Factor Pricing Models

Egon Zakrajišek
Division of Monetary Affairs
Federal Reserve Board

Summer School in Financial Mathematics
Faculty of Mathematics & Physics
University of Ljubljana
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Consumption-Based Model of Asset Prices
(Cochrane, Asset Pricing, 2005)

The setup:

- Investor must decide how much to consume, how much to save, and what portfolio of assets to hold.
- \( p_t \) = price of an asset at date \( t \)
- \( x_{t+1} \) = payoff at date \( t + 1 \) from holding the asset from \( t \) to \( t + 1 \)
  - \( x_{t+1} \) is a random variable
  - Investor does not know exactly how much he will get from his investment, but can assess the probability of possible outcomes.

**Question**: What is the value at date \( t \) of a payoff \( x_{t+1} \)?
Investor’s Optimization Problem

Assume

- Utility function:
  \[ U(c_t, c_{t+1}) = u(c_t) + E_t[\beta u(c_{t+1})] \]
  - \( u(\cdot) \) = period utility function with \( u' > 0 \) and \( u'' < 0 \)
  - \( 0 < \beta < 1 \): subjective discount factor

- \( \{e_t\} = \) exogenous income process

- Costlessly buy or sell as much of the payoff \( x_{t+1} \) at price \( p_t \).

Optimization problem:

\[ \max_\omega \{u(c_t) + E_t[\beta u(c_{t+1})]\} \quad s.t. \]
\[ c_t = e_t + p_t \omega \]
\[ c_{t+1} = e_{t+1} + x_{t+1} + x_{t+1} \omega \]
First-order condition:

\[ p_t u'(c_t) = E_t[\beta u'(c_{t+1}) x_{t+1}] \]

- \( p_t u'(c_t) \): loss in utility at \( t \) if investor buys another unit of asset
- \( E_t[\beta u'(c_{t+1}) x_{t+1}] \): gain in (discounted, expected) utility from the extra payoff at \( t + 1 \)

Consumption-based pricing equation:

\[ p_t = E_t \left[ \beta \frac{u'(c_{t+1})}{u'(c_t)} x_{t+1} \right] \]

- Relates \( p_t \)—an endogenous variable—to consumption and payoffs, two other endogenous variables.
Define stochastic discount factor (SDF):

\[ m_{t+1} \equiv \beta \frac{u'(c_{t+1})}{u'(c_t)} \]

Basic asset pricing formula:

\[ p_t = E_t(m_{t+1}x_{t+1}) \quad \text{(or) } p = E(mx) \]

- SDF is also called the marginal rate of substitution or the pricing kernel.
- Gross returns are payoffs with a price of one:

\[ 1 = E(m_{t+1}R_{t+1}) \quad \text{where} \quad R_{t+1} \equiv \frac{x_{t+1}}{p_t} \]
The (gross) risk-free rate $R_f$ is known ahead of time (i.e., at date $t$):

$$p = E(mx) \Rightarrow 1 = E(mR_f) = E(m)R_f \Rightarrow R_f = \frac{1}{E(m)}$$

- If a risk-free security is not traded, we can define $R_f = 1/E(m)$ as the shadow risk-free rate.
- If one introduced a risk-free security with return $R_f = 1/E(m)$, investors would be just indifferent to buying or selling it.
Rewrite the standard asset pricing equation:

\[ p = E(mx) = E(m)E(x) + \text{cov}(m, x) \]

Using the definition of the risk-free rate:

\[ p = \frac{E(x)}{R_f} + \text{cov}(m, x) \Rightarrow p_t = \frac{E(x_{t+1})}{R_f} + \frac{\text{cov}[\beta u'(c_{t+1}), x_{t+1}]}{u'(c_t)} \]

