The Choquet integral as a continuous certainty equivalent

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Main aim of the paper

We present a simple axiomatization of the existence of a continuous certainty equivalent functional $C$ for a total preorder $\sim$ on the space of all nonnegative random variables on a common probability space which can be represented as the Choquet integral with respect to a concave probability distortion.

The axiomatization is based on a condition presented by Parker [Glasgow Mathematical Journal, 1996]. Such a condition, which will be referred to as wide translation invariance throughout this paper, generalizes the concept of a comonotone additive functional.

We take advantage of the fact that the Choquet integral (and more generally any monotone, positively homogeneous and comonotone subadditive premium functional) is uniformly continuous with respect to the norm topology on the space of all the nonnegative bounded measurable functions.
We present a simple axiomatization of the existence of a continuous certainty equivalent functional $C$ for a total preorder $\preceq$ on the space of all nonnegative random variables on a common probability space which can be represented as the Choquet integral with respect to a concave probability distortion.

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We take advantage of the fact that the Choquet integral (and more generally any monotone, positively homogeneous and comonotone subadditive premium functional) is uniformly continuous with respect to the norm topology on the space of all the nonnegative bounded measurable functions.
Let \((\Omega, \mathcal{F}, \mathcal{P})\) be a probability space, and denote by \(\mathbb{1}_F\) the indicator function of any subset \(F\) of \(\Omega\). In the sequel, we shall denote by \(\mathbb{R}_+\) the set of all nonnegative real numbers.

For the sake of brevity, for every \(\lambda \in \mathbb{R}_+\) we identify \(\lambda\) with the constant function \(\lambda \mathbb{1}_\Omega\).

Denote by \(L^\infty_+ (\Omega, \mathcal{F}, \mathcal{P})\) the space of all the nonnegative bounded random variables on \((\Omega, \mathcal{F}, \mathcal{P})\).

Denote by \(S_X(t) = 1 - F_X(t) = \mathcal{P}(\{\omega \in \Omega : X(\omega) > t\})\) the decumulative distribution function of any random variable \(X \in L^\infty_+ (\Omega, \mathcal{F}, \mathcal{P})\).
Notation and preliminaries

Let \((\Omega, \mathcal{F}, \mathcal{P})\) be a probability space, and denote by \(\mathbb{1}_F\) the indicator function of any subset \(F\) of \(\Omega\). In the sequel, we shall denote by \(\mathbb{R}_+\) the set of all nonnegative real numbers. For the sake of brevity, for every \(\lambda \in \mathbb{R}_+\) we identify \(\lambda\) with the constant function \(\lambda \mathbb{1}_\Omega\).

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shortname

Choquet certainty equivalent
A functional $\mathcal{C} : L_+^\infty(\Omega, \mathcal{F}, \mathcal{P}) \rightarrow \mathbb{R}_+$ is said to be

1. **Monotone** if $\mathcal{C}(X) \leq \mathcal{C}(Y)$ for all $X, Y \in L_+^\infty(\Omega, \mathcal{F}, \mathcal{P})$ such that $X \leq Y$;

2. **Monotone with respect to first order stochastic dominance** if $\mathcal{C}(X) \leq \mathcal{C}(Y)$ for all $X, Y \in L_+^\infty(\Omega, \mathcal{F}, \mathcal{P})$ such that $S_X(t) \leq S_Y(t)$ for all $t \in \mathbb{R}_+$;

3. **Monotone with respect to stop-loss order** if $\mathcal{C}(X) \leq \mathcal{C}(Y)$ for all $X, Y \in L_+^\infty(\Omega, \mathcal{F}, \mathcal{P})$ such that $\mathbb{E}[(X-d)_+] \leq \mathbb{E}[(Y-d)_+]$ for all $d \in \mathbb{R}_+$;

4. **Translation Invariant** if $\mathcal{C}(X + \lambda) = \mathcal{C}(X) + \lambda$ for all $X \in L_+^\infty(\Omega, \mathcal{F}, \mathcal{P})$ and $\lambda \in \mathbb{R}_+$;

5. **Widely Translation Invariant** if $\mathcal{C}(X + c\mathbb{1}_F) = \mathcal{C}(X) + c\mathcal{C}(\mathbb{1}_F)$ for all $X \in L_+^\infty(\Omega, \mathcal{F}, \mathcal{P})$, $c \in \mathbb{R}_+$ and $F \in \mathcal{F}$ such that $\{X > 0\} \subseteq F$;

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A functional $C : L^\infty_+(\Omega, \mathcal{F}, \mathcal{P}) \rightarrow \mathbb{R}_+$ is said to be

1. **Monotone** if $C(X) \leq C(Y)$ for all $X, Y \in L^\infty_+(\Omega, \mathcal{F}, \mathcal{P})$ such that $X \leq Y$;

2. **Monotone with respect to first order stochastic dominance** if $C(X) \leq C(Y)$ for all $X, Y \in L^\infty_+(\Omega, \mathcal{F}, \mathcal{P})$ such that $S_X(t) \leq S_Y(t)$ for all $t \in \mathbb{R}_+$;

3. **Monotone with respect to stop-loss order** if $C(X) \leq C(Y)$ for all $X, Y \in L^\infty_+(\Omega, \mathcal{F}, \mathcal{P})$ such that $E[(X - d)_+] \leq E[(Y - d)_+]$ for all $d \in \mathbb{R}_+$;

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Properties of real functionals

A functional $C : L_+^\infty(\Omega, F, P) \rightarrow \mathbb{R}_+$ is said to be

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**Choquet certainty equivalent**
Comonotone Additive if $C(X + Y) = C(X) + C(Y)$ for all comonotone $X, Y \in L^\infty_+(\Omega, \mathcal{F}, P)$ (i.e., for all $X, Y \in L^\infty_+(\Omega, \mathcal{F}, P)$ such that $(X(\omega_1) - X(\omega_2))(Y(\omega_1) - Y(\omega_2)) \geq 0$ for all $\omega_1, \omega_2 \in \Omega$).

