Optimal stopping and American options
Chapter 4: Price functions. Numerical methods

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Summer School on Financial Mathematics
Ljubiana, September 2009
Optimal stopping and stochastic differential equations

Outline

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Variational inequality
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Variational inequality

Numerical methods
Consider a stochastic differential equation

\[ dX_t = b(t, X_t)dt + \sigma(t, X_t).dW_t, \]

(1)

where \( W = (W_t^{1}, \ldots, W_t^{l})_{0 \leq t \leq T} \) is a standard \( l \)-dimensional Brownian motion with respect to a filtration \( \mathbb{F} = (\mathcal{F}_t)_{0 \leq t \leq T} \), defined on a probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \), \( b \) is a continuous function from \([0, T] \times \mathbb{R}^d \) into \( \mathbb{R}^d \), \( \sigma \) a continuous function from \([0, T] \times \mathbb{R}^d \) into the space \( \mathcal{M}_{d,l} \) of real matrices with \( d \) rows and \( l \) columns. We assume that the functions \( b \) and \( \sigma \) satisfy a Lipschitz condition with respect to \( x \), which is uniform over time.
We then have existence and uniqueness of a strong solution for equation (1). For \((t, x) \in [0, T] \times \mathbb{R}^d\), let \((X_{s}^{t, x})_{t \leq s \leq T}\) be the unique solution of (1) on the time interval \([t, T]\), such that \(X_{t}^{t, x} = x\).
We then have existence and uniqueness of a strong solution for equation (1). For \((t, x) \in [0, T] \times \mathbb{R}^d\), let \((X_{s}^{t,x})_{t \leq s \leq T}\) be the unique solution of (1) on the time interval \([t, T]\), such that \(X_{t}^{t,x} = x\).

Let \(r : [0, T] \times \mathbb{R}^d \to \mathbb{R}\) be a continuous nonnegative function. We are interested in the optimal stopping problem with reward process

\[
Z_t = e^{-\int_0^t r(s,X_s)ds}f(t, X_t),
\]

where \(X\) is a solution of (1) (with \(X_0\) deterministic), and \(f : [0, T] \times \mathbb{R}^d \to \mathbb{R}\) a continuous nonnegative function, with \(f(t, x) \leq C(1 + |x|^k)\) for every \((t, x) \in [0, T] \times \mathbb{R}^d\), where \(C\) and \(k\) are positive constants.
Theorem

The function $F$, defined on $[0, T] \times \mathbb{R}^d$ by

$$F(t, x) = \sup_{\tau \in \mathcal{I}_{t,T}} \mathbb{E} \left( \beta_{T}^{t,x} f(\tau, X_{T}^{t,x}) \right),$$

with $\beta_{s}^{t,x} = \exp(-\int_t^s r(\theta, X_{\theta}^{t,x}) d\theta)$ is continuous and if $X$ is a solution of (1) (with $X_0$ deterministic), the process

$$(\beta_t F(t, X_t))_{0 \leq t \leq T},$$

where $\beta_t = \exp(-\int_0^t r(s, X_s) ds)$, is the Snell envelope of

$$Z = (\beta_t f(t, X_t))_{0 \leq t \leq T}.$$
The function $F$, defined on $[0, T] \times \mathbb{R}^d$ by

$$F(t, x) = \sup_{\tau \in I_{t,T}} \mathbb{E}\left(\beta^t_{\tau} f(\tau, X^t_{\tau})\right),$$

with $\beta^t_{s} = \exp\left(-\int_s^t r(\theta, X^t_{\theta}) d\theta\right)$ is continuous and if $X$ is a solution of (1) (with $X_0$ deterministic), the process

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$$Z = (\beta_t f(t, X_t))_{0 \leq t \leq T}.$$

Moreover, if the functions $r$, $b$ and $\sigma$ do not depend on time, we have

$$F(t, x) = \sup_{\tau \in I_{0,T-t}} \mathbb{E}\left(\beta^0_{\tau} f(t + \tau, X^0_{\tau})\right).$$
Call and put prices in Black-Scholes

In the Black-Scholes model, there is just one risky asset, with price $S_t$ at time $t$ and coefficients do not depend on time. Under the risk neutral probability measure (denoted by $\mathbb{P}$), we have

$$dS_t = S_t [(r - \delta)dt + \sigma dW_t],$$

where $(W_t)_{0 \leq t \leq T}$ is standard Brownian motion.
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$$S^{0,x}_t = xe^{(r-\delta-(\sigma^2/2))t+\sigma W_t}.$$
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$$S_{t}^{0,x} = xe^{(r-\delta-(\sigma^2/2))t + \sigma W_t}.$$

