A robust tree method for pricing American options with CIR stochastic interest rate

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1. The Wei algorithm and the Hilliard-Schwartz-Tucker algorithm

2. The robust tree algorithm: description and convergence

3. Numerical results
The scenarios that have been prevailing on the financial markets in the last decade suggest that equity models need to take into account for stochastic interest rate.

We consider the problem of pricing American options using lattice techniques when the underlying process follows a log-normal type diffusion process with stochastic interest rate described by the Cox, Ingersoll and Ross (CIR) model.

We provide both theoretical convergence of the algorithm and numerical results that show the robustness and stability of the method.
The continuous model

Under the risk-neutral probability measure, we consider the following dynamic for the equity asset value in the interval \([0, T]\):

\[
\begin{align*}
    dS(t) &= r(t)S(t)dt + \sigma_S S(t)dZ_S(t), \quad S(0) = S_0 > 0 \\
    dr(t) &= \kappa(\theta - r(t))dt + \sigma_r \sqrt{r(t)}dZ_r(t), \quad r(0) = r_0 > 0
\end{align*}
\]  \hspace{1cm} (1)

where

- \(\sigma_S\) is the stock price volatility;
- \(\sigma_r\) is the interest rate volatility;
- \(\kappa\) is the reversion speed;
- \(\theta\) is the long term reversion target;
- \(Z_S\) and \(Z_r\) are two correlated Brownian motions with correlation \(\rho\).

**Feller condition**

If \(r_0 > 0\) and \(2\kappa\theta \geq \sigma_r^2\), then a.s. the process \(r\) never hits 0.
Previous literature

- **Nelson and Ramaswamy** (1990) introduce a technique to approximate all 1-dimensional diffusion processes with a computationally simple binomial process by transforming the original process into a diffusion with unit variance.

- **Wei** (1996) and **Costabile, Gaudenzi, Massabò and Zanette** (2009) extend Nelson and Ramaswamy procedure and propose a bivariate recombining lattice to model the pair $(S(t), r(t))_t$ for the pricing of American put options.

- **Hilliard, Schwartz and Tucker** (1996), propose a recombining bivariate lattice for $(S(t), r(t))_t$ that differs from Wei procedure for the transformations used in the algorithm.
The Wei algorithm

- **First step:** transform $S$ and $r$ into diffusions with unit variance by
  \[ X = (\log S)/\sigma_S \quad \text{and} \quad R = 2\sqrt{r}/\sigma_r. \]  
  Hence, by Itô’s lemma we have
  \[
  dX(t) = \mu_X(R(t))dt + dZ_S(t), \quad \mu_X(R) = \frac{\sigma^2_r R^2/4 - \sigma^2_s/2}{\sigma_s},
  \]
  \[
  dR(t) = \mu_R(R(t))dt + dZ_r(t), \quad \mu_R(R) = \frac{\kappa(4\theta - R^2\sigma_r^2) - \sigma_r^2}{2R\sigma_r^2}.
  \]

- **Second step:** the process $Y = \frac{X - \rho R}{\sqrt{1 - \rho^2}}$ is orthogonal to $R$
  \[
  dY(t) = \mu_Y(R(t))dt + dZ_Y(t), \quad \mu_Y(R) = \frac{\mu_X(R) - \rho \mu_R(R)}{\sqrt{1 - \rho^2}}.
  \]
The Wei algorithm

Third step:

- Wei model $R$ and $Y$ separately as two independent binomial processes following Nelson-Ramaswamy procedure and then merge the two structures into a 3-dimensional tree.
- The interval $[0, T]$ is divided into $N$ intervals of length $h = \frac{T}{N}$.

\[
R_{i,k} = R_0 + (2k - i)\sqrt{h} \rightarrow \begin{cases} 
R_{i+1,k_u(i,k)}, & p_{i,k} \\
R_{i+1,k_d(i,k)}, & 1 - p_{i,k}
\end{cases} \quad (3)
\]

\[
Y_{i,j} = R_0 + (2j - i)\sqrt{h} \rightarrow \begin{cases} 
Y_{i+1,j_u(i,j,k)}, & \hat{p}_{i,j,k} \\
Y_{i+1,j_d(i,j,k)}, & 1 - \hat{p}_{i,j,k}
\end{cases} \quad (4)
\]

for every $k, j = 0, \ldots, i$. 

