BIDERIVATIONS AND COMMUTING LINEAR MAPS ON LIE ALGEBRAS

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Abstract. Let L be a Lie algebra over a commutative unital ring containing $\frac{1}{2}$. If L is perfect and centerless, then every skew-symmetric biderivation $\delta : L\times L \to L$ is of the form $\delta(x, y) = \gamma([x, y])$ for all $x, y \in L$, where $\gamma \in \mathrm{Cent}(L)$, the centroid of L. Under a milder assumption that $[c, [L, L]] = \{0\}$ implies $c = 0$, every commuting linear map from L to L lies in $\mathrm{Cent}(L)$. These two results are special cases of our main theorems which concern biderivations and commuting linear maps having their ranges in an L-module over a field. We provide a variety of examples, some of them showing the necessity of our assumptions and some of them showing that our results cover several results from the literature.

1. Introduction

Let L be a Lie algebra over a commutative unital ring F containing $\frac{1}{2}$ and let M be a module over L. Recall that a linear map $d : L \to M$ is a *derivation* if

$$d([x, y]) = x \cdot d(y) - y \cdot d(x) \text{ for all } x, y \in L.$$

We will say that a bilinear map $\delta : L \times L \to M$ is a *skew-symmetric biderivation* if $\delta(x, y) = -\delta(y, x)$ for all $x, y \in L$ and

$$\delta([x, y], z) = x \cdot \delta(y, z) - y \cdot \delta(x, z) \text{ for all } x, y, z \in L.$$

That is, $x \mapsto \delta(x, z)$ is a derivation for every $z \in L$ (and hence, since δ is skew-symmetric, $x \mapsto \delta(z, x)$ is also a derivation). Next, we will say that a linear map $f : L \to M$ is a *commuting linear map* if $x \cdot f(x) = 0$ for all $x \in L$. This condition readily implies that $x \cdot f(y) = -y \cdot f(x)$ for all $x, y \in L$, which shows that $\delta(x, y) = x \cdot f(y)$ is a skew-symmetric biderivation. Thus, skew-symmetric biderivations may be viewed as a generalization of commuting linear maps.

If F is a field, and $M = L$ with $x \cdot y = [x, y]$, the above definitions coincide with the usual ones. Generalizations involving modules, which we propose, are, on the one hand, interesting in their own right, and, on the other hand, suit the methods that we will employ.

The study of commuting maps and (skew-symmetric) biderivations has its roots in associative ring theory [2, 5], where it has turned out to be influential and far-reaching – see [3] and [4]. An interest in studying these maps on Lie algebras over fields has grown more recently [6, 11, 9, 10, 12, 13, 14]. We will cover most of the results from these papers by using a general but simple approach. The following notion will be of crucial importance: we define the *centroid of* M, and denote it by $\mathrm{Cent}(M)$, as the space of all L-module homomorphisms from L to

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M, where L is viewed as an L-module under the adjoint action. Thus, a linear map $\gamma : L \to M$ belongs to $\mathrm{Cent}(M)$ if

$$\gamma([x,y]) = x \cdot \gamma(y) \quad \text{for all } x, y \in L.$$

If F is a field, and $M = L$, this is the usual centroid of L, which can be often computed by using known results (e.g., [1] and [8]). The connection between the centroid and the theme of this paper is straightforward: if $\gamma \in \mathrm{Cent}(M)$, then γ is a commuting linear map, and, moreover, the map $\delta(x,y) = \gamma([x,y])$ is a skew-symmetric biderivation. Under appropriate assumptions, we will show that there are no other commuting linear maps and skew-symmetric biderivations than these. To describe our results more specifically, we need some further notation and conventions. For any subset S of L, we set

$$Z_M(S) = \{v \in M \mid S \cdot v = \{0\}\}.$$

Note that $Z = Z_L(L)$ is just the center of L. As usual, we write L' for $[L, L]$.

Throughout the paper, we assume that all our Lie algebras are over a commutative unital ring F containing $\frac{1}{2}$. This assumption will not be repeated in the statements of the results.

In principle, a description of skew-symmetric biderivations from $L \times L$ to M implies a description of commuting linear maps from L to M. However, we will describe commuting linear maps under milder assumptions, so we will treat them separately to some extent. Sect. 2 is devoted to biderivations, and Sect. 3 to commuting linear maps.

In Sect. 2, we first obtain a general formula for skew-symmetric biderivations from $L \times L$ to M (Lemma 2.1). Then we derive our main result, Theorem 2.3, stating that every biderivation arises from the centroid (as above) provided that L is perfect and $Z_M(L) = \{0\}$. We also provide an algorithm for describing skew-symmetric biderivations, and show that a number of known results can be deduced from Theorem 2.3. At the end of the section, we show by an example that restricting ourselves to biderivations that are skew-symmetric is really necessary.

The main result of Sect. 3, Theorem 3.2, states that already $Z_M(L') = \{0\}$ implies that every commuting linear map from L to M belongs to $\mathrm{Cent}(M)$. An algorithm for describing commuting linear maps on L is also provided and several examples given.

We conclude the introduction by a remark on the possible meaning of our results in a wider context. It is a fact that the description of additive commuting maps on prime rings [2] eventually led to the theory of functional identities on noncommutative rings [4]. Linear commuting maps on Lie algebras can therefore be viewed as a testing case for developing the theory of functional identities on Lie algebras. So far, with a partial exception [11] (where, however, only finite dimensional Lie algebras were treated), their description was known only for some special examples of Lie algebras over fields. The fact that the present paper contains a description for a large class of Lie algebras, which includes all simple Lie algebras, seems promising.

2. Biderivations

We start with the crucial lemma. The main idea of its proof, i.e., computing $\delta([x,y],[z,w])$ in two different ways, is well-known; it was first used in associative rings [5], and later also in Lie algebras [6]. The formula (2.2) from the middle of the proof is actually known. However, the final result that we will derive using this approach is, to the best of our knowledge, new.

