Cointegration

Basic Ideas and Key results

Egon Zakrajšek
Division of Monetary Affairs
Federal Reserve Board

Summer School in Financial Mathematics
Faculty of Mathematics & Physics
University of Ljubljana
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Motivation

Economic theory often suggests that certain pairs of economic or financial variables should be linked by a long-run economic relationship.

Arbitrage arguments imply that the $I(1)$ prices of certain financial time series are linked.

The framework of cointegration deals with regression models with $I(1)$ data.

Idea Behind Cointegration

Many economic or financial time series appear to be $I(1)$:

- $I(1)$ variables tend to diverge as $T \to \infty$, because their unconditional variances are proportional to T.
- Thus, it may seem that $I(1)$ variables could never be expected to obey any sort of long-run equilibrium relationship.

It is possible for two (or more) variables to be $I(1)$, and yet a certain linear combination of those variables to be $I(0)$!

If that is the case, the $I(1)$ variables are said to be cointegrated:

- If two or more $I(1)$ variables are cointegrated, they must obey an equilibrium relationship in the long-run, although they may diverge substantially from that equilibrium in the short run.

Some Examples

- The permanent income hypothesis (PIH) implies cointegration between consumption and income.
- Money demand models imply cointegration between money, nominal income, prices, and interest rates.
- Growth theory models imply cointegration between income, consumption, and investment.
- Purchasing power parity (PPP) implies cointegration between the nominal exchange rate and foreign and domestic prices.
- The Fisher equation implies cointegration between nominal interest rates and inflation.
- The expectations hypothesis of the term structure implies cointegration between nominal interest rates at different maturities.

Cointegration

Consider two time series $y_{1,t}$ and $y_{2,t}$, known to be $I(1)$:

- Let y_1 and y_2 denote T-vectors:

$$\underset{(T\times 1)}{y_1} = \begin{bmatrix} y_{1,1} \\ y_{1,2} \\ \vdots \\ y_{1,T} \end{bmatrix} \qquad \underset{(T\times 1)}{y_2} = \begin{bmatrix} y_{2,1} \\ y_{2,2} \\ \vdots \\ y_{2,T} \end{bmatrix}$$

- $y_{1,t}$ and $y_{2,t}$ will be cointegrated if there exists a vector $\eta \equiv (1, \eta_2)'$ such that, when $y_{1,t}$ and $y_{2,t}$ are in equilibrium:

$$[y_1 \; y_2]\eta \equiv y_1 - \eta_2 y_2 = 0$$

- The η is called a cointegrating vector.
- It is clearly not unique!

Cointegration

Realistically, $y_{1,t}$ and $y_{2,t}$ are likely to be changing over time systematically as well as stochastically.

Cointegrating relationship:

$$Y\eta = X\beta$$

- $Y = [y_1\ y_2]$
- $X =$ nonstochastic matrix (i.e., constant, trends)

Cointegration

Relationship $[y_1\ y_2]\eta = 0$ cannot be expected to hold exactly for all t.

Equilibrium error:

$$\nu_t = y_t'\eta - x_t'\beta$$

In the general case when $Y = [y_1\ y_2 \dots y_m]$, the m series $y_{1,t}, y_{2,t}, \dots, y_{m,t}$ are said to be cointegrated if there exists a vector η such that the equilibrium error ν_t is $I(0)$.

Common Trends

Consider the following two trend-stationary $I(1)$ series:

$$
\begin{aligned}
y_{1,t} &= \alpha_1 + \beta_1 t + u_{1,t} \\
y_{2,t} &= \alpha_2 + \beta_2 t + u_{2,t}
\end{aligned}
$$

- $\{u_{1,t}\}_{t=-\infty}^{\infty}$ and $\{u_{2,t}\}_{t=-\infty}^{\infty}$ are white noise processes.