**Proposition**

Let $\psi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be a continuous function with polynomial growth. The value at time $t$ of an American option with payoff process $Z_t = \psi(S_t)$ is given by $V(t, S_t)$, where

$$V(t, x) = \sup_{\tau \in \mathcal{T}_0, T-t} \mathbb{E}e^{-r\tau} \psi \left(xe^{(r-\delta-(\sigma^2/2))\tau + \sigma W_\tau}\right).$$
Call-put symmetry

Let

\[ C(t, x; K, r, \delta) = \sup_{\tau \in T_0, T-t} \mathbb{E} e^{-r\tau} \left( x e^{(r-\delta-\frac{\sigma^2}{2})\tau + \sigma W_\tau} - K \right) + \]

and

\[ P(t, x; K, r, \delta) = \sup_{\tau \in T_0, T-t} \mathbb{E} e^{-r\tau} \left( K - x e^{(r-\delta-\frac{\sigma^2}{2})\tau + \sigma W_\tau} \right) + . \]

**Proposition**

We have

\[ C(t, x; K, r, \delta) = P(t, K; x, \delta, r) = x P(t, K/x; 1, \delta, r). \]
Proof

For \( \tau \in I_0, T-t \), let

\[
C^\tau(t, x; K, r, \delta) = \mathbb{E} e^{-r\tau} \left( x e^{(r-\delta-\frac{\sigma^2}{2})\tau+\sigma W_\tau} - K \right) +
\]

We have, with the notation \( \hat{W}_t = W_t - \sigma t \),

\[
C^\tau(t, x; K, r, \delta) = \mathbb{E} e^{-\delta \tau} e^{\sigma \hat{W}_\tau-(\sigma^2/2)\tau} \left( x - K e^{(\delta-r+\frac{\sigma^2}{2})\tau-\sigma \hat{W}_\tau} \right) +
\]

\[
= \mathbb{E} e^{-\delta \tau} e^{\sigma \hat{W}_\tau-(\sigma^2/2)\tau} \left( x - K e^{(\delta-r-\frac{\sigma^2}{2})\tau-\sigma \hat{W}_\tau} \right) +,
\]

where the last equality comes from the fact that \( (e^{\sigma W_t-(\sigma^2/2)t})_{t\geq 0} \) is a martingale.
Therefore,

\[ C^\tau(t, x; K, r, \delta) = \hat{\mathbb{E}}e^{-\delta \tau} \left( x - Ke^{(\delta - r - \frac{\sigma^2}{2})\tau - \sigma \hat{W}_\tau} \right) + , \]

where the probability \( \hat{\mathbb{P}} \) is defined by

\[ d\hat{\mathbb{P}}/d\mathbb{P} = e^{\sigma \hat{W}_T - (\sigma^2/2)T}. \]
Therefore,

\[ C^\tau(t, x; K, r, \delta) = \hat{\mathbb{E}} e^{-\delta \tau} \left( x - Ke^{(\delta - r - \frac{\sigma^2}{2})\tau - \sigma \hat{W}_\tau} \right) +, \]

where the probability \( \hat{\mathbb{P}} \) is defined by \( \frac{d\hat{\mathbb{P}}}{d\mathbb{P}} = e^{\sigma \hat{W}_T - (\sigma^2/2)T} \).

Under probability \( \hat{\mathbb{P}} \), the process \( (\hat{W}_t)_{0 \leq t \leq T} \) is a standard Brownian motion, as well as, by symmetry, the process \( (-\hat{W}_t)_{0 \leq t \leq T} \).

Hence,

\[ C(t, x; K, r, \delta) = P(t, K; x, \delta, r). \]
The put price

\[ P(t, x) = \sup_{\tau \in [0, T - t]} \mathbb{E} e^{-r \tau} \psi \left( xe^{(r - \delta - \frac{\sigma^2}{2})\tau + \sigma W_\tau} \right), \]

with

\[ \psi(x) = (K - x)_+. \]
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We assume \( r > 0 \) since, if \( r = 0 \), the American put is equivalent to the European put.
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For every \( x \in [0, +\infty) \), \( t \mapsto P(t, x) \) is a nonincreasing function.
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The put price

\[ P(t, x) = \sup_{\tau \in \mathcal{T}_{0, T-t}} \mathbb{E} e^{-r \tau} \psi \left( xe^{(r-\delta-\frac{\sigma^2}{2})\tau + \sigma W_{\tau}} \right), \]

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For every \( (t, x) \in [0, T] \times [0, +\infty) \), \( P(t, x) \geq \psi(x) = P(T, x) \).
Proposition