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- The transition probabilities $p_{i,k}$ and $\hat{p}_{i,j,k}$ are defined so that the local consistency conditions are satisfied at each time step.
- To this end, one has to take into account that in some regions of the tree it may happen that "**multiple jumps**" are needed.
- Starting from $R_{i,k}$ at time step $i$, the process may jump at time step $i+1$ to the value $R_{i+1,k_u}$ or $R_{i+1,k_d}$, with $k_d$ and $k_u$ defined as

$$k_d = \max\{k^*: R_{i,k} + \mu_R(R_{i,k})h \geq R_{i+1,k^*}\} \quad \text{and} \quad k_u = k_d + 1,$$

Similarly we define $j_u$ and $j_d$ for the discrete approximation of $Y$. 

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The Wei algorithm

Figure: Standard jumps and multiple jumps for the discrete approximation of the process $R$. 

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The expressions of the transition probabilities are

\[ p_{i,k} = \begin{cases} 0 & \text{if} \ p_{i,k}^0 = 0 \\ \frac{\mu R_i(R_{i,k}) h + R_i,k - R_{i+1,k_d}}{R_{i+1,k_u} - R_{i+1,k_d}} & \text{otherwise} \end{cases} \land 1 \]  

(5)

and

\[ \hat{p}_{i,j,k} = \begin{cases} 0 & \text{if} \ \hat{p}_{i,j,k}^0 = 0 \\ \frac{\mu Y_i(R_{i,k}) h + Y_{i,j} - Y_{i+1,j_d}}{Y_{i+1,j_u} - Y_{i+1,j_d}} & \text{otherwise} \end{cases} \land 1. \]  

(6)

At each time step \( i (i = 0, ..., N) \), the bivariate tree has \((i + 1)^2\) nodes called \((i, j, k)\) corresponding to the values \(R_i,k\) and \(Y_{i,j}\). Starting from \((i, j, k)\) the process may reach one of the four nodes

\[ (i + 1, j_u, k_u), \quad \text{with probability} \quad q_{i,j_u,k_u} = \hat{p}_{i,j,k} p_{i,k}, \]

\[ (i + 1, j_u, k_d), \quad \text{with probability} \quad q_{i,j_u,k_d} = \hat{p}_{i,j,k} (1 - p_{i,k}), \]

\[ (i + 1, j_d, k_u), \quad \text{with probability} \quad q_{i,j_d,k_u} = (1 - \hat{p}_{i,j,k}) p_{i,k}, \]

\[ (i + 1, j_d, k_d), \quad \text{with probability} \quad q_{i,j_d,k_d} = (1 - \hat{p}_{i,j,k})(1 - p_{i,k}). \]
The Wei algorithm

- **Fourth step:** at each node of the tree convert \( R \) and \( Y \) back to \( r \) and \( S \) respectively using

\[
S_{i,j,k} = e^{\sigma S} (\sqrt{1-\rho^2} Y_{i,j} + \rho R_{i,k})
\]

(7)

and

\[
r_{i,k} = \begin{cases} 
\frac{R_{i,k}^2 \sigma_r^2}{4}, & \text{if } R_{i,k} > 0 \\
0, & \text{otherwise}
\end{cases}
\]

(8)

- The price at time 0 of an American put option is obtained proceeding backwardly.
The price at time 0 of an American Put option with maturity $T$ and strike $K$ can be computed by the following backward dynamic programming equations

$$
\begin{align*}
\nu_{N,j,k} &= (K - S_{n,j,k})_+ \\
\nu_{i,j,k} &= \max((K - S_{i,j,k})_+, e^{-r_i,k\Delta T} \left[ \begin{array}{c}
\nu_{i+1,j_u,k_u} + \\
\nu_{i+1,j_d,k_d} + \nu_{i+1,j_d,k_u} + \nu_{i+1,j_d,k_d}
\end{array} \right]),
\end{align*}
$$

where $\nu_{i,j,k}, i = 0, \ldots, n$ and $j, k = 0, \ldots, i$, provide the American option price at every node $(i, j, k)$ of the tree structure.
Hilliard, Schwartz and Tucker propose a procedure similar to Wei that differs for the transformations employed:

1. The transformations that turn the original processes into constant variance processes are

\[ X = \frac{\log S}{\sigma_S}, \quad R = 2\sqrt{r}; \]  

(9)

2. The transformations that remove the correlation between \( X \) and \( R \) are

\[ X_1 = \sigma_r X + R, \quad X_2 = \sigma_r X - R. \]  