Lemma 2.1. *Let L be a Lie algebra and let M be an L-module. If $\delta : L \times L \to M$ is a skew-symmetric biderivation, then*

$$\delta(u, [x, y]) - u \cdot \delta(x, y) \in Z_M(L'), \forall u, x, y \in L.$$

Proof. For any $x, y, z, w \in L$, we have

$$\begin{aligned}
&\delta([x, y], [z, w]) \\
=&x \cdot \delta(y, [z, w]) - y \cdot \delta(x, [z, w]) \\
=& - x \cdot \delta([z, w], y) + y \cdot \delta([z, w], x) \\
=& - x \cdot (z \cdot \delta(w, y)) + x \cdot (w \cdot \delta(z, y)) + y \cdot (z \cdot \delta(w, x)) - y \cdot (w \cdot \delta(z, x)).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\delta([x, y], [z, w]) \\
=& - \delta([z, w], [x, y]) \\
=& - z \cdot \delta(w, [x, y]) + w \cdot \delta(z, [x, y]) \\
=&z \cdot \delta([x, y], w) - w \cdot \delta([x, y], z) \\
=&z \cdot (x \cdot \delta(y, w)) - z \cdot (y \cdot \delta(x, w)) - w \cdot (x \cdot \delta(y, z)) + w \cdot (y \cdot \delta(x, z)).
\end{aligned}$$

Comparing both relations and using the assumption that δ is skew-symmetric, we obtain

$$[x, z] \cdot \delta(y, w) + [y, w] \cdot \delta(x, z) = [x, w] \cdot \delta(y, z) + [y, z] \cdot \delta(x, w) \tag{2.1}$$

Note that $\delta(y, y) = 0$ since δ is skew-symmetric and $\mathrm{char}(F) \neq 2$. Writing y for w and x for z in (2.1) therefore gives

$$[x, y] \cdot \delta(y, x) + [y, x] \cdot \delta(x, y) = 0,$$

that is, $2[x, y] \cdot \delta(x, y) = 0$. Consequently,

$$[x, y] \cdot \delta(x, y) = 0.$$

A linearization on x yields

$$[x, y]\delta(z, y) + [z, y]\delta(x, y) = 0.$$

Further, linearizing this relation on y we get

$$[x, y]\delta(z, w) + [x, w]\delta(z, y) + [z, y]\delta(x, w) + [z, w]\delta(x, y) = 0.$$

Now, rewrite (2.1) so that the roles of y and z are replaced:

$$[x, y] \cdot \delta(z, w) + [z, w] \cdot \delta(x, y) - [x, w] \cdot \delta(z, y) - [z, y] \cdot \delta(x, w) = 0.$$

Summing up the last two relations we get

$$2\big([x, y] \cdot \delta(z, w) + [z, w] \cdot \delta(x, y)\big) = 0,$$

which implies

$$[x, y] \cdot \delta(z, w) = [w, z] \cdot \delta(x, y). \tag{2.2}$$

In particular,

$$[[x, u], y] \cdot \delta(z, w) = [w, z] \cdot \delta([x, u], y). \tag{2.3}$$

On the other hand, by the Jacobi identity,

$$[[x, u], y] \cdot \delta(z, w) = [[x, y], u] \cdot \delta(z, w) + [[y, u], x] \cdot \delta(z, w),$$

and hence, by (2.2),

$$[[x,u],y]\cdot\delta(z,w)=[w,z]\cdot\delta([x,y],u)+[w,z]\cdot\delta([y,u],x). \tag{2.4}$$

Comparing (2.3) and (2.4) we get

$$[w,z]\cdot\big(\delta([x,y],u)+\delta([y,u],x)+\delta([u,x],y)\big)=0. \tag{2.5}$$

Now, applying that δ is a skew-symmetric biderivation it follows from (2.5) by a direct calculation that

$$2[w,z]\cdot\big(x\cdot\delta(y,u)+y\cdot\delta(u,x)+u\cdot\delta(x,y)\big)=0.$$

Since

$$x\cdot\delta(y,u)+y\cdot\delta(u,x)=x\cdot\delta(y,u)-y\cdot\delta(x,u)=\delta([x,y],u)=-\delta(u,[x,y]),$$

the desired conclusion follows. □

More can be said if L is perfect.

Lemma 2.2. *Let L be a perfect Lie algebra and let M be an L-module. If $\delta : L\times L\to M$ is a skew-symmetric biderivation, then*

$$\delta(u,[x,y])=u\cdot\delta(x,y),\forall u,x,y\in L. \tag{2.6}$$

Proof. For any $u,v,x,y\in L$, we have

$$\begin{aligned}&\delta([u,v],[x,y])-[u,v]\cdot\delta(x,y)\\ =&u\cdot\delta(v,[x,y])-v\cdot\delta(u,[x,y])-u\cdot(v\cdot\delta(x,y))+v\cdot(u\cdot\delta(x,y))\\ =&u\cdot\Big(\delta(v,[x,y])-v\cdot\delta(x,y)\Big)-v\cdot\Big(\delta(u,[x,y])-u\cdot\delta(x,y)\Big).\end{aligned}$$

Lemma 2.1, together with assumption that $L=L'$, yields (2.6). □

We are now in a position to prove our fundamental theorem.

Theorem 2.3. *Let L be a perfect Lie algebra and let M be an L-module such that $Z_M(L)=\{0\}$. Then every skew-symmetric biderivation $\delta : L\times L\to M$ is of the form $\delta(x,y)=\gamma([x,y])$ where $\gamma\in\mathrm{Cent}(M)$.*

Proof. Define $\gamma : L=L'\to M$ by

$$\gamma([x,y])=\delta(x,y),\forall x,y\in L. \tag{2.7}$$

Let us show that Lemma 2.2 implies that γ is well-defined. Indeed, assuming that $\sum_i[x_i,y_i]=0$, we have

$$0=\delta\Big(u,\sum_i[x_i,y_i]\Big)=\sum_i\delta(u,[x_i,y_i])=u\cdot\Big(\sum_i\delta(x_i,y_i)\Big),$$

and hence $\sum_i\delta(x_i,y_i)=0$ follows from $Z_M(L)=\{0\}$. We can now write (2.6) as $\delta(u,v)=u\cdot\gamma(v)$ for all $u,v\in L$. Together with (2.7) this shows that $\gamma\in\mathrm{Cent}(M)$. □

From now on we consider skew-symmetric biderivations on a Lie algebra L, that is, skew-symmetric biderivations from $L\times L$ to L. We first record an immediate corollary to Theorem 2.3.