1. For every \((t, x) \in [0, T] \times [0, +\infty)\), we have

\[
P(t, x) = \sup_{\tau \in \mathcal{T}_{0, 1}} \mathbb{E} e^{-r\tau(T-t)} \psi \left( xe^{(r-\delta-\frac{\sigma^2}{2})\tau(T-t) + \sigma \sqrt{T-t} W_\tau} \right).
\]

2. For every \(t \in [0, T]\), and for \(x, y \geq 0\),

\[
|P(t, x) - P(t, y)| \leq |x - y|.
\]

3. There exits a positive constant \(C\) such that, for \(x \in [0, +\infty[\), and for \(t, s \in [0, T]\),

\[
|P(t, x) - P(s, x)| \leq C \left| \sqrt{T-t} - \sqrt{T-s} \right|.
\]
Proof

The first equality is a consequence of the scaling property of Brownian motion. Indeed, let $\tilde{F}_s = F(T-t)_s$. We have $\tau \in T_0, T-t$ if and only if $\tau/(T-t) \in \tilde{T}_{0,1}$, where $\tilde{T}_{0,1}$ is the set of all stopping times with respect to the filtration $(\tilde{F}_s)_{0 \leq s \leq 1}$, with values in $[0, 1]$. Therefore,

$$P(t, x) = \sup_{\tau \in \tilde{T}_{0,1}} \mathbb{E} e^{-r\tau(T-t)} \psi \left( x e^{(r-\delta-\frac{\sigma^2}{2})\tau(T-t)+\sigma W_{\tau(T-t)}} \right)$$

Now, observe that $(\tilde{F}_s)_{0 \leq s \leq 1}$ is the (completion of) the natural filtration of the process $(W_s(T-t))_{0 \leq s \leq 1}$ and that $(W_s(T-t))_{0 \leq s \leq 1}$ has the same law as $(\sqrt{T-t} W_s)_{0 \leq s \leq 1}$. 
For the second statement, note that, for \( \tau \in \mathcal{T}_0, T-t \),

\[
\left| \psi \left( x e^{(r-\delta-\frac{\sigma^2}{2})\tau + \sigma W_\tau} \right) - \psi \left( y e^{(r-\delta-\frac{\sigma^2}{2})\tau + \sigma W_\tau} \right) \right| \leq |x - y| e^{(r-\delta-\frac{\sigma^2}{2})\tau + \sigma W_\tau},
\]

where we have used the Lipschitz property of \( \psi \). The desired inequality can now be derived from \( \delta \geq 0 \) and \( \mathbb{E} e^{\sigma W_\tau - \frac{\sigma^2 \tau}{2}} = 1 \).

The third part of the proposition can be deduced in a similar way from (2).
Remark

It follows from the Lipschitz properties of $P$, as given by Proposition 3, that the first order partial derivatives of $P$ (in the sense of distributions) are locally bounded on the open set $(0, T) \times (0, +\infty)$. More precisely, we have

$$
\|\partial P/\partial x\|_{L^\infty([0, T] \times [0, +\infty))} \leq 1
$$

and, for $t \in [0, T[$,

$$
\|\partial P/\partial t(t, \cdot)\|_{L^\infty([0, +\infty))} \leq \frac{C}{\sqrt{T-t}}.
$$
Introduce the so-called *Dynkin operator* $\mathcal{D}$, associated with the SDE:

$$
\mathcal{D} = \frac{\partial}{\partial t} + \frac{1}{2} \sum_{i=1}^{d} \sum_{j=1}^{d} a_{ij}(t, x) \frac{\partial^{2}}{\partial x_{i} \partial x_{j}} + \sum_{i=1}^{d} b_{i}(t, x) \frac{\partial}{\partial x_{i}},
$$

where the matrix $a(t, x) = (a_{ij}(t, x))_{1 \leq i, j \leq d}$ is the product of the matrix $\sigma(t, x)$ with its transpose:

$$
a(t, x) = \sigma(t, x) \sigma^{*}(t, x).
$$
If $X$ is a solution of the SDE on the interval $[0, T]$ and if 
\[ \beta_t = \exp \left( - \int_0^t r(s, X_s) \, ds \right), \]
we have, for a function $F$ of class $C^{1,2}$ on $[0, T] \times \mathbb{R}^d$,

\[ \beta_t F(t, X_t) = F(0, X_0) + \int_0^t \beta_s \nabla F(s, X_s) \cdot \sigma(s, X_s) \, dW_s \]
\[ + \int_0^t \beta_s (\mathcal{D} F - rF)(s, X_s) \, ds, \]

