Preddoktorski izpit (Predoctoral exam)

25. aprila 2021 (25th April 2021)

Čas reševanja je 150 minut. Število možnih točk je 24. Za pozitivno oceno morate doseči vsaj 12 točk. Če ni izrecno povedano drugače, morate vse odgovore utemeljiti. Veliko uspeha!

The exam duration is 150 minutes. The total number of available marks is 24. To pass the exam, you must achieve a total of at least 12 marks. All answers must be justified, unless explicitly instructed otherwise. Good luck!

Rešite **vse štiri** od 1. do 4. naloge.
Answer **all four** of questions 1–4.

1. naloga (3 točke)

Fibonaccijeva števila so določena s pravili $F_0 = 0$, $F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$, kjer je $n \geq 2$. Poiščite vsa taka naravna števila $n \geq 0$, da je $F_n = n^2$.
The Fibonacci numbers are defined by $F_0 = 0$, $F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. Find all natural numbers $n \geq 0$ such that $F_n = n^2$.

2. naloga (3 točke)

Naj bo $f\colon [0,\infty) \to \mathbb{R}$ taka zvezno odvedljiva funkcija, da je f' *absolutno integrabilna* (t.j. $\int_0^\infty |f'(x)|\, dx < \infty$).

(a) Dokažite, da obstaja limita $\lim_{x\to\infty} f(x)$.

(b) Izračunajte limito $\lim_{x\to\infty} f(x)$ v primeru, ko je f absolutno integrabilna.

Let $f\colon [0,\infty) \to \mathbb{R}$ be a continuously differentiable function such that f' is *absolutely integrable* (i.e. $\int_0^\infty |f'(x)|\, dx < \infty$).

(a) Prove that the limit $\lim_{x\to\infty} f(x)$ exists.

(b) Calculate $\lim_{x\to\infty} f(x)$ in the case that f is absolutely integrable.

3. naloga (3 točke)

Poiščite vse homomorfizme kolobarjev iz $\mathbb{Z}_6$ v $\mathbb{Z}_{12}$. (Pozor: ni zahtevano, da homomorfizem kolobarjev slika enoto v enoto.)
Find all ring homomorphisms from $\mathbb{Z}_6$ to $\mathbb{Z}_{12}$. (In this question, there is no requirement that ring homomorphisms preserve the multiplicative identity 1.)

4. naloga (3 točke)

Naj bo A realna kvadratna matrika.

(a) Naj bo vsota elementov v vsaki vrsti enaka c. Dokažite, da je c lastna vrednost matrike A.

(b) Ali je c tudi lastna vrednost, če je vsota elementov v vsakem stolpcu enaka c? (Vsote vrst niso specificirane.)

Let A be a real square matrix.

(a) Suppose that the sum of elements in every row is c. Prove that c is an eigenvalue of the matrix.

(b) Is c also an eigenvalue if the sum of the elements in every column is c? (The sums of the rows are unspecified.)

Rešite **natanko dve** od 5. do 11. naloge.
Answer **exactly two** of questions 5–11.

5. naloga (6 točk)

Naj bo $(X, \mathcal{A}, \mu)$ merljiv prostor z neskončno mero. Naj za vsako merljivo podmnožico $A \subseteq X$ z neskončno mero obstaja taka merljiva množica $B \subseteq A$, da je $0 < \mu(B) < \infty$. Dokažite, da ne obstajata različni števili $p, q \in [1, \infty]$, da je $L^q(\mu) = L^p(\mu)$.

Let $(X, \mathcal{A}, \mu)$ be a measure space with infinite measure. Suppose that every subset $A \subseteq X$ with infinite measure has a measurable subset $B \subseteq A$ such that $0 < \mu(B) < \infty$. Prove that there do not exist distinct numbers $p, q \in [1, \infty]$ such that $L^q(\mu) = L^p(\mu)$.

6. naloga (6 točk)

Naj bo G končna grupa in H podgrupa v G. Naj bo U unija vseh konjugirank podgrupe H v G, torej unija vseh gHg^{-1}, kjer $g \in G$.

(a) Naj bo $\{g_1, g_2, \ldots, g_n\}$ množica predstavnikov vseh levih odsekov H v G. Pokaži, da velja

$$U \setminus \{1\} = \bigcup_{i=1}^{n} (g_i H g_i^{-1} \setminus \{1\}).$$

(b) Naj bo H podgrupa v G, ki vsebuje vsaj en element iz vsakega konjugiranega razreda v G. Pokaži, da je $H = G$.

(c) Naj bo G podgrupa v S_n, ki naravno deluje na množico $\{1, 2, \ldots, n\}$, kjer je $n \geq 2$. Recimo, da ima to delovanje eno samo orbito. Pokaži, da obstaja element grupe G, ki nima fiksne točke glede na to delovanje.

Let G be a finite group and H a subgroup of G. Let U be the union of all conjugate subgroups of H in G, that is, the union of all gHg^{-1}, where $g \in G$.

(a) Let $\{g_1, g_2, \ldots, g_n\}$ be a set of representatives for all left cosets of H in G. Show that

$$U \setminus \{1\} = \bigcup_{i=1}^{n} (g_i H g_i^{-1} \setminus \{1\}).$$

(b) Let H be a subgroup of G containing at least one element from every conjugacy class in G. Prove that $H = G$.