Corollary 2.4. *If L is a perfect and centerless Lie algebra (in particular, if L is simple), then every skew-symmetric biderivation δ on L is of the form $\delta(x,y)=\gamma([x,y])$ where $\gamma\in\mathrm{Cent}(L)$.*

Our goal now is to examine concrete situations to which our results are applicable. To this end, we need some auxiliary notions and results.

Let L be an arbitrary Lie algebra. Note that any skew-symmetric bilinear map $\delta : L \times L \to Z = Z_L(L)$ with $\delta(L, L') = 0$ is a skew-symmetric biderivation. We will call it a *trivial biderivation* on L. It is clear that every skew-symmetric biderivation having the range in Z is automatically trivial.

If $\delta : L \times L \to L$ is an arbitrary skew-symmetric biderivation, for any $x, y \in L, z \in Z$ we have

$$0 = \delta([z, x], y) = [z, \delta(x, y)] + [\delta(z, y), x] = [\delta(z, y), x],$$

hence $\delta(Z, L) \subset Z$. Consequently, setting $\bar{L} = L/Z$ we can define a skew-symmetric biderivation $\bar{\delta} : \bar{L} \times \bar{L} \to \bar{L}$ by

$$\bar{\delta}(\bar{x}, \bar{y}) = \overline{\delta(x, y)},$$

where $\bar{x} = x + Z \in \bar{L}$ for $x \in L$.

Lemma 2.5. *Let L be a Lie algebra. Up to trivial biderivations on L, the map $\delta \to \bar{\delta}$ is a 1-1 map from skew-symmetric biderivations on L to skew-symmetric biderivations on $\bar{L}$.*

Proof. Let δ_1, δ_2 be skew-symmetric biderivations on L such that $\bar{\delta}_1 = \bar{\delta}_2$. Then $\delta = \delta_1 - \delta_2$ is a skew-symmetric biderivation with $\delta(L, L) \subset Z$. Thus, δ is a trivial biderivation on L. □

Any skew-symmetric biderivation $\delta : L \times L \to L$ satisfying $\delta(L', L') = 0$ with range in $Z_L(L')$ will be called a *special biderivation.*

Example 2.6. In this example, F is a field. Every Lie algebra L with nontrivial center and such that the codimension of L' in L is greater than 1 has nonzero special biderivations. Indeed, by the codimension assumption, there exists a nonzero skew-symmetric bilinear functional $\omega : L \times L \to F$ such that $\omega(L, L') = \{0\}$, and taking any nonzero $z_0 \in Z$, we have that $\delta(x, y) = \omega(x, y)z_0$ is a nonzero special biderivation which is also trivial.

The next example shows that there exist special biderivations δ that are not of the form $\delta(x, y) = \gamma([x, y]) + \delta_0(x, y)$ where $\gamma \in \mathrm{Cent}(L)$ and δ_0 is a trivial biderivation.

Example 2.7. In this example, F is a field of characteristic 0. Let $\mathcal{F}$ be the free Lie algebra in variables x_1, x_2, x_3. Denote by $\mathcal{I}$ the ideal of $\mathcal{F}$ generated by all elements of the form $[[x_1, f_1], f_2] + [[x_1, f_2], f_1]$ where $f_1, f_2 \in \mathcal{F}$, and by $\mathcal{J}$ the ideal of $\mathcal{F}$ generated by all elements of the form $[[x_1, g_1], g_2]$ where $g_1, g_2 \in \mathcal{F}'$.

Set $L = \mathcal{F}/(\mathcal{I} + \mathcal{J})$. It is clear that $a = \bar{x}_1 = x_1 + \mathcal{I} + \mathcal{J}$ is a nonzero element in L. Define the bilinear map $\delta(x, y) = [[a, x], y]$ for all $x, y \in L$. It is easy to see that δ is a special biderivation on L.

Assume there are $\gamma \in \mathrm{Cent}(L)$ and a trivial biderivation δ_0 on L such that

$$\delta(x, y) = \gamma([x, y]) + \delta_0(x, y).$$

For $x, y, u \in L$, we then have

$$\begin{aligned} &[[a, [x, y]], u] = \delta([x, y], u) \\ =&\gamma([[x, y], u]) + \delta_0([x, y], u) = [\gamma([x, y]), u] + \delta_0([x, y], u) \\ =&[\delta(x, y), u] + \delta_0([x, y], u) = [[[a, x], y], u] + \delta_0([x, y], u), \end{aligned}$$

yielding that $[[[a,y],x],u] \in Z$, the center of L. We claim that this is not the case. Specifically, we will show that

$$[[[[\bar{x}_1,\bar{x}_2],\bar{x}_3],\bar{x}_3],\bar{x}_3] \neq 0, \tag{2.8}$$

so that $[[\bar{x}_1,\bar{x}_2],\bar{x}_3],\bar{x}_3] \notin Z$.

As usual, we can define monomials, the degrees $\deg_1, \deg_2, \deg_3$ of a monomial with respect to x_1, x_2, x_3 respectively, and homogeneous elements in $\mathcal{F}$. Then every element in $\mathcal{I}$ (or $\mathcal{J}$) can be written as a sum of homogeneous elements in $\mathcal{I}$ (or $\mathcal{J}$). It is easy to see that there are no nonzero elements in $\mathcal{J}$ with $\deg_1 = 1, \deg_2 = 1$ and $\deg_3 = 3$. The only homogeneous elements with $\deg_1 = 1, \deg_2 = 1$ and $\deg_3 = 3$ in $\mathcal{I}$ are

$$\begin{aligned}
&[[[[x_1,x_2],x_3] + [[x_1,x_3],x_2],x_3],x_3] \\
&= [[[[x_1,x_2],x_3],x_3],x_3] + [[[[x_1,x_3],x_2],x_3],x_3], \\
&[[[x_1,x_3],x_3],[x_2,x_3]] = [[[x_1,x_3],x_3],x_2],x_3] - [[[x_1,x_3],x_3],x_3],x_2], \\
&[[[[x_1,x_3],x_3],x_2],x_3].
\end{aligned}$$

Since

$$\begin{aligned}
&[[[[x_1,x_2],x_3],x_3],x_3], \quad [[[[x_1,x_3],x_2],x_3],x_3], \\
&[[[[x_1,x_3],x_3],x_2],x_3], \quad [[[[x_1,x_3],x_3],x_3],x_2],
\end{aligned}$$

are linearly independent in $\mathcal{F}$, it follows that $[[[[x_1,x_2],x_3],x_3],x_3]$ does not lie in $\mathcal{I}$, and hence neither in $\mathcal{I}+\mathcal{J}$. This proves (2.8). Consequently, δ is not of the form $\delta(x,y) = \gamma([x,y]) + \delta_0(x,y)$. □

Every skew-symmetric biderivation $\delta : L \times L \to L$ satisfies

$$\delta(u,[x,y]) = [x,\delta(u,y)] + [\delta(u,x),y] \in L', \forall x,y,u \in L. \tag{2.9}$$

Thus we have a skew-symmetric biderivation $\delta' : L' \times L' \to L'$ by restricting δ to $L' \times L'$.