Linear combination of $y_{1,t}$ and $y_{2,t}$ with $\eta = (1, -\eta)'$:

$$
\nu_t = (\alpha_1 - \eta\alpha_2) + (\beta_1 - \eta\beta_2)t + u_{1,t} - \eta u_{2,t}
$$

- ν_t, in general, is still $I(1)$.
- The only way the ν_t series can be made $I(0)$ is if $\eta = \beta_1/\beta_2$.
- The effect of combining the two series is to remove the common linear trend.

Common Trends

Consider the following two independent random walk processes:

$$
\begin{aligned}
y_{1,t} &= w_{1,t} \\
y_{2,t} &= w_{2,t}
\end{aligned}
$$

- $w_{i,t} = w_{i,t-1} + \epsilon_{i,t}$
- $\epsilon_{i,t}$, $i = 1, 2$, are two independent white noise processes.

Any linear combination of $y_{1,t}$ and $y_{2,t}$ must involve random walks $w_{1,t}$ and $w_{2,t}$:

- $y_{1,t}$ and $y_{2,t}$ cannot be cointegrated unless $w_{1,t} = w_{2,t}$
- Once again, $y_{1,t}$ and $y_{2,t}$ must have a common trend.

Common Trends

In a bivariate case, the end result is that if $y_{1,t}$ and $y_{2,t}$ are cointegrated, then they must share exactly one common stochastic or deterministic trend.

This observation, readily generalizes to multivariate cointegration:

- A set of m series that are cointegrated can be written as a covariance-stationary component plus a linear combination of a smaller set of common trends.
- The effect of cointegration is to purge these common trends from the resultant series.

Obvious econometric questions:

- How to estimate the cointegrating vector η?
- How to test whether two or more variables are in fact cointegrated?

Estimating Cointegrating Vectors

Cointegrating relationship between m series $(y_{1,t}, y_{2,t}, \ldots, y_{m,t})$ can, in general, be written as

$$\nu_t = y_t'\eta - x_t'\beta$$

Rewrite the above equation as a linear regression and use OLS to estimate η:

$$y_1 = X\beta + Y^*\eta^* + \nu$$

- Coefficient of y_1 is arbitrarily normalized to unity.
- $Y^* = [y_2\, y_3 \ldots y_m]$.
- Parameter vector η^* is equal to minus the $(m-1)$ free elements of the parameter vector η.

Estimating Cointegrating Vectors

Two potentially serious problems:

- Endogeneity: If $(y_{1,t}, y_{2,t}, \ldots, y_{m,t})$ are cointegrated, they are surely determined jointly. The error term ν_t will almost certainly be correlated with the regressors!
- Spurious regression: We are regressing an $I(1)$ variable on one or more other $I(1)$ variables!

Nevertheless, OLS may be used to obtain a consistent estimate of a normalized cointegrating vector η.

Properties of the OLS Estimator of η^*

Important caveats (see Stock (1987) and Phillips (1991)):

- $T(\hat{\eta}^* - \eta^*)$ converges in distribution to a nonnormal RV not necessarily centered at zero.
- The OLS estimator $\hat{\eta}^*$ is consistent and converges to η^* at rate T instead of the usual rate $\sqrt{T}$. That is, $\hat{\eta}^*$ is super consistent.
- Asymptotically, there is no simultaneity bias.
- In general, the asymptotic distribution of $T(\hat{\eta}^* - \eta^*)$ is asymptotically biased and nonnormal. The usual OLS standard errors are not correct.
- Even though the asymptotic bias goes to zero as $T \to \infty$, $\hat{\eta}^*$ may be substantially biased in finite samples.
- The OLS estimator $\hat{\eta}^*$ is also not efficient.

Dynamic OLS (DOLS)

Asymptotically more efficient estimates of η^* may be obtained by DOLS (see Stock & Watson 1993):

$$y_1 = X\beta + Y^*\eta^* + \sum_{j=-p}^{p} \Delta Y^*_{-j}\gamma_j + \nu$$

- DOLS specification simply adds p leads and p lags of the first difference of Y^* to the standard cointegrating regression.
- The addition of leads and lags removes the deleterious effects that short-run dynamics of the equilibrium process ν_t have on the estimate of the cointegrating vector η^*.