Comonotone Subadditive if $C(X + Y) \leq C(X) + C(Y)$ for all comonotone $X, Y \in L^\infty_+(\Omega, \mathcal{F}, P)$;

Sublinear if $C$ is positively homogeneous (i.e., $C(\lambda X) = \lambda C(X)$ for every $\lambda \in \mathbb{R}_+$ and $X \in L^\infty_+(\Omega, \mathcal{F}, P)$) and subadditive (i.e., $C(X + Y) \leq C(X) + C(Y)$ for all $X, Y \in L^\infty_+(\Omega, \mathcal{F}, P)$).

A real functional $C$ is said to satisfy the normalization condition if $C(\mathbb{1}_\Omega) = 1$.

It is clear that if a real functional $C$ on $L^\infty_+(\Omega, \mathcal{F}, P)$ satisfies the normalization condition and is widely translation invariant then it is translation invariant.
Comonotone Additive if $C(X + Y) = C(X) + C(Y)$ for all comonotone $X, Y \in L_+^\infty(\Omega, \mathcal{F}, \mathcal{P})$ (i.e., for all $X, Y \in L_+^\infty(\Omega, \mathcal{F}, \mathcal{P})$ such that $(X(\omega_1) - X(\omega_2))(Y(\omega_1) - Y(\omega_2)) \geq 0$ for all $\omega_1, \omega_2 \in \Omega$).

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A real functional $C$ is said to satisfy the normalization condition if $C(1_{\Omega}) = 1$.

It is clear that if a real functional $C$ on $L_+^\infty(\Omega, \mathcal{F}, \mathcal{P})$ satisfies the normalization condition and is widely translation invariant then it is translation invariant.
**Comonotone Additive** if \( C(X + Y) = C(X) + C(Y) \) for all comonotone \( X, Y \in L_+^\infty(\Omega, \mathcal{F}, \mathcal{P}) \) (i.e., for all \( X, Y \in L_+^\infty(\Omega, \mathcal{F}, \mathcal{P}) \) such that \( (X(\omega_1) - X(\omega_2))(Y(\omega_1) - Y(\omega_2)) \geq 0 \) for all \( \omega_1, \omega_2 \in \Omega \)).

**Comonotone Subadditive** if \( C(X + Y) \leq C(X) + C(Y) \) for all comonotone \( X, Y \in L_+^\infty(\Omega, \mathcal{F}, \mathcal{P}) \);

**Sublinear** if \( C \) is positively homogeneous (i.e., \( C(\lambda X) = \lambda C(X) \) for every \( \lambda \in \mathbb{R}_+ \) and \( X \in L_+^\infty(\Omega, \mathcal{F}, \mathcal{P}) \)) and subadditive (i.e., \( C(X + Y) \leq C(X) + C(Y) \) for all \( X, Y \in L_+^\infty(\Omega, \mathcal{F}, \mathcal{P}) \)).

A real functional \( C \) is said to satisfy the **normalization condition** if \( C(1_\Omega) = 1 \).

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Comonotone Subadditive if $C(X + Y) \leq C(X) + C(Y)$ for all comonotone $X, Y \in L^\infty_+(\Omega, \mathcal{F}, \mathcal{P})$;

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A real functional $C$ is said to satisfy the normalization condition if $C(1_{\Omega}) = 1$.

It is clear that if a real functional $C$ on $L^\infty_+(\Omega, \mathcal{F}, \mathcal{P})$ satisfies the normalization condition and is widely translation invariant then it is translation invariant.
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7 **Comonotone Subadditive** if $C(X + Y) \leq C(X) + C(Y)$ for all comonotone $X, Y \in L_+^\infty(\Omega, \mathcal{F}, \mathcal{P})$;

8 **Sublinear** if $C$ is positively homogeneous (i.e., $C(\lambda X) = \lambda C(X)$ for every $\lambda \in \mathbb{R}_+$ and $X \in L_+^\infty(\Omega, \mathcal{F}, \mathcal{P})$) and subadditive (i.e., $C(X + Y) \leq C(X) + C(Y)$ for all $X, Y \in L_+^\infty(\Omega, \mathcal{F}, \mathcal{P})$).

A real functional $C$ is said to satisfy the *normalization condition* if $C(\mathbb{1}_{\Omega}) = 1$.

It is clear that if a real functional $C$ on $L_+^\infty(\Omega, \mathcal{F}, \mathcal{P})$ satisfies the normalization condition and is widely translation invariant then it is translation invariant.
Properties of preorders

Recall that a binary relation \( \succeq \) on a set \( A \) is said to be a \textit{preorder} if it is reflexive and transitive.

A preorder \( \succeq \) on \( L_+^\infty(\Omega, \mathcal{F}, \mathcal{P}) \) is said to be

1. \textbf{Monotone} if \( X \succeq Y \) for all \( X, Y \in L_+^\infty(\Omega, \mathcal{F}, \mathcal{P}) \) such that \( X \leq Y \);

2. \textbf{Monotone with respect to first order stochastic dominance} if \( X \succeq Y \) for all \( X, Y \in L_+^\infty(\Omega, \mathcal{F}, \mathcal{P}) \) such that \( S_X(t) \leq S_Y(t) \) for all \( t \in \mathbb{R}_+ \);

3. \textbf{Monotone with respect to stop-loss order} if \( X \succeq Y \) for all \( X, Y \in L_+^\infty(\Omega, \mathcal{F}, \mathcal{C}) \) such that \( \mathbb{E}[(X - d)_+] \leq \mathbb{E}[(Y - d)_+] \) for all \( d \in \mathbb{R}_+ \);

4. \textbf{Homothetic} if, for every \( X, Y \in L_+^\infty(\Omega, \mathcal{F}, \mathcal{P}) \), and \( \lambda \in \mathbb{R}_{++} \), \( X \succeq Y \) is equivalent to \( \lambda X \succeq \lambda Y \);

5. \textbf{Translation Invariant} if \( X + \lambda \sim Y + \lambda \) for all \( X, Y \in L_+^\infty(\Omega, \mathcal{F}, \mathcal{P}) \) such that \( X \sim Y \) and \( \lambda \in \mathbb{R}_+ \);

6. \textbf{Subadditive} \( X \succeq \lambda \) and \( Y \succeq \mu \) imply \( X + Y \succeq \lambda + \mu \) for all \( X, Y \in L_+^\infty(\Omega, \mathcal{F}, \mathcal{P}) \) and \( \lambda, \mu \in \mathbb{R}_+ \).
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Recall that a binary relation $\succeq$ on a set $A$ is said to be a preorder if it is reflexive and transitive.