(c) Let G be a subgroup of S_n, with its natural action on the set $\{1, 2, \ldots, n\}$, where $n \geq 2$. Suppose that this action has only one orbit. Show that there exists an element of the group G whose action on $\{1, 2, \ldots, n\}$ has no fixed points.

7. naloga (6 točk)

Naj bo M poljubna množica. Za metriko $d\colon M \times M \to [0, +\infty)$ pravimo, da je *ultrametrika*, če za poljubne $x, y, z \in M$ velja $d(x, z) \leq \max\{d(x, y), d(y, z)\}$.

(a) Na isti množici M poiščite en primer metrike, ki je ultrametrika in en primer metrike, ki ni ultrametrika,

(b) Naj bo B odprta krogla v ultrametričnem prostoru (M, d). Določite vsa središča B ($x \in M$ je *središče* B, če je $B = B(x, r)$ za nek $r > 0$).

(c) Dokažite, da v ultrametričnem prostoru lahko poljuben zaprto kroglo s polmerom s predstavimo kot disjunktno unijo odprtih krogel s polmerom r za poljuben $r \in (0, s]$.

Let M be any set. An *ultrametric* on M is a metric $d\colon M \times M \to [0, +\infty)$ which satisfies $d(x, z) \leq \max\{d(x, y), d(y, z)\}$ for every $x, y, z \in M$.

(a) Give an example of a metric that is also an ultrametric, and an example of a metric that is not an ultrametric, both on the same set M.

(b) Let B be an open ball in an ultrametric space (M, d). Describe all centres of B. ($x \in M$ is a *centre* of B if $B = B(x, r)$ for some $r > 0$)

(c) Prove that in an ultrametric space every closed ball with radius s can be decomposed as a disjoint union of open balls with radius r, for any $r \in (0, s]$.

8. naloga (6 točk)

Za dana naravna števila m, n in k izračunajte spodnjo vsoto. Rezultat utemeljite s kombinatoričnim premislekom.

$$\sum_{i=0}^{k} \binom{m}{i} \binom{n}{k-i}.$$

Given natural numbers m, n and k, calculate the sum above. Explain the result in combinatorial terms.

9. naloga (6 točk)

Naj bosta A in B 1-dimenzionalni tabeli naravnih števil z dolžino n.

(a) Opišite algoritem s časovno zahtevnostjo $O(n \log n)$, ki vrne **true**, če je B permutacija tabele A, sicer vrne **false**.

(b) Prilagodite algoritem, da bo izračunal tudi parnost permutacije (če obstaja), še vedno s časovno zahtevnostjo $O(n \log n)$.

Let A and B be 1-dimensional integer matrices each of size n.

(a) Describe an algorithm with time complexity $O(n \log n)$ that returns **true** if B is a permutation of A and **false** otherwise.

(b) Adapt the algorithm to further compute the parity of the permutation (if one exists), again with time complexity $O(n \log n)$.

10. naloga (6 točk)

Naj bo $A \in \mathbb{R}^{m\times n}$, $m \geq n$, polnega ranga in $\boldsymbol{b} \in \mathbb{R}^m$. Denimo, da poznate razcep $A = QR$, kjer je $Q \in \mathbb{R}^{m\times m}$ ortogonalna matrika in $R \in \mathbb{R}^{m\times n}$ kvazi zgornja trikotna matrika.

(a) Dokažite da je
$$\min_{\boldsymbol{x}\in\mathbb{R}^n} \|A\boldsymbol{x} - \boldsymbol{b}\|_2 = \min_{\boldsymbol{x}\in\mathbb{R}^n} \|R\boldsymbol{x} - Q^T\boldsymbol{b}\|_2.$$

(b) Zapišite algoritem za izračun $\boldsymbol{x} \in \mathbb{R}^n$, ki minimizira $\|A\boldsymbol{x} - \boldsymbol{b}\|_2$. Ali rešitev vedno obstaja? Ali je enolična?

(c) Preštejte število operacij algoritma iz prejšnje točke.

Let $A \in \mathbb{R}^{m\times n}$ where $m \geq n$ be of full rank, and let $\boldsymbol{b} \in \mathbb{R}^m$. Suppose that we know a factorisation $A = QR$, where $Q \in \mathbb{R}^{m\times m}$ is an orthogonal matrix and $R \in \mathbb{R}^{m\times n}$ is a quasi-upper-triangular matrix.

(a) Prove that:
$$\min_{\boldsymbol{x}\in\mathbb{R}^n} \|A\boldsymbol{x} - \boldsymbol{b}\|_2 = \min_{\boldsymbol{x}\in\mathbb{R}^n} \|R\boldsymbol{x} - Q^T\boldsymbol{b}\|_2.$$

(b) Write an algorithm to calculate $\boldsymbol{x} \in \mathbb{R}^n$ that minimises $\|A\boldsymbol{x} - \boldsymbol{b}\|_2$. Does a solution always exist? Is it unique?

(c) Count the number of operations used by the algorithm from the previous point.

11. naloga (6 točk)

Žrebamo z vračanjem iz žare, ki vsebuje a belih, b rdečih in c črnih kroglic, dokler prvič ne izžrebamo bele po tem, ko smo že izžrebali kroglice obeh drugih barv. Naj bo X število belih krogel, ki smo jih izžrebali. Izračunajte $E(X)$.

We draw with replacement from an urn that contains a white, b red and c black balls, until we draw the first white ball that appears after both other colours have already been seen. Let X be the number of white balls drawn. Calculate $E(X)$.