Properties of DOLS Estimator of η^*

- The DOLS estimator $\hat{\eta}^*$ is consistent, asymptotically normally distributed, and efficient.
- Asymptotically valid standard errors for the individual elements of $\hat{\eta}^*$ are given by their corresponding HAC (e.g., Newey-West) standard errors.
- If T is not large relative to $p(m-1)$, there may be so many additional regressors that finite-sample properties of the DOLS estimator $\hat{\eta}^*$ will actually be quite poor.

Testing for Cointegration

Basic idea (Engle & Granger 1987):

- If $(y_{1,t}, y_{2,t}, \ldots, y_{m,t})$ are cointegrated, the true equilibrium error process ν_t must be $I(0)$.
- If they are not cointegrated, then ν_t must be $I(1)$.
- Test the null hypothesis of no cointegration against the alternative of cointegration by performing a unit root test on the equilibrium error process ν_t.

Residual-Based Cointegration Tests

Engle-Granger (EG) 2-step procedure:

- Choose the normalization (i.e., $\eta_1 = 1$).
- Form the cointegrating residual $\nu_t = Y_t' \eta^*$.
- Test whether or not ν_t has a unit root—that is, is an $I(1)$ process.
- Rejection of the null hypothesis at a pre-specified significance level implies that the m series are cointegrated.

Two cases to consider:

- The proposed cointegrating vector η is pre-specified.
- The proposed cointegrating vector η is estimated from the data and an estimate of the cointegrating residual is formed:

$$\hat{\nu}_t = Y_t' \hat{\eta}$$

Tests using the pre-specified cointegrating vector are much more powerful!

Testing for Cointegration when η^* is Unknown

ADF test:

$$\Delta\hat{\nu}_t = (\alpha - 1)\hat{\nu}_t + \sum_{j=1}^{p} \theta_j \Delta\hat{\nu}_{t-j} + e_t$$

- $\hat{\nu}_t = y_{1t} - X'\hat{\beta} - Y'_{2t}\hat{\nu}$
- $\hat{\eta}$ = OLS or DOLS estimate of the cointegrating vector η

Testing for Cointegration when η^* is Unknown

Key results:

- Phillips and Ouliaris (1990) show that residual-based unit root tests applied to the estimated cointegrating residuals do not have the usual Dickey-Fuller distributions under the null hypothesis of no-cointegration.
- Because of the spurious regression phenomenon under the null hypothesis, the distribution of these tests have asymptotic distributions that depend on:
  - Deterministic terms in the regression used to estimate η.
  - $(m-1) =$ number of varibles Y_{2t}.
- These distributions are known as Phillips-Ouliaris distributions and critical values have been tabulated.

Error-Correction Model

Consider:

- $Y_t = [y_{1t}\ y_{2t}]' =$ bivariate $I(1)$ process
- Y_t is cointegrated with $\eta = [1\ \ -\eta_2]'$

Error-Correction Model (ECM) (Engle & Granger (1987)):

$$\begin{aligned}\Delta y_{1t} &= \alpha_1 + \lambda_1[y_{1t-1} - \eta y_{2t-1}] \\ &\quad + \sum_j \beta_{1j}\Delta y_{1t-j} + \sum_j \gamma_{1j}\Delta y_{2t-j} + e_{1t} \\ \Delta y_{2t} &= \alpha_2 + \lambda_2[y_{1t-1} - \eta y_{2t-1}] \\ &\quad + \sum_j \beta_{2j}\Delta y_{1t-j} + \sum_j \gamma_{2j}\Delta y_{2t-j} + e_{2t}\end{aligned}$$

Error Correction Model

- ECM links the long-run equilibrium relationship between y_{1t} and y_{2t} implied by cointegration with the short-run dynamic adjustment mechanism that describes how the two series react when they move out of long-run equilibrium.
- Parameters λ_1 and λ_2 measure the speed of adjustment of y_{1t} and y_{2t} to the long-run equilibrium, respectively.