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2. **Monotone with respect to first order stochastic dominance** if $X \succeq Y$ for all $X, Y \in L^\infty_+(\Omega, \mathcal{F}, \mathcal{P})$ such that $S_X(t) \leq S_Y(t)$ for all $t \in \mathbb{R}_+$;

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**Choquet certainty equivalent**
Properties of preorders

Recall that a binary relation \( \succeq \) on a set \( A \) is said to be a *preorder* if it is reflexive and transitive.

A preorder \( \succeq \) on \( L_+^\infty(\Omega, \mathcal{F}, \mathcal{P}) \) is said to be

1. **Monotone** if \( X \succeq Y \) for all \( X, Y \in L_+^\infty(\Omega, \mathcal{F}, \mathcal{P}) \) such that \( X \leq Y \);
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3. **Monotone with respect to stop-loss order** if \( X \succeq Y \) for all \( X, Y \in L_+^\infty(\Omega, \mathcal{F}, \mathcal{C}) \) such that \( \mathbb{E}[(X - d)_+] \leq \mathbb{E}[(Y - d)_+] \) for all \( d \in \mathbb{R}_+ \);
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shortname Choquet certainty equivalent
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We shall study the case of a total preorder \( \preceq \) on \( L^\infty_+(\Omega, \mathcal{F}, \mathcal{P}) \). This means that for any pair \( X, Y \in L^\infty_+(\Omega, \mathcal{F}, \mathcal{P}) \), either \( X \preceq Y \) or \( Y \preceq X \).

The strict part \( \prec \) and the symmetric part \( \sim \) of \( \preceq \) are defined in the usual way. Under our assumptions, \( \sim \) is a binary equivalence relation.

**Definition**

We say that a functional \( C : L^\infty_+(\Omega, \mathcal{F}, \mathcal{P}) \to \mathbb{R} \) is a certainty equivalence functional for a total preorder \( \preceq \) on \( L^\infty_+(\Omega, \mathcal{F}, \mathcal{P}) \) if the following conditions are met:

(i) \( C \) is a utility functional for \( \preceq \) (i.e., for every \( X, Y \in L^\infty_+(\Omega, \mathcal{F}, \mathcal{P}) \), \( X \preceq Y \) if and only if \( C(X) \leq C(Y) \));

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Conditions (i) and (ii) above imply that:

(iii) $X \sim C(X)$ for every $X \in L_+^\infty(\Omega, \mathcal{F}, \mathcal{P})$.

Condition (iii) means that $C(X)$ is a certainty equivalent of $X$ for every $X \in L_+^\infty(\Omega, \mathcal{F}, \mathcal{P})$.

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A real functional $\mathcal{C}$ on $L_+^\infty(\Omega,\mathcal{F},\mathcal{P})$ is the Choquet integral with respect to a probability distortion $g$ (i.e., $g$ is a real-valued, nondecreasing and nonnegative function on $[0,1]$ such that $g(0) = 0$ and $g(1) = 1$) if, for every $X \in L_+^\infty(\Omega,\mathcal{F},\mathcal{P})$,

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\mathcal{C}(X) = \int X dg \circ \mathcal{P} = \int_0^{+\infty} g(S_X(t))dt.
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If the above probability distortion $g$ is concave, then the corresponding Choquet integral is sublinear. If a total preorder $\preceq$ on $L_+^\infty(\Omega,\mathcal{F},\mathcal{P})$ admits a continuous certainty equivalent functional $\mathcal{C}$ which can be represented as the Choquet integral with respect to a concave probability distortion then $\preceq$ is risk loving (see e.g. Suijs and Borm, 1999), in the sense that $E(X) \preceq X$ for all $X \in L_+^\infty(\Omega,\mathcal{F},\mathcal{P})$, i.e., any stochastic payoff $X$ is weakly preferred to its expectation (see also Bosi and Zuanon, 2003).
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Uniform continuity of comonotone subadditive functionals

We recall that the norm on $L_+^\infty(\Omega, \mathcal{F})$ is defined as follows:

$$\|X\| = \sup_{\omega \in \Omega} |X(\omega)|.$$

The following lemma slightly improves Lemma 6 in Parker [Glasgow Mathematical Journal, 1996].

**Lemma**

If $C$ is a monotone, positively homogeneous and comonotone subadditive premium functional on $L_+^\infty(\Omega, \mathcal{F})$, then $C$ is uniformly continuous with respect to the norm topology on $L_+^\infty(\Omega, \mathcal{F})$.

**Outline of the proof.** For any two real-valued functions $X, Y \in L_+^\infty(\Omega, \mathcal{F})$ we have that

$$C(X) - C(Y) \leq C(\|X - Y\| \mathbb{1}_\Omega + Y) - C(Y) \leq \|X - Y\| C(\mathbb{1}_\Omega) + C(Y) - C(Y) = \|X - Y\| C(\mathbb{1}_\Omega).$$
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**Lemma**

If $C$ is a monotone, positively homogeneous and comonotone subadditive premium functional on $L^\infty_+(\Omega, \mathcal{F})$, then $C$ is uniformly continuous with respect to the norm topology on $L^\infty_+(\Omega, \mathcal{F})$.

**Outline of the proof.** For any two real-valued functions $X, Y \in L^\infty_+(\Omega, \mathcal{F})$ we have that

$$C(X) - C(Y) \leq C(\| X - Y \| \mathbb{1}_\Omega + Y) - C(Y)$$

$$\leq \| X - Y \| C(\mathbb{1}_\Omega) + C(Y) - C(Y)$$

$$= \| X - Y \| C(\mathbb{1}_\Omega).$$
Characterization of the Choquet integral as a continuous certainty equivalent

Lemma

Let $C$ be a positively homogeneous real functional on $L^\infty_+(\Omega, \mathcal{F}, P)$ and assume that $C$ satisfies the normalization condition. Then the following conditions are equivalent:

(i) There exists a concave probability distortion $g$ such that

$$C(X) = \int X \, d g \circ P$$

for all $X \in L^\infty_+(\Omega, \mathcal{F}, P)$;