with the notation
\[ \nabla F(s, X_s) \cdot \sigma(s, X_s) \, dW_s = \sum_{i=1}^d \frac{\partial F}{\partial x_i}(s, X_s) \sum_{j=1}^d \sigma_{ij}(s, X_s) dW_s^j. \]
For the process \((\beta_t F(t, X_t))_{0 \leq t \leq T}\) to be the Snell envelope of the discounted payoff process \((\beta_t f(t, X_t))_{0 \leq t \leq T}\), we need

\[
D F - r F \leq 0
\]

\[
F \geq f
\]

\[
F(T, \cdot) = f(T, \cdot)
\]

\[
D F - r F = 0 \text{ on the set } \{F > f\}
\]

In summary

\[
\max(D F - r F, f - F) = 0
\]

\[
F(T, \cdot) = f(T, \cdot)
\]
For the process \((\beta_t F(t, X_t))_{0 \leq t \leq T}\) to be the Snell envelope of the discounted payoff process \((\beta_t f(t, X_t))_{0 \leq t \leq T}\), we need

\[\mathcal{D}F - rF \leq 0,\]
For the process $\left( \beta_t F(t, X_t) \right)_{0 \leq t \leq T}$ to be the Snell envelope of the discounted payoff process $\left( \beta_t f(t, X_t) \right)_{0 \leq t \leq T}$, we need

$$\mathcal{D}F - rF \leq 0,$$

$$F \geq f, \quad F(T, \cdot) = f(T, \cdot).$$
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\[\mathcal{D}F - rF \leq 0,\]

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\[\mathcal{D}F - rF = 0 \quad \text{on the set} \quad \{F > f\}\]

In summary

\[
\begin{cases}
\max (\mathcal{D}F - rF, f - F) = 0 \\
F(T, \cdot) = f(T, \cdot).
\end{cases}
\]
The American put price

In the Black-Scholes model. If we set $X_t = \log(S_t)$, we have

$$dX_t = \mu dt + \sigma dW_t,$$

(3)

with $\mu = r - \delta - \frac{\sigma^2}{2}$. Denote by $X^x$ the solution of (3) with $X_0^x = x$, so that $X^x_t = x + \mu t + \sigma W_t$. The Dynkin operator is given by

$$\mathcal{D} = \frac{\partial}{\partial t} + \frac{\sigma^2}{2} \frac{\partial^2}{\partial x^2} + \mu \frac{\partial}{\partial x}.$$

American put price: $P(t, x) = F(t, \log x)$, where the function $F$ is defined by

$$F(t, x) = \sup_{\tau \in \mathcal{T}_0, \tau - t} \mathbb{E} e^{-r\tau} f(X^x_\tau), \text{ with } f(x) = (K - e^x)_+.$$
**Theorem**

1. The partial derivatives \( \partial F / \partial x \), \( \partial F / \partial t \) and \( \partial^2 F / \partial x^2 \) are locally bounded. More precisely, \( \partial F / \partial x \) is uniformly bounded on and there exists \( C_1 > 0 \) such that

\[
\forall t \in [0, T), \quad \left\| \frac{\partial F}{\partial t} (t, \cdot) \right\|_{L^\infty(\mathbb{R})} + \left\| \frac{\partial^2 F}{\partial x^2} (t, \cdot) \right\|_{L^\infty(\mathbb{R})} \leq \frac{C_1}{\sqrt{T - t}}.
\]

2. The function \( F \) satisfies the variational inequality

\[
\max (\mathcal{D}F(t, x) - r(t, x)F, f(x) - F(t, x)) = 0,
\]

\( dtdx \) a.e. in \((0, T) \times \mathbb{R}\), with terminal condition \( F(T, \cdot) = f \).

**Corollary**

The function \( \partial F / \partial x \) is continuous on the set \([0, T) \times \mathbb{R}\).
Perpetual put

\[ P_\infty(x) = \sup_{\tau \in \mathcal{I}_0,+\infty} \mathbb{E}e^{-r\tau} \psi \left( x e^{(r-\delta-\frac{\sigma^2}{2})\tau+\sigma W_\tau} \right), \]

with \( \psi(x) = (K - x)_+ \) and \( \mathcal{I}_0,+\infty \) the set of all finite stopping times. We have

\[ P_\infty(x) = \begin{cases} 
  K - x, & \text{if } x \leq x^* \\
  (K - x^*)(x/x^*)^{-\gamma}, & \text{if } x > x^*, \end{cases} \]

with \( x^* = K \gamma/(1 + \gamma) \) and

\[ \gamma = \frac{1}{\sigma^2} \left[ \left( r - \delta - \frac{\sigma^2}{2} \right) + \sqrt{\left( r - \delta - \frac{\sigma^2}{2} \right)^2 + 2r\sigma^2} \right]. \]
Exercise boundary