(10)
Structural differences with Wei method and HST method:

1. Transformations similar to Wei and HST are only used to set up the state-space of the discrete approximation of the pair \((S, r)\);

2. The probabilistic structure for the discrete approximation of both \(S\) and \(r\) as individual diffusions is defined directly on the original processes;

3. The transition probabilities of the bivariate lattice take into consideration the covariance structure.
The robust tree method: description of the algorithm

- **Discrete approximation of the process $r$:**
  we construct the state-space of the discrete approximation by transforming the computationally simple lattice for $R = \frac{2\sqrt{r}}{\sigma_r}$ by means of the map
  
  $$
  r_{i,k} = \begin{cases} 
  \frac{R_{i,k}^2 \sigma_r^2}{4}, & \text{if } R_{i,k} > 0 \\
  0, & \text{otherwise}
  \end{cases} \quad \forall i, k.
  $$

- **From the node $(i, k)$ the process may jump to**
  
  $$(i + 1, k_u(i, k)), \quad \text{with probability } p_{i,k},$$
  $$(i + 1, k_d(i, k)), \quad \text{with probability } 1 - p_{i,k},$$

  with $p_{i,k} = 0 \lor \frac{\mu_r(r_{i,k})h + r_{i,k} - r_{i+1,k_d(i,k)}}{r_{i+1,k_u(i,k)} - r_{i+1,k_d(i,k)}} \land 1,$

  and where $k_u(i, k), k_d(i, k)$ denote the “multiple jumps”.

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The robust tree method: description of the algorithm

- **Discrete approximation of the process $S$:**
  
  We first consider a computationally simple tree defined as
  
  $$U_{i,j} = U_0 + (2j - i)\sqrt{h}, \quad U_0 = \frac{1}{\sigma_S} \log S_0$$  
  (11)
  
  and then we build a tree for $S$ by means of
  
  $$S_{i,j} = e^{\sigma_S U_{i,j}}, \quad \forall i, j.$$  
  (12)

- **From the node $$(i, j)$$ the process may jump to**
  
  $$(i + 1, j_u(i, j, k)),$$ with probability $\hat{p}_{i,j,k},$
  
  $$(i + 1, j_d(i, j, k)),$$ with probability $1 - \hat{p}_{i,j,k},$
  
  with $\hat{p}_{i,j,k} = 0 \lor \frac{\mu_S(S_{i,j}, r_i, k)h + S_{i,j} - S_{i+1,j_d(i,j,k)}}{S_{i+1,j_u(i,j,k)} - S_{i+1,j_d(i,j,k)}} \land 1,$$
  
  and where $j_u(i, j, k), j_d(i, j, k)$ denote the “multiple jumps”.

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It is worth to say that the transformation does not seem to be natural in order to describe the evolution of the pair. But in fact we only need to set up the state-space of the Markov chain that we want to approximate the continuous time process \((S, r)\). What is important now is to define the transition probabilities in order to link the tree with the diffusion pair \((S, r)\).
The robust tree method: description of the algorithm

- Discrete approximation of the pair \((S, r)\):
  at time step \(i\), starting from a node \((i, j, k)\) corresponding to the values \((S_{i,j}, r_{i,k})\) and considering “multiple jumps”, the process may reach one of the four nodes

\[
\begin{align*}
(i + 1, j_u, k_u), & \quad \text{with probability} \quad q_{i,j_u,k_u}, \\
(i + 1, j_u, k_d), & \quad \text{with probability} \quad q_{i,j_u,k_d}, \\
(i + 1, j_d, k_u), & \quad \text{with probability} \quad q_{i,j_d,k_u}, \\
(i + 1, j_d, k_d), & \quad \text{with probability} \quad q_{i,j_d,k_d}.
\end{align*}
\]
The robust tree method: description of the algorithm