(ii) $C$ is uniformly continuous in the norm topology on $L^\infty_+(\Omega, \mathcal{F})$, monotone with respect to stop loss order and for every finite chain $F_1 \subseteq F_2 \subseteq ... \subseteq F_n$ in $\mathcal{F}$,

$$C\left(\sum_{i=1}^{n} 1_{F_i}\right) = \sum_{i=1}^{n} C(1_{F_i});$$

(iii) $C$ is monotone with respect to stop loss order and widely translation invariant.
Characterization of the Choquet integral as a continuous certainty equivalent

Lemma

Let $\mathcal{C}$ be a positively homogeneous real functional on $L^\infty_+(\Omega, \mathcal{F}, \mathcal{P})$ and assume that $\mathcal{C}$ satisfies the normalization condition. Then the following conditions are equivalent:

(i) There exists a concave probability distortion $g$ such that $\mathcal{C}(X) = \int X dg \circ \mathcal{P}$ for all $X \in L^\infty_+(\Omega, \mathcal{F}, \mathcal{P})$;

(ii) $\mathcal{C}$ is uniformly continuous in the norm topology on $L^\infty_+(\Omega, \mathcal{F})$, monotone with respect to stop loss order and for every finite chain $F_1 \subseteq F_2 \subseteq ... \subseteq F_n$ in $\mathcal{F}$,

$$\mathcal{C}(\sum_{i=1}^n 1_{F_i}) = \sum_{i=1}^n \mathcal{C}(1_{F_i});$$

(iii) $\mathcal{C}$ is monotone with respect to stop loss order and widely translation invariant.
Lemma

Let $C$ be a positively homogeneous real functional on $L^\infty_+(\Omega, \mathcal{F}, \mathcal{P})$ and assume that $C$ satisfies the normalization condition. Then the following conditions are equivalent:

(i) There exists a concave probability distortion $g$ such that $C(X) = \int Xdg \circ \mathcal{P}$ for all $X \in L^\infty_+(\Omega, \mathcal{F}, \mathcal{P})$;

(ii) $C$ is uniformly continuous in the norm topology on $L^\infty_+(\Omega, \mathcal{F})$, monotone with respect to stop loss order and for every finite chain $F_1 \subseteq F_2 \subseteq \ldots \subseteq F_n$ in $\mathcal{F}$,

$$C\left(\sum_{i=1}^{n} \mathbf{1}_{F_i}\right) = \sum_{i=1}^{n} C(\mathbf{1}_{F_i});$$

(iii) $C$ is monotone with respect to stop loss order and widely translation invariant.
Characterization of the Choquet integral as a continuous certainty equivalent

Lemma

Let $\mathcal{C}$ be a positively homogeneous real functional on $L_+^\infty(\Omega, \mathcal{F}, P)$ and assume that $\mathcal{C}$ satisfies the normalization condition. Then the following conditions are equivalent:

(i) There exists a concave probability distortion $g$ such that $\mathcal{C}(X) = \int X d\mathcal{P}$ for all $X \in L_+^\infty(\Omega, \mathcal{F}, P)$;
(ii) $\mathcal{C}$ is uniformly continuous in the norm topology on $L_+^\infty(\Omega, \mathcal{F})$, monotone with respect to stop loss order and for every finite chain $F_1 \subseteq F_2 \subseteq \ldots \subseteq F_n$ in $\mathcal{F}$,

$$\mathcal{C}\left(\sum_{i=1}^{n} 1_{F_i}\right) = \sum_{i=1}^{n} \mathcal{C}(1_{F_i});$$

(iii) $\mathcal{C}$ is monotone with respect to stop loss order and widely translation invariant.
Lemma

Let $C$ be a positively homogeneous real functional on $L^\infty_+(\Omega, \mathcal{F}, P)$ and assume that $C$ satisfies the normalization condition. Then the following conditions are equivalent:

(i) There exists a concave probability distortion $g$ such that $C(X) = \int Xdg \circ P$ for all $X \in L^\infty_+(\Omega, \mathcal{F}, P)$;

(ii) $C$ is uniformly continuous in the norm topology on $L^\infty_+(\Omega, \mathcal{F})$, monotone with respect to stop loss order and for every finite chain $F_1 \subseteq F_2 \subseteq \ldots \subseteq F_n$ in $\mathcal{F}$,

$$C\left(\sum_{i=1}^n 1_{F_i}\right) = \sum_{i=1}^n C(1_{F_i});$$

(iii) $C$ is monotone with respect to stop loss order and widely translation invariant.

Lemma

Let $C$ be the Choquet integral on $L_+^\infty(\Omega, \mathcal{F}, \mathcal{P})$ with respect to a probability distortion $g$. If $C$ is subadditive, then $g$ is concave.

A total preorder $\preceq$ on $L_+^\infty(\Omega, \mathcal{F}, \mathcal{P})$ is said to be continuous in the norm topology on $L_+^\infty(\Omega, \mathcal{F})$ if the upper contour set $U_{\preceq}(X) = \{ Y \in L_+^\infty(\Omega, \mathcal{F}, \mathcal{P}) : X \preceq Y \}$ and the lower contour set $L_{\preceq}(X) = \{ Y \in L_+^\infty(\Omega, \mathcal{F}, \mathcal{P}) : Y \preceq X \}$ are closed for of every element $X \in L_+^\infty(\Omega, \mathcal{F}, \mathcal{P})$ in the norm topology on $L_+^\infty(\Omega, \mathcal{F})$. 

shortname

Choquet certainty equivalent

**Lemma**

Let $C$ be the Choquet integral on $L_{+}^{\infty}(\Omega, \mathcal{F}, \mathcal{P})$ with respect to a probability distortion $g$. If $C$ is subadditive, then $g$ is concave.

A total preorder $\preceq$ on $L_{+}^{\infty}(\Omega, \mathcal{F}, \mathcal{P})$ is said to be continuous in the norm topology on $L_{+}^{\infty}(\Omega, \mathcal{F})$ if the upper contour set $U_{\preceq}(X) = \{Y \in L_{+}^{\infty}(\Omega, \mathcal{F}, \mathcal{P}) : X \preceq Y\}$ and the lower contour set $L_{\preceq}(X) = \{Y \in L_{+}^{\infty}(\Omega, \mathcal{F}, \mathcal{P}) : Y \preceq X\}$ are closed for of every element $X \in L_{+}^{\infty}(\Omega, \mathcal{F}, \mathcal{P})$ in the norm topology on $L_{+}^{\infty}(\Omega, \mathcal{F})$. 