Lemma 2.8. *Let L be a centerless Lie algebra.*

(a) *Up to a special biderivation, any skew-symmetric biderivation δ on L is an extension of a unique skew-symmetric biderivation on L'.*

(b) *If L' is further perfect, then L has no nonzero special biderivation.*

Proof. (a) Let δ_1, δ_2 be biderivations on L such that $\delta_1' = \delta_2'$. Let $\delta = \delta_1 - \delta_2$. Then $\delta(L', L') = 0$. Taking $u, y \in L'$ in (2.9) we see that

$$[\delta(u,x),y] = 0, \forall x \in L, y,u \in L',$$

i.e., $\delta(L,L') \subset Z_L(L')$. Applying this to Lemma 2.1, we see that $[L,\delta(L,L)] \subset Z_L(L')$.

Using (2.2) we have

$$0 = [[L,L],\delta(L,L')] = [\delta(L,L),[L,L']].$$

Then for any $x,y,z,u,v \in L$ we have

$$0 = [[[x,y],z],\delta(u,v)] = [[[x,y],\delta(u,v)],z] + [[x,y],[z,\delta(u,v)]] = [[[x,y],\delta(u,v)],z].$$

Since L is centerless, we deduce that $\delta(L,L) \subset Z_L(L')$. Thus, δ is a special biderivation of L.

(b) Let δ be a special biderivation of L. From the beginning of the proof of (a) we see that $\delta(L,L') \subset Z_L(L')$. Since L is perfect and centerless, it follows that $\delta(L,L') = 0$.

Assuming that $\delta \neq 0$, there are $x_1, x_2 \in L$ such that $\delta(x_1, x_2) = z_{12} \neq 0$. Since L is centerless we can find $x_3 \in L$ so that $[x_3, z_{12}] = z \neq 0$. Let $\delta(x_1, x_3) = z_{13}, \delta(x_2, x_3) = z_{23}$. From

$$\begin{aligned} 0 &= \delta([x_1, x_3], x_2) = [x_1, \delta(x_3, x_2)] - z = -[x_1, z_{23}] - z, \\ 0 &= \delta([x_1, x_2], x_3) = [z_{13}, x_2] + [x_1, z_{23}], \\ 0 &= \delta([x_2, x_3], x_1) = -[x_2, z_{13}] + z, \end{aligned}$$

we deduce that $-z = [x_1, z_{23}] = -[z_{13}, x_2] = z$, a contradiction. Thus, $\delta = 0$. $\square$

Let us explain an algorithm for finding all skew-symmetric biderivations on a Lie algebra L using Lemmas 2.5 and 2.8. We have a sequence of quotient Lie algebras:

$$L_{(1)} = L, L_{(2)} = L_{(1)}/Z(L_{(1)}), \cdots, L_{(r+1)} = L_{(r)}/Z(L_{(r)}). \tag{2.10}$$

If there is $r \in \mathbb{N}$ such that $Z(L_{(r)}) = 0$, then repeatedly applying Lemma 2.5 to the above sequence backward, we reduce the problem of finding skew-symmetric biderivations on L to the problem of finding skew-symmetric biderivations on the centerless Lie algebra $L_{(r+1)}$. If $L_{(r+1)}$ is also perfect, then using Corollary 2.4 we have all biderivations on $L_{(r+1)}$. We are done in this case. If $L_{(r+1)}$ is not perfect, using Lemma 2.8 we reduce the problem of finding skew-symmetric biderivations on $L_{(r+1)}$ to the problem of finding skew-symmetric biderivations on the Lie algebra $L'_{(r+1)}$. Now we repeat the procedure based on (2.10) with L replaced by $L'_{(r+1)}$, and continue this algorithm.

We will now apply our results to concrete examples.

Example 2.9. Let L be a simple Lie algebra over an algebraically closed field F of characteristic not 2 such that $\mathrm{Card}(F) > \dim(L)$. As is well-known, every L-module endomorphism of L is a scalar map [8]. Therefore, by Corollary 2.4, every skew-symmetric biderivation δ of L is of the form $\delta(x, y) = \lambda[x, y]$, $x, y \in L$, for some $\lambda \in F$.

This example, in particular, covers the results in [6]. $\square$

In the next examples (2.12-2.15), take F to be a field of caracteristic 0.

Example 2.10. For $a, b \in F$ with $(a, b) \notin \mathbb{Z} \times \{0, -1\}$, the Lie algebra

$$W(a, b) = \mathrm{span}_F\{L_m, I_m \mid m \in \mathbb{Z}\}$$

is an infinite-dimensional Lie algebra over F equipped with the following brackets:

$$\begin{aligned} [L_m, L_n] &= (n - m)L_{m+n}, \\ [L_m, I_n] &= (n + a + bm)I_{m+n}, \\ [I_m, I_n] &= 0, \end{aligned}$$

for all $m, n \in \mathbb{Z}$. These Lie algebras $W(a, b)$ are perfect and centerless. Let $L = \oplus_{i\in\mathbb{Z}} FL_i$ and $I = \oplus_{i\in\mathbb{Z}} FI_i$. Then I is an ideal of $W(a, b)$ and L is a subalgebra which is isomorphic to the Witt algebra. As indecomposable L-modules, L and I are not isomorphic, and L-module homomorphisms of L (and I) are scalar maps.