The transition probabilities used in the backward procedure are now computed from the matching conditions on the conditional mean and on the conditional covariance:

\[
\begin{align*}
q_{i,j_u,k_u} + q_{i,j_u,k_d} &= \hat{p}_{i,j,k} \\
q_{i,j_u,k_u} + q_{i,j_d,k_u} &= p_{i,k} \\
q_{i,j_u,k_u} + q_{i,j_d,k_u} + q_{i,j_u,k_d} + q_{i,j_d,k_d} &= 1 \\
m_{i,j_u,k_u}q_{i,j_u,k_u} + m_{i,j_u,k_d}q_{i,j_u,k_d} + m_{i,j_d,k_u}q_{i,j_d,k_u} + m_{i,j_d,k_d}q_{i,j_d,k_d} &= A_{i,j,k}
\end{align*}
\]

where

\[
A_{i,j,k} = \rho \sigma_r \sigma_S \sqrt{r_{i,k}} S_{i,j} h
\]

and

\[
\begin{align*}
m_{i,j_u,k_u} &= (S_{i+1,j_u} - S_{i,j})(r_{i+1,k_u} - r_{i,k}) \\
m_{i,j_u,k_d} &= (S_{i+1,j_u} - S_{i,j})(r_{i+1,k_d} - r_{i,k}) \\
m_{i,j_d,k_u} &= (S_{i+1,j_d} - S_{i,j})(r_{i+1,k_u} - r_{i,k}) \\
m_{i,j_d,k_d} &= (S_{i+1,j_d} - S_{i,j})(r_{i+1,k_d} - r_{i,k})
\end{align*}
\]
The robust tree method: convergence

- Using standard techniques it is possible to prove the theoretical convergence on the space of the continuous functions $C([0, T]; [0, +\infty) \times [0, +\infty))$ of the tree method to the pair $(S, r)$ solution to the SDE (1).
- The idea of the proof is standard, see e.g. Nelson and Ramaswamy or also classical books such as Billingsley, Ethier and Kurtz or Stroock and Varadhan.
The robust tree method: convergence

In order to state the main result we set \((S_i^h, r_i^h)_{i=0,\ldots,N}\) the Markov chain running on the bivariate lattice, that is defined as follows

- \(S_0^h = S_0\) and \(r_0^h = r_0\);
- at time \(ih\) the state-space for the pair \((S_i^h, r_i^h)\) is
  \[
  \{(S_{i,j}, r_{i,k}) : j, k = 0, \ldots, i\};
  \]
- the transition law in \(\mathbb{R}^2\) from time \(ih\) to time \((i + 1)h\) is
  \[
  \Pi_h(S_i, j, r_i, k; dx) = q_{i,ju,k} \delta\{(S_{i+1,ju}, r_{i+1,ku})\} + q_{i,ju,kd} \delta\{(S_{i+1,ju}, r_{i+1,kd})\} + q_{i,jd,ku} \delta\{(S_{i+1,jd}, r_{i+1,ku})\} + q_{i,jd,kd} \delta\{(S_{i+1,jd}, r_{i+1,kd})\},
  \]
where \(\delta\{a\}\) denotes the Dirac mass in \(a \in \mathbb{R}^2\).
We now set \((\bar{S}_t^h, \bar{r}_t^h)_{i\in[0,T]}\) as the continuous process defined as

\[
\bar{S}_t^h = S_i^h + \frac{t - ih}{h} (S_{i+1}^h - S_i^h) \quad \text{and} \quad \bar{r}_t^h = r_i^h + \frac{t - ih}{h} (r_{i+1}^h - r_i^h),
\]

for \(t \in [ih, (i + 1)h]\).

**Theorem**

The Markov process \((\bar{S}_t^h, \bar{r}_t^h)_{t\in[0,T]}\) converges in law on the space of the continuous functions \(C([0, T]; [0, +\infty) \times [0, +\infty))\) endowed with its Borel \(\sigma\)-algebra to the diffusion process \((S_t, r_t)_{t\in[0,T]}\).
Proof of the Theorem

The proof relies in checking that for \( r_* > 0 \) and \( S_* > 0 \) fixed, setting \( A_* = \{(i, j, k) : r_{i,k} \leq r_*, S_{i,j} \leq S_*\} \), then the following properties hold:

- convergence of the local drift:

\[
\lim_{h \to 0} \sup_{(i,j,k) \in A_*} \frac{1}{h} |M_{i,j,k}^S(1) - (\mu_S)_{i,j,k}h| = 0,
\]

\[
\lim_{h \to 0} \sup_{(i,j,k) \in A_*} \frac{1}{h} |M_{i,j,k}^r(1) - (\mu_r)_{i,j,k}h| = 0,
\]

where

\[
M_{i,j,k}^S(1) = \mathbb{E}(S_{i+1}^h - S_i^h | (S_i^h, r_i^h) = (S_{i,j}, r_{i,k}))
\]

\[
M_{i,j,k}^r(1) = \mathbb{E}(r_{i+1}^h - r_i^h | (S_i^h, r_i^h) = (S_{i,j}, r_{i,k}))
\]
Proof of the Theorem