**Lemma**

Let $C$ be the Choquet integral on $L_+^\infty(\Omega, \mathcal{F}, \mathcal{P})$ with respect to a probability distortion $g$. If $C$ is subadditive, then $g$ is concave.

A total preorder $\preceq$ on $L_+^\infty(\Omega, \mathcal{F}, \mathcal{P})$ is said to be *continuous in the norm topology on $L_+^\infty(\Omega, \mathcal{F})$* if the upper contour set $U_{\preceq}(X) = \{Y \in L_+^\infty(\Omega, \mathcal{F}, \mathcal{P}) : X \preceq Y\}$ and the lower contour set $L_{\preceq}(X) = \{Y \in L_+^\infty(\Omega, \mathcal{F}, \mathcal{P}) : Y \preceq X\}$ are closed for of every element $X \in L_+^\infty(\Omega, \mathcal{F}, \mathcal{P})$ in the norm topology on $L_+^\infty(\Omega, \mathcal{F})$. 
Theorem

Let $\succeq$ be a total preorder on $L_+^\infty(\Omega, \mathcal{F}, \mathcal{P})$. Then the following conditions are equivalent:

(i) There exists a concave probability distortion $g$ such that $\mathcal{C}(X) = \int Xdg \circ \mathcal{P}$ for all $X \in L_+^\infty(\Omega, \mathcal{F}, \mathcal{P})$ is a certainty equivalence functional for $\succeq$;

(ii) The following conditions are verified:

(a) $\succeq$ is homothetic;
(b) $\succeq$ is continuous in the norm topology on $L_+^\infty(\Omega, \mathcal{F})$;
(c) $\succeq$ is monotone with respect to stop loss order;
(d) $\succeq$ is comonotone subadditive;
(e) the following condition holds for every finite chain $F_1 \subseteq F_2 \subseteq \ldots \subseteq F_n$ in $\mathcal{F}$ and for every $\lambda_1, \ldots, \lambda_n \in \mathbb{R}_+$:

\[
\text{if } 1_{F_i} \sim \lambda_i \text{ for every } i \in \{1, \ldots, n\} \text{ then } \sum_{i=1}^{n} 1_{F_i} \sim \sum_{i=1}^{n} \lambda_i.
\]
Theorem

Let \( \preceq \) be a total preorder on \( L_\infty^+(\Omega, \mathcal{F}, \mathcal{P}) \). Then the following conditions are equivalent:

(i) There exists a concave probability distortion \( g \) such that \( C(X) = \int X dg \circ \mathcal{P} \) for all \( X \in L_\infty^+(\Omega, \mathcal{F}, \mathcal{P}) \) is a certainty equivalence functional for \( \preceq \);

(ii) The following conditions are verified:

(a) \( \preceq \) is homothetic;
(b) \( \preceq \) is continuous in the norm topology on \( L_\infty^+(\Omega, \mathcal{F}) \);
(c) \( \preceq \) is monotone with respect to stop loss order;
(d) \( \preceq \) is comonotone subadditive;
(e) the following condition holds for every finite chain \( F_1 \subseteq F_2 \subseteq \ldots \subseteq F_n \) in \( \mathcal{F} \) and for every \( \lambda_1, \ldots, \lambda_n \in \mathbb{R}_+ \):

\[
\text{if } \mathbbm{1}_{F_i} \sim \lambda_i \text{ for every } i \in \{1, \ldots, n\} \text{ then } \sum_{i=1}^{n} \mathbbm{1}_{F_i} \sim \sum_{i=1}^{n} \lambda_i.
\]
Theorem

Let \( \preceq \) be a total preorder on \( L_+^\infty(\Omega, \mathcal{F}, \mathcal{P}) \). Then the following conditions are equivalent:

(i) There exists a concave probability distortion \( g \) such that \( C(X) = \int Xdg \circ \mathcal{P} \) for all \( X \in L_+^\infty(\Omega, \mathcal{F}, \mathcal{P}) \) is a certainty equivalence functional for \( \preceq \);

(ii) The following conditions are verified:

(a) \( \preceq \) is homothetic;

(b) \( \preceq \) is continuous in the norm topology on \( L_+^\infty(\Omega, \mathcal{F}) \);

(c) \( \preceq \) is monotone with respect to stop loss order;

(d) \( \preceq \) is comonotone subadditive;

(e) the following condition holds for every finite chain \( F_1 \subseteq F_2 \subseteq \ldots \subseteq F_n \) in \( \mathcal{F} \) and for every \( \lambda_1, \ldots, \lambda_n \in \mathbb{R}_+ \):

\[
\text{if } 1_{F_i} \sim \lambda_i \text{ for every } i \in \{1, \ldots, n\} \text{ then } \sum_{i=1}^{n} 1_{F_i} \sim \sum_{i=1}^{n} \lambda_i. \]
**Theorem**

Let \( \preceq \) be a total preorder on \( L_+^\infty(\Omega, \mathcal{F}, \mathcal{P}) \). Then the following conditions are equivalent:

(i) There exists a concave probability distortion \( g \) such that \( C(X) = \int Xdg \circ \mathcal{P} \) for all \( X \in L_+^\infty(\Omega, \mathcal{F}, \mathcal{P}) \) is a certainty equivalence functional for \( \preceq \);

(ii) The following conditions are verified:

(a) \( \preceq \) is homothetic;

(b) \( \preceq \) is continuous in the norm topology on \( L_+^\infty(\Omega, \mathcal{F}) \);

(c) \( \preceq \) is monotone with respect to stop loss order;

(d) \( \preceq \) is comonotone subadditive;

(e) the following condition holds for every finite chain \( F_1 \subseteq F_2 \subseteq \ldots \subseteq F_n \) in \( \mathcal{F} \) and for every \( \lambda_1, \ldots, \lambda_n \in \mathbb{R}_+ \):

\[
\text{if } 1_{F_i} \sim \lambda_i \text{ for every } i \in \{1, \ldots, n\} \text{ then } \sum_{i=1}^{n} 1_{F_i} \sim \sum_{i=1}^{n} \lambda_i.
\]
Theorem

Let \( \succcurlyeq \) be a total preorder on \( L^\infty_+(\Omega, \mathcal{F}, \mathcal{P}) \). Then the following conditions are equivalent:

(i) There exists a concave probability distortion \( g \) such that \( C(X) = \int X dg \circ \mathcal{P} \) for all \( X \in L^\infty_+(\Omega, \mathcal{F}, \mathcal{P}) \) is a certainty equivalence functional for \( \succcurlyeq \);

(ii) The following conditions are verified:

(a) \( \succcurlyeq \) is homothetic;

(b) \( \succcurlyeq \) is continuous in the norm topology on \( L^\infty_+(\Omega, \mathcal{F}) \);

(c) \( \succcurlyeq \) is monotone with respect to stop loss order;

(d) \( \succcurlyeq \) is comonotone subadditive;

(e) the following condition holds for every finite chain \( F_1 \subseteq F_2 \subseteq ... \subseteq F_n \) in \( \mathcal{F} \) and for every \( \lambda_1, ..., \lambda_n \in \mathbb{R}_+ \):

\[
\text{if } 1_{F_i} \sim \lambda_i \text{ for every } i \in \{1, ..., n\} \text{ then } \sum_{i=1}^{n} 1_{F_i} \sim \sum_{i=1}^{n} \lambda_i.\
\]

shortname Choquet certainty equivalent
Let $\preceq$ be a total preorder on $L_+^\infty(\Omega, \mathcal{F}, \mathcal{P})$. Then the following conditions are equivalent:

(i) There exists a concave probability distortion $g$ such that $C(X) = \int X dg \circ \mathcal{P}$ for all $X \in L_+^\infty(\Omega, \mathcal{F}, \mathcal{P})$ is a certainty equivalence functional for $\preceq$;

(ii) The following conditions are verified:

(a) $\preceq$ is homothetic;
(b) $\preceq$ is continuous in the norm topology on $L_+^\infty(\Omega, \mathcal{F})$;
(c) $\preceq$ is monotone with respect to stop loss order;
(d) $\preceq$ is comonotone subadditive;
(e) the following condition holds for every finite chain $F_1 \subseteq F_2 \subseteq \ldots \subseteq F_n$ in $\mathcal{F}$ and for every $\lambda_1, \ldots, \lambda_n \in \mathbb{R}_+$:

$$\text{if } 1_{F_i} \sim \lambda_i \text{ for every } i \in \{1, \ldots, n\} \text{ then } \sum_{i=1}^{n} 1_{F_i} \sim \sum_{i=1}^{n} \lambda_i.$$
Let \( \preceq \) be a total preorder on \( L_+^\infty(\Omega, \mathcal{F}, \mathcal{P}) \). Then the following conditions are equivalent:

(i) There exists a concave probability distortion \( g \) such that \( \mathcal{C}(X) = \int X \circ g \circ \mathcal{P} \) for all \( X \in L_+^\infty(\Omega, \mathcal{F}, \mathcal{P}) \) is a certainty equivalence functional for \( \preceq \);

(ii) The following conditions are verified:
   (a) \( \preceq \) is homothetic;
   (b) \( \preceq \) is continuous in the norm topology on \( L_+^\infty(\Omega, \mathcal{F}) \);
   (c) \( \preceq \) is monotone with respect to stop loss order;
   (d) \( \preceq \) is comonotone subadditive;
   (e) the following condition holds for every finite chain \( F_1 \subseteq F_2 \subseteq \ldots \subseteq F_n \) in \( \mathcal{F} \) and for every \( \lambda_1, \ldots, \lambda_n \in \mathbb{R}_+ \):

\[
\text{if } 1_{F_i} \sim \lambda_i \text{ for every } i \in \{1, \ldots, n\} \text{ then } \sum_{i=1}^n 1_{F_i} \sim \sum_{i=1}^n \lambda_i.
\]

shortname
Choquet certainty equivalent
Main result

**Theorem**

Let $\succeq$ be a total preorder on $L^\infty_+ (\Omega, \mathcal{F}, \mathcal{P})$. Then the following conditions are equivalent:

(i) There exists a concave probability distortion $g$ such that 
$$C(X) = \int X dg \circ \mathcal{P} \text{ for all } X \in L^\infty_+ (\Omega, \mathcal{F}, \mathcal{P})$$
is a certainty equivalence functional for $\succeq$;

(ii) The following conditions are verified:

(a) $\succeq$ is homothetic;
(b) $\succeq$ is continuous in the norm topology on $L^\infty_+ (\Omega, \mathcal{F})$;
(c) $\succeq$ is monotone with respect to stop loss order;
(d) $\succeq$ is comonotone subadditive;
(e) the following condition holds for every finite chain $F_1 \subseteq F_2 \subseteq ... \subseteq F_n$ in $\mathcal{F}$ and for every $\lambda_1 , ..., \lambda_n \in \mathbb{R}_+$:

$$\text{if } 1_{F_i} \sim \lambda_i \text{ for every } i \in \{1, ..., n\} \text{ then } \sum_{i=1}^{n} 1_{F_i} \sim \sum_{i=1}^{n} \lambda_i.$$
Theorem

Let \( \sim \) be a total preorder on \( L_+^\infty(\Omega, \mathcal{F}, \mathcal{P}) \). Then the following conditions are equivalent:

(i) There exists a concave probability distortion \( g \) such that
\[
C(X) = \int X \, dg \circ \mathcal{P}
\]
for all \( X \in L_+^\infty(\Omega, \mathcal{F}, \mathcal{P}) \) is a certainty equivalence functional for \( \sim \);

(ii) The following conditions are verified:
(a) \( \sim \) is homothetic;
(b) \( \sim \) is continuous in the norm topology on \( L_+^\infty(\Omega, \mathcal{F}) \);
(c) \( \sim \) is monotone with respect to stop loss order;
(d) \( \sim \) is comonotone subadditive;
(e) the following condition holds for every finite chain \( F_1 \subseteq F_2 \subseteq \ldots \subseteq F_n \) in \( \mathcal{F} \) and for every \( \lambda_1, \ldots, \lambda_n \in \mathbb{R}_+ \):

\[
\text{if } \mathbb{1}_{F_i} \sim \lambda_i \text{ for every } i \in \{1, \ldots, n\} \text{ then } \sum_{i=1}^{n} \mathbb{1}_{F_i} \sim \sum_{i=1}^{n} \lambda_i.
\]
Theorem (Continuation)

(iii) $\preceq$ is widely translation invariant and in addition satisfies the above condition (a), (b), (c) and (d).