Let $\gamma \in \mathrm{Cent}(W(a, b))$. Then

$$\gamma(I) \subset \gamma([L, I]) = [\gamma(L), I] \subset I,$$

and

$$\gamma([L_m, I_n]) = [\gamma(L_m), I_n] = [L_m, \gamma(I_n)].$$

Since γ can be considered as an L-module homomorphism, we see that $\gamma(L) \subset L$. There exist $c_1, c_2 \in F$ such that $\gamma(L_m) = c_1 L_m, \gamma(I_m) = c_2 I_m$. Furthermore, $c_2 = c_1$. So the centroid of $W(a,b)$ consists of scalar maps from $W(a,b)$ to $W(a,b)$. Corollary 2.4 therefore tells us that for every skew-symmetric biderivation δ of $W(a,b)$ there exists $\lambda \in F$ such that $\delta(x,y) = \lambda[x,y]$, $x, y \in W(a,b)$. □

Example 2.11. Consider the Lie algebra $W = W(0,0)$ which is defined using the same brackets as in Example 2.10 with $a = b = 0$. This Lie algebra has center $Z = FI_0$, and $\bar{W} = W/Z$ is perfect and centerless. By similar arguments we see that any map in $\mathrm{Cent}(\bar{W})$ is a scalar map. Corollary 2.4 therefore tells us that for every skew-symmetric biderivation δ of $\bar{W}$ there exists $\lambda \in F$ such that $\delta(x,y) = \lambda[x,y]$, $x, y \in \bar{W}$. Since W is perfect, it follows from Lemma 2.5 that every skew-symmetric biderivation δ of W is of the form $\delta(x,y) = \lambda[x,y]$, $x, y \in W$, for some $\lambda \in F$. □

In the next example we will derive the usual conclusion that all skew-symmetric biderivations of the Lie algebra L in question are of the form $\delta(x,y) = \lambda[x,y]$ with $\lambda \in F$. It is interesting, however, that this will be derived from the description of skew-symmetric biderivations of $\bar{L} = L/Z$ which is more involved (that is, the centroid of $\bar{L}$ contains more than just scalar maps).

Example 2.12. Consider the Lie algebra

$$\tilde{W}(0,-1) = \mathrm{span}_F\{L_m, I_m, c_1, c_2, c_3 \,|\, m \in \mathbb{Z}\}$$

equipped with the following brackets:

$$\begin{aligned}
[L_m, L_n] =& (n-m)L_{m+n} + \delta_{m,-n}\frac{m^3-m}{12}c_1,\\
[L_m, I_n] =& (n-m)I_{m+n} + \delta_{m,-n}\frac{m^3-m}{12}c_2,\\
[I_m, I_n] =& \delta_{m,-n}\frac{m^3-m}{12}c_3,
\end{aligned}$$

and c_i being central elements. In fact, the center Z of $\tilde{W}(0,-1)$ is equal to $\mathrm{span_F}\{\mathrm{c_1}, \mathrm{c_2}, \mathrm{c_3}\}$. The Lie algebra $\tilde{W}(0,-1)/Z = W(0,-1)$ is defined using the same brackets as in Example 2.10 with $(a,b) = (0,-1)$. The Lie algebra $W(0,-1)$ is perfect and centerless. Let $L = \oplus_{i\in\mathbb{Z}} FL_i \subset W(0,-1)$ and $I = \oplus_{i\in\mathbb{Z}} FI_i \subset W(0,-1)$. Then I is an ideal of $W(0,-1)$ and L is a subalgebra. As L-modules, L and I are isomorphic.

Let $\gamma \in \mathrm{Cent}(W(0,-1))$. Then $\gamma(I) \subset \gamma([L,I]) = [\gamma(L), I] \subset I$. There are $a, b, c \in F$ such that $\gamma(L_m) = aL_m + bI_m, \gamma(I_m) = cI_m$. Since

$$\gamma([L_m, I_n]) = [\gamma(L_m), I_n] = [L_m, \gamma(I_n)],$$

we deduce that $c = a$. We denote this element in $\mathrm{Cent}(W(0,-1))$ by $\gamma_{a,b}$. So $\mathrm{Cent}(W(0,-1)) = \{\gamma_{a,b} \,|\, a, b \in F\}$. From Corollary 2.4 we see that for any skew-symmetric biderivation δ of $W(0,-1)$, there are $a, b \in F$ such that

$$\delta(x,y) = \gamma_{a,b}([x,y]), \forall x, y \in W(0,-1).$$

It is easy to see that there is $g_a \in \mathrm{Cent}(\tilde{W}(0,-1))$ such that $\bar{g}_a = \gamma_{a,0}$ (actually g_a is the scalar map determined by a).

Next suppose there exists a skew-symmetric biderivation h on $\tilde{W}(0,-1)$ such that

$$\bar{h}(\bar{x}, \bar{y}) = \gamma_{0,b}([\bar{x}, \bar{y}]).$$

Then $h(L, I) \subset Z$, and further

$$h(L, I) = h(L', I) = [h(L, I), L] + [L, h(L, I)] = 0.$$

We may assume that

$$h(L_m, L_n) = (n-m)bI_{m+n} + C_{m,n}, \text{ where } C_{m,n} \in Z.$$

From

$$\begin{aligned}
&(n-m)\left((r-m-n)bI_{m+n+r} + C_{m+n,r}\right) = h([L_m, L_n], L_r) \\
=&[h(L_m, L_r), L_n] + [L_m, h(L_n, L_r)] \\
=&(r-m)b((n-r-m)I_{m+n+r} + \delta_{m+n,-r}\frac{n-n^3}{12}c_2) \\
&+ (r-n)b((n+r-m)I_{m+n+r} + \delta_{m+n,-r}\frac{m^3-m}{12}c_2),
\end{aligned}$$

we obtain that

$$(n-m)C_{m+n,r} = (r-m)b\delta_{m+n,-r}\frac{n-n^3}{12}c_2 + (r-n)b\delta_{m+n,-r}\frac{m^3-m}{12}c_2.$$

Letting $n = -m - r$ we have

$$-(r+2m)C_{-r,r} = \big((r-m)((m+r)^3 - m - r) - (2r+m)(m^3-m))\big)bc_2/12.$$

Note that $C_{-r,r} = 0$. Since m is arbitrary, it follows that $b = 0$. Thus, if $b \neq 0$, there is no biderivation h on $\tilde{W}(0, -1)$ such that $\bar{h}(\bar{x}, \bar{y}) = \gamma_{0,b}([\bar{x}, \bar{y}])$.

By Lemma 2.5, every skew-symmetric biderivation δ of $\tilde{W}(0, -1)$ is of the form $\delta(x, y) = \lambda[x, y]$, $x, y \in \tilde{W}(0, -1)$, for some $\lambda \in F$.