- convergence of the local diffusion coefficient:

\[
\lim_{h \to 0} \sup_{(i,j,k) \in A_*} \frac{1}{h} |\mathcal{M}^{S}_{i,j,k}(2) - \sigma_S^2 S_{i,j}^2 h| = 0,
\]

\[
\lim_{h \to 0} \sup_{(i,j,k) \in A_*} \frac{1}{h} |\mathcal{M}^{r}_{i,j,k}(2) - \sigma_r^2 r_{i,k} h| = 0,
\]

\[
\lim_{h \to 0} \sup_{(i,j,k) \in A_*} \frac{1}{h} |\mathcal{M}^{S,r}_{i,j,k} - \rho \sigma r \sigma_S S_{i,j} \sqrt{r_{i,k}} h| = 0,
\]

where

\[
\mathcal{M}^{S}_{i,j,k}(2) = \mathbb{E}((S_{i+1}^h - S_i^h)^2 | (S_i^h, r_i^h) = (S_{i,j}, r_{i,k})),
\]

\[
\mathcal{M}^{r}_{i,j,k}(2) = \mathbb{E}((r_{i+1}^h - r_i^h)^2 | (S_i^h, r_i^h) = (S_{i,j}, r_{i,k})),
\]

\[
\mathcal{M}^{S,r}_{i,j,k} = \mathbb{E}((S_{i+1}^h - S_i^h)(r_{i+1}^h - r_i^h) | (S_i^h, r_i^h) = (S_{i,j}, r_{i,k})).
\]
Proof of the Theorem

- fast convergence to 0 of the fourth order local moment

\[
\lim_{h \to 0} \sup_{(i,j,k) \in A_*} \frac{1}{h} M_{i,j,k}^S(4) = 0,
\]

\[
\lim_{h \to 0} \sup_{(i,j,k) \in A_*} \frac{1}{h} M_{i,j,k}^r(4) = 0,
\]

where

\[
M_{i,j,k}^S(4) = \mathbb{E}((S^h_{i+1} - S^h_i)^4 | (S^h_i, r^h_i) = (S_{i,j}, r_{i,k})),
\]

\[
M_{i,j,k}^r(4) = \mathbb{E}((r^h_{i+1} - r^h_i)^4 | (S^h_i, r^h_i) = (S_{i,j}, r_{i,k})).
\]
We compare the performance of our lattice (ACZ) with the procedures of Wei (WEI) and of Hilliard, Schwartz and Tucker (HST) for the computation of European and American put option prices.

In order to study the numerical robustness of the algorithms we choose different values for $\sigma_r$: 0.08, 0.5, 1, 3.

Let us remark that for $\sigma_r = 0.5, 1, 3$, the Feller condition $2\kappa\theta \geq \sigma_r^2$ is not satisfied at all.

The other parameters of the option contracts are: $K = 100$, $T = 1, 2$, $S_0 = 100$, $r_0 = 0.06$, $\theta = 0.01$, $\kappa = 0.5$ and $\rho = -0.25$. 
### Table: European put options with $T = 1$, $S_0 = 100$, $K = 100$, $\sigma_S = 0.25$, $r_0 = 0.06$, $\theta = 0.1$, $\kappa = 0.5$, $\rho = -0.25$, $\sigma_r$ varying.

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<th>$\sigma_r$</th>
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<th>ACZ</th>
<th>MC Benchmark</th>
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Table: European put options with $T = 2$, $S_0 = 100$, $K = 100$, $\sigma_S = 0.25$, $r_0 = 0.06$, $\theta = 0.1$, $\kappa = 0.5$, $\rho = -0.25$, $\sigma_r$ varying.
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Table: American put options with \( T = 1, S_0 = 100, K = 100, \sigma_S = 0.25, r_0 = 0.06, \theta = 0.1, \kappa = 0.5, \rho = -0.25, \sigma_r \) varying.

*E. Appolloni, **L. Caramellino, ***A. Zanette
A robust tree method for pricing American options with CIR stochastic interest rate
### Numerical results

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**Table:** American put options with $T = 2$, $S_0 = 100$, $K = 10$, $\sigma_S = 0.25$, $r_0 = 0.06$, $\theta = 0.1$, $\kappa = 0.5$, $\rho = -0.25$, $\sigma_r$ varying.