Proof. (i) $\Rightarrow$ (ii). The proof of this implication is an easy consequence of the implication (i) $\Rightarrow$ (ii) in the lemma above.

(ii) $\Rightarrow$ (iii). Since the total preorder $\preceq$ on $L^\infty(\Omega, \mathcal{F}, \mathcal{P})$ is homothetic and continuous, there exists a unique nonnegative, positively homogeneous and norm continuous certainty functional $C$ for $\preceq$. It is immediate to check that condition (e) implies that the certainty equivalent $C$ satisfies the displayed condition (ii) of the lemma above. Since the implication (ii) $\Rightarrow$ (iii) of the lemma guarantees that $C$ is widely translation invariant, this immediately implies that also $\preceq$ is widely translation invariant.
Theorem (Continuation)

(iii) $\succeq$ is widely translation invariant and in addition satisfies the above condition (a), (b), (c) and (d).

Proof. (i) $\Rightarrow$ (ii). The proof of this implication is an easy consequence of the implication (i) $\Rightarrow$ (ii) in the lemma above.

(ii) $\Rightarrow$ (iii). Since the total preorder $\succeq$ on $L^\infty(\Omega, \mathcal{F}, \mathcal{P})$ is homothetic and continuous, there exists a unique nonnegative, positively homogeneous and norm continuous certainty functional $C$ for $\succeq$. It is immediate to check that condition (e) implies that the certainty equivalent $C$ satisfies the displayed condition (ii) of the lemma above. Since the implication (ii) $\Rightarrow$ (iii) of the lemma guarantees that $C$ is widely translation invariant, this immediately implies that also $\succeq$ is widely translation invariant.
We have already noticed that there exists a nonnegative, positively homogeneous and norm continuous certainty functional $C$ for $\succeq$. By condition (c), we immediately realize that $C$ is monotone with respect to stop-loss order. Let us now show that comonotone subadditivity of $\succeq$ implies that $C$ is comonotone subadditive. By contraposition, assume that there exist comonotone $X, Y \in L_+^\infty(\Omega, \mathcal{F})$ such that $C(X) + C(Y) < C(X + Y)$. Then there exist two positive real numbers $\lambda, \mu$ such that $C(X) < \lambda$, $C(Y) < \mu$, $\lambda + \mu < C(X + Y)$. Therefore we have that $X \prec \lambda$, $Y \prec \mu$, but $\lambda + \mu \prec X + Y$, contradicting comonotone subadditivity of $\succeq$. By a lemma above, $C$ is uniformly continuous. The present implication follows from the implication (iii) $\Rightarrow$ (i) in the lemma.
In the following corollary we are concerned with the case when the probability space \((\Omega, \mathcal{F}, \mathcal{P})\) is atomless.

**Corollary**

Let \(\preceq\) be a total preorder on \(L^\infty_+(\Omega, \mathcal{F}, \mathcal{P})\) and assume that the probability space \((\Omega, \mathcal{F}, \mathcal{P})\) is atomless. Then the following conditions are equivalent:

(i) There exists a concave probability distortion \(g\) such that \(\mathcal{C}(X) = \int Xdg \circ \mathcal{P}\) for all \(X \in L^\infty_+(\Omega, \mathcal{F}, \mathcal{P})\) is a certainty equivalence functional for \(\preceq\);

(ii) The following conditions are verified:

(a) \(\preceq\) is homothetic;
(b) \(\preceq\) is continuous in the norm topology on \(L^\infty_+(\Omega, \mathcal{F})\);
(c) \(\preceq\) is monotone with respect to first order stochastic dominance;
(d) \(\preceq\) is translation invariant;
In the following corollary we are concerned with the case when the probability space \((\Omega, \mathcal{F}, P)\) is atomless.

**Corollary**

Let \(\preceq\) be a total preorder on \(L^\infty_+(\Omega, \mathcal{F}, \mathcal{P})\) and assume that the probability space \((\Omega, \mathcal{F}, \mathcal{P})\) is atomless. Then the following conditions are equivalent:

(i) There exists a concave probability distortion \(g\) such that
\[
C(X) = \int X d g \circ P
\]
for all \(X \in L^\infty_+(\Omega, \mathcal{F}, \mathcal{P})\) is a certainty equivalence functional for \(\preceq\);

(ii) The following conditions are verified:
   (a) \(\preceq\) is homothetic;
   (b) \(\preceq\) is continuous in the norm topology on \(L^\infty_+(\Omega, \mathcal{F})\);
   (c) \(\preceq\) is monotone with respect to first order stochastic dominance;
   (d) \(\preceq\) is translation invariant;
In the following corollary we are concerned with the case when the probability space \((\Omega, \mathcal{F}, \mathcal{P})\) is atomless.

**Corollary**

Let \(\prec\) be a total preorder on \(L^\infty_+ (\Omega, \mathcal{F}, \mathcal{P})\) and assume that the probability space \((\Omega, \mathcal{F}, \mathcal{P})\) is atomless. Then the following conditions are equivalent:

(i) There exists a concave probability distortion \(g\) such that \(\mathcal{C}(X) = \int X d g \circ \mathcal{P}\) for all \(X \in L^\infty_+ (\Omega, \mathcal{F}, \mathcal{P})\) is a certainty equivalence functional for \(\prec\);

(ii) The following conditions are verified:
   (a) \(\prec\) is homothetic;
   (b) \(\prec\) is continuous in the norm topology on \(L^\infty_+ (\Omega, \mathcal{F})\);
   (c) \(\prec\) is monotone with respect to first order stochastic dominance;
   (d) \(\prec\) is translation invariant;
In the following corollary we are concerned with the case when the probability space \((\Omega, \mathcal{F}, \mathcal{P})\) is atomless.

