil p, q, za katera velja $p^3 + 1 = q^2$.

Find all pairs of prime numbers p, q that satisfy $p^3 + 1 = q^2$.

2. naloga (3 točke/marks)

Koliko ničel ima spodnja preslikava med pozitivnimi realnimi števili x?

$$f(x) \;=\; \ln x - \frac{x}{3}$$

How many zeros does the above function have among positive real numbers x?

3. naloga (3 točke/marks)

Poiščite tako zaporedje realnih števil (x_n), da vrsta $\sum_{n=1}^{\infty} x_n$ konvergira, vrsta $\sum_{n=1}^{\infty} {x_n}^2$ pa divergira. Ali je možno, da so vsi x_n positivni?

Find a sequence of real numbers (x_n), such that the series $\sum_{n=1}^{\infty} x_n$ converges, but the series $\sum_{n=1}^{\infty} {x_n}^2$ diverges. Is it possible for all x_n to be positive?

4. naloga (3 točke/marks)

Za katera realna števila a obstaja taka simetrična realna matrika A v spodnji obliki, da velja $A^2 = I$? Kaj pa za kompleksna števila a?

$$A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$$

For which real numbers a does there exist a real symmetric matrix A as above such that $A^2 = I$? What about for complex numbers a?

5. naloga (3 točke/marks)

Naj bosta M in N podgrupi edinki grupe G, za kateri je $M \cap N = \{1\}$. Pokažite, da vsak $m \in M$ komutira z vsakim $n \in N$.

Let M and N be normal subgroups of a group G such that $M \cap N = \{1\}$. Show that every $m \in M$ commutes with every $n \in N$.

6. naloga (3 točke/marks)

Naj bo $f\colon X \to Y$ zvezna funkcija med metričnima prostoroma X in Y in naj bo (x_n) zaporedje v X. Katere izmed naslednjih trditev so pravilne?

1. Če je zaporedje (x_n) konvergentno, potem je tudi zaporedje $(f(x_n))$ konvergentno.
2. Če je zaporedje $(f(x_n))$ konvergentno, potem je tudi zaporedje (x_n) konvergentno.
3. Če je zaporedje (x_n) Cauchyjevo, potem je tudi zaporedje $(f(x_n))$ Cauchyjevo.

Let $f\colon X \to Y$ be a continuous function between metric spaces X and Y and let (x_n) be a sequence in X. Which of the following statements are correct?

1. If the sequence (x_n) converges then the sequence $(f(x_n))$ also converges.
2. If the sequence $(f(x_n))$ converges then the sequence (x_n) also converges.
3. If (x_n) is a Cauchy sequence then $(f(x_n))$ is also a Cauchy sequence.

7. naloga (3 točke/marks)

Rešite *natanko eno* od naslednjih nalog.
Answer *exactly one* of the following questions.

1. V simetrični grupi S_6:

   (a) Koliko je elementov reda 6?

   (b) Koliko je elementov x za katere je x^6 identična permutacija?

   In the symmetric group S_6:

   (a) How many elements of order 6 are there?

   (b) For how many elements x does it hold that x^6 is the identity permutation?

2. Računalnik ima primitivno operacijo seštevanja, ki sešteje števili v podanih spominskih celicah X in Y ter zapiše rezultat v spominsko celico Z. (Možno je, da so X, Y in Z ista spominska celica.) Računalnik lahko vzporedno izvede neomejeno število operacij seštevanja, dokler vzporedne operacije uporabljajo različne spominske celice. Pokažite, da lahko računalnik sešteje nabor A, sestavljen iz n spominskih celic, v času $O(\log\ n)$. (Postopek lahko spreminja vrednosti v A.)

   A computer has a primitive addition operation that adds the numbers in two given memory cells X and Y and writes the result into a memory cell Z. (It is possible for X, Y and Z to be the same memory cell.) The computer is able to perform an unlimited number of addition operations in parallel as long as parallel operations use different memory cells. Show that the computer can sum an array A of n memory cells in time $O(\log\ n)$. (The procedure is allowed to change the values in A.)

3. Strok graha običajno vsebuje pet zrn, v povprečju pa eden od destetih vsebuje le štiri.

   (a) Katero je najbližje celo število odstotkov verjetnosti, da je v stotih strokih z vrta 500 zrn?

   (b) Koliko strokov morate nabrati, da imate najmanj 500 zrn z verjetnostjo približno 0,9 ? (Namig: kumulativna porazdelitvena funkcija Φ za normalno porazdelitev s povprečjem 0 in standardnim odklonom 1 zadošča $\Phi(1.3) \approx 0,90$.)

   Pea pods usually contain five peas, but one in ten pods on average contains only four.

   (a) What is the nearest integer percentage to the probability that one hundred pods from the garden contain 500 peas?

   (b) How many pods must one pick in order that the probability of obtaining at least 500 peas is approximately 0.9 ? (Hint: the cumulative distribution function Φ for a normal distribution with mean 0 and standard deviation 1 satisfies $\Phi(1.3) \approx 0.90$.)

8. naloga (3 točke/marks)

Rešite *natanko eno* od naslednjih nalog.
Answer *exactly one* of the following questions.

1. Naj zvezna funkcija $f\colon \mathbb{R} \to \mathbb{R}$ za vsak $x \in (0, \infty) \cap \mathbb{Q}$ zadošča

$$\int_{-x}^{x} f(t)\, dt = 0.$$

   Dokažite, da je f liha funkcija.

