Level Densities for General β-ensembles: An Operator-valued Free Probability Perspective

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Abstract

Random point processes corresponding to β-ensembles for arbitrary $\beta > 0$, or, equivalently, log gases at inverse temperature β, are being subject to intense study. The orthogonal, unitary, and symplectic ensembles ($\beta = 1, 2$ or 4, respectively) are now well understood, but other values of β are believed to also be relevant in theory (e.g. relevant for the study of self-adjoint and Schrödinger operators) and applications (e.g. in logistics). For certain rational values of β, β-ensembles are related to Jack polynomials, but for general β much less is known. In a seminal article, Dumitriu and Edelman (2002) constructed tridiagonal random matrix models for general β-Hermite and β-Laguerre ensembles and defined open problems for research on general β-ensembles, including finding a unified formula for the level density in general β-case. In general, level density is defined as distribution of a random eigenvalue of an ensemble (by the Wigner semicircular law, the limiting distribution of the eigenvalue is semicircular). We exploit the product nature of Dumitriu and Edelman's construction of tridiagonal random matrix models and derive the formula for the level density in the general β-case depending on the multivariate Fuss-Narayana polynomials and concepts from operator-valued free probability theory. We study perturbation invariability of the level densities (Wang and Yan, 2005; Kozhan, 2017) and discuss extensions to problems of sampling general β-ensembles (referring to Li and Menon, 2013; Olver et al., 2015; Srakar and Verbic, 2020) and limiting entropy in β-ensembles related point processes (Mészaros, 2020).

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