This and the previous two examples recover all results in [10]. □

Example 2.13. The Schrödinger-Virasoro Lie algebra S is the infinite-dimensional Lie algebra with F-basis $\{L_m,\, Y_p\,,\, M_n \,|\, m,\, n, p \in \mathbb{Z}\}$ and Lie brackets,

$$\begin{aligned}
&[L_m,\, L_n] = (n-m)L_{n+m}, \\
&[L_m,\, Y_p] = (p - \frac{m}{2})Y_{p+m}, \\
&[L_m,\, M_n] = nM_{n+m}, \\
&[Y_p,\, Y_n] = (n-p)M_{n+p}\,, \\
&[Y_p,\, M_n] = [M_n,\, M_m] = 0.
\end{aligned}$$

We know that S is perfect with center $Z = FM_0$, and that $\bar{S} = S/Z$ is perfect and centerless. Let $L = \oplus_{i\in\mathbb{Z}}FL_i$, $Y = \oplus_{i\in\mathbb{Z}}FY_i$ and $M = \oplus_{i\in\mathbb{Z}\setminus\{0\}}FM_i$.

Let $\gamma \in \mathrm{Cent}(\bar{S})$. Since L, M, Y are not isomorphic to each other as indecomposable L-modules, there are $a, b, c \in F$ such that

$$\gamma(L_m) = aL_m, \;\; \gamma(M_m) = bM_m, \;\; \gamma(Y_m) = bY_m.$$

From

$$\begin{aligned}
\gamma([L_m, M_n]) &= [\gamma(L_m), M_n] = [L_m, \gamma(M_n)], \\
\gamma([L_m, Y_n]) &= [\gamma(L_m), Y_n] = [L_m, \gamma(Y_n)],
\end{aligned}$$

we deduce that $a = b = c$. Thus γ is a scalar map. By Corollary 2.4, for any skew-symmetric biderivation δ of $\bar{S}$, there is $\lambda \in F$ such that $\delta(x, y) = \lambda[x, y]$ for all $x, y \in \bar{S}$. Since S is perfect, Lemma 2.5 implies that every skew-symmetric biderivation δ is of the form $\delta(x, y) = \lambda[x, y]$, $x, y \in S$, for some $\lambda \in F$.

This example recovers all results in [13]. □

Example 2.14. Let $q \in F$. The Block Lie algebra $B(q)$ is the Lie algebra over F with a basis $\{L_{m,i} \mid m, i \in \mathbb{Z}\}$ subject to the following Lie brackets

$$[L_{m,i}, L_{n,j}] = (n(i+q) - m(j+q))L_{m+n,i+j}, \ \forall\, i, j, m, n \in \mathbb{Z}.$$

Some of these Lie algebras are not perfect or centerless. Anyway $B(q)'/Z$ is a simple Lie algebra. So for any skew-symmetric biderivation δ of $B(q)'/Z$, there is $\lambda \in F$ such that $\delta(x, y) = \lambda[x, y]$ for all $x, y \in B(q)'/Z$.

Now we use Lemma 2.5. If $q \neq 0$, for any skew-symmetric biderivation δ of $B(q)'$, there is $\lambda \in F$ such that $\delta(x, y) = \lambda[x, y]$ for all $x, y \in B(q)'$. Using Lemma 2.8, if $q \neq 0$, there is no special biderivations, hence, for any skew-symmetric biderivation δ of $B(q)$ there is $\lambda \in F$ such that $\delta(x, y) = \lambda[x, y]$ for all $x, y \in B(q)$.

If $q = 0$, we know that $B(0) = B(0)' \oplus FL_{0,0}$ as ideals. It is easy to see that for any skew-symmetric biderivation δ of $B(0)$ there is $\lambda \in F$ such that $\delta(x, y) = \lambda[x, y]$ for all $x, y \in B(0)$.

This example recovers the results on skew-symmetric biderivations in [9]. $\square$

We have thus seen that our methods cover a variety of results from the literature. On the other hand, obviously they do not work for Lie algebras L such that the sequence (2.10) does not terminate.

We close the section by discussing general, not necessarily skew-symmetric, biderivations. By a *biderivation* we mean, of course, a bilinear map $\delta : L \times L \to M$ such that the maps $x \mapsto \delta(x, z)$ and $x \mapsto \delta(z, x)$ are derivations for every $z \in L$. Biderivations that are skew-symmetric seem to be more natural than others, in particular because of their connection to the centroid and commuting linear maps. However, general biderivations are also mathematically challenging and therefore deserve some attention.

Every biderivation δ can be written as the sum of a symmetric biderivation and a skew-symmetric biderivation, namely,

$$\delta(x, y) = \frac{\delta(x, y) + \delta(y, x)}{2} + \frac{\delta(x, y) - \delta(y, x)}{2}.$$

We can therefore focus on symmetric biderivations. Finding examples on Abelian Lie algebras L is trivial: every symmetric bilinear map from $L \times L$ to L is a symmetric biderivation. On the other hand, in a Lie (or associative) algebra with trivial (or small) center, one normally expects that a symmetric biderivation is always 0. In view of Theorem 2.3, it is tempting to conjecture that there are no nonzero symmetric biderivations from $L \times L$ to M if L is perfect and $Z_M(L) = \{0\}$. However, the following example (which admittedly came as a surprise to us) shows that this is not the case – not even when $L = \mathfrak{sl}_2$!

Example 2.15. For convenience we take a basis d_1, d_0, d_{-1} of the Lie algebra $\mathfrak{sl}_2$ over F subject to the brackets

$$[d_i, d_j] = (j - i)d_{i+j}, \forall i, j \in \{0, 1, -1\},$$

where $d_k = 0$ if $k \notin \{0, 1, -1\}$. For $a, b \in F$, let $M(a, b) = \oplus_{i \in \mathbb{Z}} F v_i$ be the $\mathfrak{sl}_2$-module given by

$$d_i \cdot v_j = (j + a + bi)v_{i+j}.$$

Note that $M(a, b)$ is a simple $\mathfrak{sl}_2$-module if and only if $a \notin \mathbb{Z}$ or $b \notin \mathbb{Z}$. Also, $Z_{M(a,0)}(\mathfrak{sl}_2) = \operatorname{span}\{v_{-a}\}$ if $a \in \mathbb{Z}$, and otherwise $Z_{M(a,0)}(\mathfrak{sl}_2) = \{0\}$.