   Let $f\colon \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying

$$\int_{-x}^{x} f(t)\, dt = 0 \ ,$$

   for every $x \in (0, \infty) \cap \mathbb{Q}$. Prove that f is an odd function.

2. Hodnik je širok 2 metra in dolg n metrov. Tlakujemo ga z ploščicami dolgimi 2 metra in širokimi 1 meter. Ploščice lahko dobimo iz neomejene zaloge v dveh barvah. Koliko je različnih načinov tlakovanja hodnika?

   A corridor is 2 meters wide and n meters long. It is to be tiled with tiles that are 2 meters long and 1 meter wide. The tiles come in two colours and there is an unlimited supply of each colour. How many different ways of tiling the corridor are there?

3. Slučajni spremenljivki X, Y sta definirani na sledeč način: najprej izberemo X enakomerno na $[0, 2]$, potem pa pri izbranem X izberemo Y enakomerno na $[0, 2X]$.

   (a) Izračunajte pogojno gostoto spremenljivke X (glede na Y).

   (b) Izračunajte brezpogojno gostoto Y.

   Random variables X, Y are defined in the following manner: first one chooses X uniformly on $[0, 2]$ and then, with X being chosen, Y is chosen uniformly on $[0, 2X]$.

   (a) Compute the conditional density of X conditioned on Y.

   (b) Compute the unconditional density of Y.

9. naloga (3 točke/marks)

Rešite *natanko eno* od naslednjih nalog.
Answer *exactly one* of the following questions.

1. Naj bo $f\colon \mathbb{R} \to \mathbb{R}$ odvedljiva funkcija z omejenim odvodom. Dokažite, da za vsak $x \in \mathbb{R}$ zaporedje $(a_n)_{n\in\mathbb{N}}$, podano s spodnjim splošnim členom, konvergira.

$$a_n = \sum_{k=1}^{n} f\left(x + \frac{1}{k^2}\right) - nf(x)$$

Let $f\colon \mathbb{R} \to \mathbb{R}$ be a differentiable function with bounded derivative. Prove that, for every $x \in \mathbb{R}$, the sequence $(a_n)_{n\in\mathbb{N}}$ defined above converges.

2. Ali so spodnje trditve o veliki O notaciji za preslikave na naravnih številih resnične? Odgovore utemeljite.

   (a) n^{n+1} je $O(n^n)$.
   (b) $(n+1)^n$ je $O(n^n)$.
   (c) Če so $f, g, h\colon \mathbb{N} \to \mathbb{N}$ takšne, da je $g(n)$ v $O(h(n))$, potem je $f(g(n))$ v $O(f(h(n)))$.

   Are the following statements about big-O notation for functions on the natural numbers true or false? Justify your answers.

   (a) n^{n+1} is $O(n^n)$.
   (b) $(n+1)^n$ is $O(n^n)$.
   (c) If $f, g, h\colon \mathbb{N} \to \mathbb{N}$ are such that $g(n)$ is $O(h(n))$ then it follows that $f(g(n))$ is $O(f(h(n)))$.

3. Naj bosta $U_1, U_2 \in \mathbb{R}^{n\times n}$ realni zgornje trikotni matriki velikosti $n \times n$ in U_1 nesingularna. Rešujemo matrični sistem linearnih enačb $U_1 X = U_2$.

   (a) Dokažite, da je X zgornja trikotna matrika.
   (b) Preštejte število osnovnih aritmetičnih operacij za **ekonomičen** izračun matrike X.

   Let $U_1, U_2 \in \mathbb{R}^{n\times n}$ be real upper triangular matrices of size $n \times n$ with U_1 nonsingular. We are solving the matrix system of linear equations $U_1 X = U_2$.

   (a) Prove that X is an upper triangular matrix.
   (b) Count the number of basic arithmetic operations required to **efficiently** calculate the matrix X.

10. naloga (3 točke/marks)

Rešite *natanko eno* od naslednjih nalog.
Answer *exactly one* of the following questions.

1. Za katere $a \in \mathbb{Z}_3$ je $\mathbb{Z}_3[X]/(X^3 + aX^2 + 1)$ polje?

   For which $a \in \mathbb{Z}_3$ is $\mathbb{Z}_3[X]/(X^3 + aX^2 + 1)$ a field?

2. Naj bo G r-regularni dvodelni graf na n vozliščih s premerom 3. Denimo, da G nima ciklov dolžine 4. Izrazite n z r. Ali obstaja takšen graf z $r = 3$?

   Let G be an r-regular bipartite graph on n vertices of diameter 3. Suppose that G has no cycles of length 4. Express n in terms of r. Does there exist such a graph with $r = 3$?

3. Določite realni števili $a, b \in \mathbb{R}$ v predpisu funkcije $f(x) = x^3 - ax^2 + b$ tako, da bo zaporedje približkov po Newtonovi metodi

   $$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1, \ldots,$$

   konvergiralo k limitni vrednosti 1 z redom konvergence 3 za vsak dovolj dober začetni približek x_0.

   Find real numbers $a, b \in \mathbb{R}$ such that when Newton's method

   $$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1, \ldots,$$

   is applied to the function $f(x) = x^3 - ax^2 + b$ the resulting sequence of approximations converges to the limiting value 1 with rate of convergence 3 for every sufficiently good initial approximation x_0.