**Corollary**

Let \(\preceq\) be a total preorder on \(L_+^{\infty}(\Omega, \mathcal{F}, \mathcal{P})\) and assume that the probability space \((\Omega, \mathcal{F}, \mathcal{P})\) is atomless. Then the following conditions are equivalent:

(i) There exists a concave probability distortion \(g\) such that \(\mathcal{C}(X) = \int Xdg \circ \mathcal{P}\) for all \(X \in L_+^{\infty}(\Omega, \mathcal{F}, \mathcal{P})\) is a certainty equivalence functional for \(\preceq\);

(ii) The following conditions are verified:
   (a) \(\preceq\) is homothetic;
   (b) \(\preceq\) is continuous in the norm topology on \(L_+^{\infty}(\Omega, \mathcal{F})\);
   (c) \(\preceq\) is monotone with respect to first order stochastic dominance;
   (d) \(\preceq\) is translation invariant;
In the following corollary we are concerned with the case when the probability space \((\Omega, \mathcal{F}, \mathcal{P})\) is atomless.

**Corollary**

Let \(\preceq\) be a total preorder on \(L^\infty_+ (\Omega, \mathcal{F}, \mathcal{P})\) and assume that the probability space \((\Omega, \mathcal{F}, \mathcal{P})\) is atomless. Then the following conditions are equivalent:

(i) There exists a concave probability distortion \(g\) such that \(C(X) = \int X \, dg \circ \mathcal{P}\) for all \(X \in L^\infty_+ (\Omega, \mathcal{F}, \mathcal{P})\) is a certainty equivalence functional for \(\preceq\);

(ii) The following conditions are verified:

(a) \(\preceq\) is homothetic;

(b) \(\preceq\) is continuous in the norm topology on \(L^\infty_+ (\Omega, \mathcal{F})\);

(c) \(\preceq\) is monotone with respect to first order stochastic dominance;

(d) \(\preceq\) is translation invariant;
In the following corollary we are concerned with the case when the probability space $(\Omega, \mathcal{F}, \mathcal{P})$ is atomless.

**Corollary**

Let $\preceq$ be a total preorder on $L_+^\infty(\Omega, \mathcal{F}, \mathcal{P})$ and assume that the probability space $(\Omega, \mathcal{F}, \mathcal{P})$ is atomless. Then the following conditions are equivalent:

(i) There exists a concave probability distortion $g$ such that $\mathcal{C}(X) = \int Xdg \circ \mathcal{P}$ for all $X \in L_+^\infty(\Omega, \mathcal{F}, \mathcal{P})$ is a certainty equivalence functional for $\preceq$;

(ii) The following conditions are verified:
   (a) $\preceq$ is homothetic;
   (b) $\preceq$ is continuous in the norm topology on $L_+^\infty(\Omega, \mathcal{F})$;
   (c) $\preceq$ is monotone with respect to first order stochastic dominance;
   (d) $\preceq$ is translation invariant;
In the following corollary we are concerned with the case when the probability space \((\Omega, \mathcal{F}, \mathcal{P})\) is atomless.

**Corollary**

Let \(\preceq\) be a total preorder on \(L^\infty_+(\Omega, \mathcal{F}, \mathcal{P})\) and assume that the probability space \((\Omega, \mathcal{F}, \mathcal{P})\) is atomless. Then the following conditions are equivalent:

(i) There exists a concave probability distortion \(g\) such that \(\mathcal{C}(X) = \int Xdg \circ \mathcal{P}\) for all \(X \in L^\infty_+(\Omega, \mathcal{F}, \mathcal{P})\) is a certainty equivalence functional for \(\preceq\);

(ii) The following conditions are verified:
   (a) \(\preceq\) is homothetic;
   (b) \(\preceq\) is continuous in the norm topology on \(L^\infty_+(\Omega, \mathcal{F})\);
   (c) \(\preceq\) is monotone with respect to first order stochastic dominance;
   (d) \(\preceq\) is translation invariant;
In the following corollary we are concerned with the case when the probability space \((\Omega, \mathcal{F}, \mathcal{P})\) is atomless.

**Corollary**

Let \(\preceq\) be a total preorder on \(L_+^\infty(\Omega, \mathcal{F}, \mathcal{P})\) and assume that the probability space \((\Omega, \mathcal{F}, \mathcal{P})\) is atomless. Then the following conditions are equivalent:

(i) There exists a concave probability distortion \(g\) such that \(\mathcal{C}(X) = \int X \, dg \circ \mathcal{P}\) for all \(X \in L_+^\infty(\Omega, \mathcal{F}, \mathcal{P})\) is a certainty equivalence functional for \(\preceq\);

(ii) The following conditions are verified:

(a) \(\preceq\) is homothetic;

(b) \(\preceq\) is continuous in the norm topology on \(L_+^\infty(\Omega, \mathcal{F})\);

(c) \(\preceq\) is monotone with respect to first order stochastic dominance;

(d) \(\preceq\) is translation invariant;
(e) $\preceq$ is subadditive;

(f) the following condition holds for every finite chain $F_1 \subseteq F_2 \subseteq ... \subseteq F_n$ in $\mathcal{F}$ and for every $\lambda_1, ..., \lambda_n \in \mathbb{R}_+$:

$$\text{if } 1_{F_i} \sim \lambda_i \text{ for every } i \in \{1, ..., n\} \text{ then } \sum_{i=1}^{n} 1_{F_i} \sim \sum_{i=1}^{n} \lambda_i.$$  

**Proof.** (i) $\Rightarrow$ (ii). Immediate. (ii) $\Rightarrow$ (i). Let $\preceq$ be a total preorder satisfying the above conditions (a) through (f). The nonnegative, positively homogeneous and norm continuous certainty functional $C$ for $\preceq$ is also subadditive and translation invariant. Since the probability space $(\Omega, \mathcal{F}, P)$ is atomless, we have that $C$ is monotone with respect to stop loss order by Song and Yan, *Insurance: Mathematics and Economics*, 2009. Further, $C$ is continuous in the norm topology on $L^\infty_+(\Omega, \mathcal{F})$ and therefore the thesis follows from the previous theorem.
Corollary (Continuation)

(e) $\preceq$ is subadditive;

(f) the following condition holds for every finite chain $F_1 \subseteq F_2 \subseteq \ldots \subseteq F_n$ in $\mathcal{F}$ and for every $\lambda_1, \ldots, \lambda_n \in \mathbb{R}_+$:

$$\text{if } 1_{F_i} \sim \lambda_i \text{ for every } i \in \{1, \ldots, n\} \text{ then } \sum_{i=1}^{n} 1_{F_i} \sim \sum_{i=1}^{n} \lambda_i.$$ 

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