Take an arbitrary $k \in \mathbb{Z}$. It is easy to see that the bilinear map $\delta_k : \mathfrak{sl}_2 \times \mathfrak{sl}_2 \to M(a, 0)$ determined by

$$\delta_k(d_m, d_n) = v_{m+n+k}, \forall i, j \in \{0, 1, -1\},$$

is a symmetric biderivation. Similarly, for any $k \in \mathbb{Z}$, $\delta'_k : \mathfrak{sl}_2 \times \mathfrak{sl}_2 \to M(a, 1)$ determined by

$$\delta'_k(d_m, d_n) = (m + n + k + a)v_{m+n+k}, \forall i, j \in \{0, 1, -1\}$$

is a symmetric biderivation. □

We leave as an open question whether or not there exists a simple Lie algebra L (over a field of characteristic not 2) that admits a nonzero symmetric biderivation from $L \times L$ to L.

3. Commuting linear maps

Let M be a module over a Lie algebra L. It is clear that every $\gamma \in \mathrm{Cent}(M)$ satisfies $x \cdot \gamma(x) = 0$ for each $x \in L$. We will show that under a mild assumption, this condition is characteristic for the centroid., i.e., commuting linear maps f from L to M belong to Cent(M).

Lemma 3.1. *Let L be a Lie algebra and let M be an L-module. If $f : L \to M$ is a commuting linear map, then*

$$[w, z] \cdot \Big(u \cdot \big(f([x, y]) - x \cdot f(y)\big)\Big) = 0$$

for all $x, y, u, w, z \in L$.

Proof. Linearizing $x \cdot f(x) = 0$ we get $x \cdot f(y) = -y \cdot f(x)$. This shows that the map $\delta : L \times L \to M$, $\delta(x, y) = x \cdot f(y)$ is a skew-symmetric biderivation. According to Lemma 2.1,

$$[w, z] \cdot \Big(\delta(u, [x, y]) - u \cdot \delta(x, y)\Big) = 0$$

for all $x, y, u, w, z \in L$. Since $\delta(x, y) = x \cdot f(y)$ and $\delta(u, [x, y]) = u \cdot f([x, y])$, the result follows. □

The following theorem follows immediately from Lemma 3.1.

Theorem 3.2. *Let L be a Lie algebra and let M be an L-module such that $Z_M(L') = \{0\}$. If $f : L \to M$ is a commuting linear map, then $f \in \mathrm{Cent}(M)$.*

In the special case where $M = L$, Theorem 3.2 gets the following form.

Corollary 3.3. *Let L be a Lie algebra such that $Z_L(L') = \{0\}$. Then every commuting linear map $f : L \to L$ belongs to* $\mathrm{Cent}(L)$.

We remark that the set of commuting linear maps of Lie algebras L was studied in [11] under the name of *quasi-centroid*. It was shown, in particular, that the quasi-centroid of L coincides with the centroid in case L is finite dimensional, centerless, and perfect [11, Theorem 5.28]. Corollary 3.3 is obviously considerably stronger. In particular, it shows that the assumption that L is finite dimensional is superfluous.

If L has a nontrivial center, then every map with the range in Z is commuting. Moreover, it does not lie in $\mathrm{Cent}(L)$ in case it does not vanish on L'. This justifies that Corollary 3.3 deals with centerless Lie algebras. However, we have assumed more than that. To justify the assumption that L' has trivial centralizer in L, consider the following example. It is a modification of an example from [7].

Example 3.4. In this example, F is a field of characteristic not 2. Let L be the Lie algebra consisting of all 4×4 matrices of the form

$$\begin{bmatrix} x_{11} & x_{12} & x_{13} & x_{14} \\ 0 & x_{11} & 0 & x_{24} \\ 0 & 0 & 0 & x_{34} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

with $x_{ij} \in F$. One easily checks that L is centerless (but L' is not), and that $f : L \to L$ given by

$$\begin{bmatrix} x_{11} & x_{12} & x_{13} & x_{14} \\ 0 & x_{11} & 0 & x_{24} \\ 0 & 0 & 0 & x_{34} \\ 0 & 0 & 0 & 0 \end{bmatrix} \mapsto \begin{bmatrix} 0 & x_{13} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_{24} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

is a linear commuting map. However, $f \notin \mathrm{Cent}(L)$ since, for example,

$$0 = f([e_{13}, e_{24}]) \neq [e_{13}, f(e_{24})] = e_{14}.$$

Here, e_{ij} are matrix units. □

As we mentioned above, every *central* linear map, i.e., a map with the range in the center Z of L, is trivially commuting. Note also that the sum of commuting maps is again commuting. Thus, if $\gamma \in \mathrm{Cent}(L)$ and μ is a central map, then $f = \gamma + \mu$ is commuting.

The next simple lemma connects the problem of describing skew-symmetric biderivations with the problem of describing commuting linear maps.

Lemma 3.5. *Let L be a Lie algebra. If every skew-symmetric biderivation δ on L is of the form $\delta(x, y) = \gamma([x, y])$, then every commuting linear map f on L is of the form $f = \gamma + \mu$, where $\gamma \in \mathrm{Cent}(L)$ and μ is a central linear map.*

Proof. If $f : L \to L$ is a commuting linear map, then $[f(x), y] = -[f(y), x]$ for all $x, y \in L$, and hence $\delta(x, y) = [f(x), y]$ is a skew-symmetric biderivation. This readily implies the conclusion of the lemma. □

Using this lemma together with the description of biderivations on various Lie algebras obtained in the preceding section, we can now also describe commuting linear maps on all these Lie algebras.

Example 3.6. Every commuting linear map f on $W(a, b)$ for $a, b \in F$, $\tilde{W}(0, -1)$, S, $\bar{S}$ and $B(q)$ for $q \in F$, is of the form $f = \gamma + \mu$, where γ lies in the centroid and μ is a central map. □

We now give an example of a commuting linear map that is not a sum of a map in $\mathrm{Cent}(L)$ and a central map. It is particularly interesting that this map is a derivation.