For $t \in [0, T)$, let

$$s^*(t) = \inf\{x \in [0, +\infty[ \mid P(t, x) > \psi(x) = (K - x)_+\}.$$

The number $s^*(t)$ is called critical price at time $t$. We have

$$x^* \leq s^*(t) < K, \quad t \in [0, T).$$

We deduce from the convexity of $x \mapsto P(t, x)$ that

$$\forall x \leq s^*(t), \quad P(t, x) = K - x$$

and

$$\forall x > s^*(t), \quad P(t, x) > (K - x)_+.$$
By translating the variational inequality satisfied by
\( F(t, x) = P(t, e^x) \) into an inequality satisfied by \( P \), we get,
\[ dtdx\text{-almost everywhere on } (0, T) \times (0, +\infty), \]

\[ \mathcal{D}P(t, x) = (\delta x - rK)1_{\{x \leq s^*(t)\}}, \]

where
\[ \mathcal{D} = \frac{\partial}{\partial t} + \frac{\sigma^2}{2}x^2\frac{\partial^2}{\partial x^2} + (r - \delta)x\frac{\partial}{\partial x} - r. \]

Using the generalized Ito formula, we get
\[ e^{-rt}P(t, S_t) = P(0, S_0) + \int_0^t e^{-ru}\sigma S_u \frac{\partial P}{\partial x}(u, S_u) dW_u \]
\[ + \int_0^t e^{-ru}(\delta S_u - rK)1_{\{S_u \leq s^*(u)\}} du. \]
Observe that the amount of risky asset in the minimal hedging strategy is given by $H_t = (\partial P/\partial x)(t, S_t)$. 
Observe that the amount of risky asset in the minimal hedging strategy is given by $H_t = (\partial P/\partial x)(t, S_t)$.

Let $t \to T$ to obtain

$$e^{-rT}(K - S_T) = P(0, S_0) + \int_0^T e^{-ru} \sigma S_u \frac{\partial P}{\partial x}(u, S_u) dW_u$$

$$+ \int_0^T e^{-ru}(\delta S_u - rK)1_{\{S_u \leq s^*(u)\}} du$$
Observe that the amount of risky asset in the minimal hedging strategy is given by $H_t = \left( \partial P / \partial x \right)(t, S_t)$. Let $t \to T$ to obtain

$$
e^{-rT}(K - S_T) = P(0, S_0) + \int_0^T e^{-ru} \sigma S_u \frac{\partial P}{\partial x}(u, S_u) dW_u$$

$$+ \int_0^T e^{-ru}(\delta S_u - rK)1\{S_u \leq s^*(u)\} du$$

Hence

$$P_e(0, S_0) = P(0, S_0) + \int_0^T e^{-ru} \mathbb{E} \left( (\delta S_u - rK)1\{S_u \leq s^*(u)\} \right) du,$$

where $P_e$ is the function price of the European put.
Early exercise premium

\[ P(t, x) = P_e(t, x) \]
\[ + \int_0^{T-t} \left( rK e^{-ru} N(d_1(x, t, u)) - \delta x e^{-\delta u} N(d_2(x, t, u)) \right) du, \]

where \( N \) is the standard normal cumulative distribution function,

\[ d_1(x, t, u) = \frac{\log(s^*(t + u)/x) - (r - \delta - \frac{\sigma^2}{2})u}{\sigma \sqrt{u}}, \]

\[ d_2(x, t, u) = \frac{\log(s^*(t + u)/x) - (r - \delta + \frac{\sigma^2}{2})u}{\sigma \sqrt{u}}. \]
Numerical methods

Many numerical methods have been implemented in the Premia software, which can be downloaded from the web-site http://www-rocq.inria.fr/mathfi/Premia/index.html

For PDE methods, see the forthcoming Encyclopedia of Quantitative Finance, Wiley, the recent book by Achdou and Pironneau, a forthcoming book by C. Schwab et al.

Integral equations for the free boundary (G. Peskir, Chen-Chadam).

For quantization methods, see http://www.quantification.finance-mathematique.com/

Monte-Carlo methods: duality (Rogers, Haugh-Kogan), Malliavin calculus, least squares regression (Longstaff-Schwartz...), Glasserman's book.
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- Monte-Carlo methods: duality (Rogers, Haugh-Kogan), Malliavin calculus, least squares regression (Longstaff-Schwartz...), Glasserman’s book.