- Marginal utility \( u'(c) \) declines as \( c \) increases (i.e., \( u'' < 0 \)).
- If \( \text{cov}(c, x) > 0 \) \( \Rightarrow p_t \downarrow \)
- If \( \text{cov}(c, x) < 0 \) \( \Rightarrow p_t \uparrow \)
- Covariance of a payoff with the SDF determines the riskiness of the asset.
- Investors do not like uncertainty about consumption!
Consider a risky asset with a gross return $R^i$:

$$1 = E(mR^i) \Rightarrow E(R^i) - R^f = -R^f \times \text{cov}(m, R^i)$$

or

$$E(R^i_{t+1}) - R^f = -\frac{\text{cov}[u'(c_{t+1}), R^i_{t+1}]}{E[u'(c_{t+1})]}$$

- All assets have an expected return equal to the risk-free rate, plus a risk adjustment.
- Assets whose returns are positively correlated with consumption must yield higher expected returns.
- Assets whose returns are negatively correlated with consumption can offer expected rates of return that are less than $R^f$. 
If \( \text{cov}(m, R^i) = 0 \) then \( E[R^i] = R_f \).

- No matter how volatile is the asset, if its return is uncorrelated with the SDF, the asset receives no risk correction. Its expected return is \( R_f \).
- Investors receive no compensation for holding idiosyncratic risk.
- Only systematic risk generates a risk correction.
Expected Return-Beta Representation

Expected return equation:

\[ E(R^i) = R^f - R^f \times \text{cov}(m, R^i) \]

\[ = R^f + \left[ \frac{\text{cov}(m, R^i)}{\text{var}(m)} \right] \times \left[ - \frac{\text{var}(m)}{E(m)} \right] \]

Beta pricing model:

\[ E(R^i) = R^f + \beta_{i,m} \lambda_m \]

- \( \beta_{i,m} = \) coefficient (i.e., beta) from the regression of \( R^i \) on \( m \)
- \( \beta_{i,m} = \) quantity of risk in asset \( i \)
- \( \lambda_m = \) price of risk
- The price of risk depends on the volatility of the SDF.
Factor pricing models replace the consumption-based expression for the SDF with a linear model:

$$m_{t+1} = a + b' f_{t+1}$$

What should one use for factors $f_{t+1}$:

- Look for variables that are good proxies for growth of aggregate marginal utility:
  $$\beta \frac{u'(c_{t+1})}{u'(c_t)} \approx a + b' f_t$$

- Sensible economic models imply that consumption is related to returns on broad-based portfolios, interest rates, economic growth, etc.

- Consumption (and marginal utility) responds to news. So variables that forecast macroeconomic outcomes can be used as factors (e.g., term spread, default-risk premium, dividend-yield).

- Factors should be (largely) unforecastable.
A Single-Factor Model

When factor $f_t$ is also a return:

$$E(R_{it} - R^f_t) = \beta_i E(f_t), \quad i = 1, \ldots, N$$

Market model (CAPM):

$$(R_{it} - R^f_t) = \alpha_i + \beta_i (R^M_t - R^f_t) + \epsilon_{it}$$

- $R^M_t = \text{market return}$
- CAPM can be evaluated by running an OLS time series regression for each $i = 1, \ldots, N$.
- Model implication: $\alpha_i = 0$ for all $i$.
- Are pricing errors jointly equal to zero assuming $E[\epsilon_{it}\epsilon_{jt}] \neq 0$?
  - Robust Wald test:
    $$\hat{\alpha}' \text{Cov}[\hat{\alpha}]^{-1} \hat{\alpha} \sim \chi^2_N$$
A Multifactor Model

General $m$-factor pricing model:

$$(R_{it} - R^f_t) = \alpha_i + \beta_{1i} f_{1t} + \cdots + \beta_{mi} f_{mt} + \epsilon_{it}$$

- $\{f_t\} = m$-dimensional (zero-mean) covariance-stationary process

Macroeconomic “surprises” model (Chen, Roll & Ross (1986)):

$$(R_{it} - R^f_t) = \alpha_i + \beta_{\pi,i} \pi^u_t + \beta_{u,i} e^u_t + \epsilon_{it}$$

- $\pi^u_t = \text{inflation “surprise”}$
- $e^u_t = \text{employment growth “surprise”}$

Macroeconomic “surprises” are obtained from an auxilliary VAR model.