Example 3.7. Let L be the Lie algebra from Example 2.7. We know that the derivation $f(x) = [a, x]$ is a commuting linear map. Suppose f is a sum of a map in $\mathrm{Cent}(L)$ and a central map. Then

$$[a, [x, y]] = [x, [a, y]] \mod Z, \forall x, y \in L,$$

and hence $[[a, x], y] \in Z$ for all $x, y \in L$. However, from Example 2.7 we know that this is not true; specifically, this follows from $[[\bar{x}_1, \bar{x}_2], \bar{x}_3] \notin Z$. Thus, f is not a sum of a map in $\mathrm{Cent}(L)$ and a central map. □

Finally, we will propose an algorithm for computing commuting linear map of a Lie algebra without using biderivations. Although we are interested in commuting linear maps from L to L, this algorithm involves commuting linear maps from L to an L-module M (and hence gives one of justifications for working in a more general framework involving modules). In view of Theorem 3.2 and Corollary 3.3, we are now interested in the case where $Z_M(L') \neq 0$.

Any linear map $f : L \to Z_M(L)$ is a commuting map. We call it a *central map*. A commuting linear map $f : L \to M$ will be called a *special commuting linear map* if $f(L') = 0$ and $f(L) \subset Z_M(L')$.

For any commuting linear map $f : L \to M$ we can define another commuting linear map $\tilde{f} : L \to M/Z_M(L')$ by

$$\tilde{f}(x) = f(x) + Z_M(L').$$

The following lemma then holds.

Corollary 3.8. *Let L be a Lie algebra and M be an L-module. Then $f \to \tilde{f}$ is, up to sums of special commuting and central linear maps, a 1-1 map from commuting linear maps from L to M to commuting linear maps from L to $M/Z_M(L')$.*

Proof. Let f_1, f_2 be commuting linear maps on L such that $\tilde{f}_1 = \tilde{f}_2$. Let $f = f_1 - f_2$. Then $f(L) \subset Z_M(L')$. From $0 = L' \cdot f(L) = -L \cdot f(L')$, we see that $f(L') \subset Z_M(L)$. By subtracting a central linear map from f we may assume that $f(L') = 0$. Now this $f : L \to M$ is a special commuting linear map. □

For an L-module M we have a sequence of quotient modules:

$$M = M_1, M_2 = M_1/Z_{M_1}(L'), \cdots, M_r = M_{r-1}/Z_{M_{r-1}}(L'). \tag{3.1}$$

If there is $r \in \mathbb{N}$ such that $Z_{M_r}(L') = 0$, we can start to find commuting maps $f_r : L \to M_r$. Then repeatedly using Corollary 3.8 we obtain all commuting maps $f : L \to M$. Let us point out that $Z_{M_r}(L') = 0$ implies, by Theorem 3.2, that every commuting map $f_r : L \to M_r$ lies in $\mathrm{Cent}(M_r)$.

We conclude the paper by an example illustrating this algorithm.

Example 3.9. Let $\mathfrak{g}$ be a finite dimensional simple Lie algebra over a field F of characteristic not 2, $F[t]$ be the polynomial algebra in t, and $n > 1$ be an integer. Consider the Lie algebra

$$L = \mathfrak{g} \otimes (tF[t]/t^{2n+1}F[t]) = \oplus_{k=1}^{2n}(\mathfrak{g} \otimes t^k).$$

Note that L is a nilpotent Lie algebra. As in (3.1), we have

$$L_1 = L,\ L_2 \simeq \mathfrak{g} \otimes (tF[t]/t^{2n-1}F[t]),\ \ L_3 \simeq \mathfrak{g} \otimes (tF[t]/t^{2n-3}F[t]),$$
$$\cdots,\ \ L_n \simeq \mathfrak{g} \otimes (tF[t]/t^3F[t]),\ \ L_{n+1} \simeq \mathfrak{g} \otimes (tF[t]/t^2F).$$

Step 1. Find all commuting maps to L_n.

For any commuting linear map $f_n : L \to L_{n+1}$ which can be any linear map, let $g_n : L \to L_n$ be a commuting map such that $\tilde{g}_n = f_n$. We may assume that $g_n(\mathfrak{g}\otimes t) \subset \mathfrak{g}\otimes t$ up to a central map since $\mathfrak{g}\otimes t^2 \subset Z_{L_n}(L')$. Let $g_n(x\otimes t) = h(x)\otimes t$. Then $h : \mathfrak{g} \to \mathfrak{g}$ is a commuting map on $\mathfrak{g}$ which has to be a scalar map by Corollary 3.3. From this one can deduce that $g_n(\mathfrak{g} \otimes t^2F[t]) \subset \mathfrak{g} \otimes t^2$.

It is clear that any special commuting map is central. Thus every commuting map on L_n is a sum of a scalar map (induced) and a central map.

Step 2. Find all commuting maps on L_{n-1}.

For any commuting map $f_{n-1} : L \to L_n$ with

$$f_{n-1}(x \otimes t + y \otimes t^2) = ax \otimes t + (h_1(x) + h_2(y)) \otimes t^2,$$

where $a \in F$ and h_1, h_2 are linear maps on $\mathfrak{g}$, let $g_{n-1} : L \to L_{n-1}$ be a commuting map such that $\tilde{g}_{n-1} = f_{n-1}$. Up to a central map we may assume that

$$g_{n-1}(x \otimes t) = ax \otimes t + h_1(x) \otimes t^2, \ \ g_{n-1}(x \otimes t^2) = h_2(x) \otimes t^2.$$

Using similar arguments as in Step 1 we deduce that $h_1(x) = bx$ for some $b \in F$ and $h_2(x) = ax$. From this one can deduce that $g_{n-1}(\mathfrak{g} \otimes t^3 F[t]) \subset \mathfrak{g} \otimes t^3$.

In this case it is also clear that any special commuting map is central. Thus every commuting map on L_{n-1} is a sum of a central map and a map g_{n-1}.

Continuing in this manner we deduce that, up to a central map, every commuting map $f : L \to L$ is of the form

$$f(x_k \otimes t^k) = x_k \otimes \sum_{j=k}^{2n} a_{j-k+1} t^j, \forall x_k \in \mathfrak{g},$$

where $a_k \in F$. One can further see that f lies in the centroid of L. □

We have seen from the above examples that our methods cover a variety of results from the literature. Theoretically, one can repeatedly use Corollaries 3.3 and 3.8 to find all commuting linear maps on various Lie algebras.

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