ALGEBRAS IN WHICH NONSCALAR ELEMENTS HAVE SMALL CENTRALIZERS

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Abstract. We describe algebras in which the centralizer of every nonscalar element is equal to the subalgebra generated by this element, and finite dimensional algebras (over perfect fields) in which the centralizer of every nonscalar element is commutative.

1. Introduction

Let A be a unital algebra over a field F. We identify F with $F \cdot 1$, the set of scalar multiples of 1. Let $C(a)$ denote the centralizer of an element $a \in A$ in A. The goal of the paper is to classify algebras in which every nonscalar element has *trivial centralizer*, i.e.,

(TC) $C(a) = F[a]$ for every $a \in A \setminus F$,

and finite dimensional algebras (over perfect fields) in which every nonscalar element has a *commutative centralizer*, i.e.,

(CC) $C(a)$ is commutative for every $a \in A \setminus F$.

In Theorem 3.1 we will show that only subalgebras of the 2×2 matrix algebra $M_2(F)$ and certain finite dimensional central division algebras satisfy (TC); in particular, every algebra satisfying (TC) is finite dimensional. The class of finite dimensional algebras satisfying (CC) is somewhat larger and will be described in Theorem 6.1. We do not consider infinite dimensional algebras satisfying (CC) as this class of algebras seems to be too broad. For example, it includes free algebras and the first Weyl algebra (cf. [3, Example 2]).

With $M_2(F)$ and the quaternions $\mathbb{H}$ presenting themselves as immediate examples of algebras satisfying (TC), we have found this condition interesting in its own right. Apparently quite similar conditions in rings have been studied in [1]; however, as it is evident from the results, these conditions are of a different nature and can occur mostly in finite rings. We also remark that a similar yet slightly less restrictive condition in groups has been considered recently in [2].

Our main motivation for studying the condition (CC) comes from the paper by Dolžan, Klep, and Moravec [3] dealing with *weakly commutative transitive* rings. These are rings in which $C(a)$ is commutative for every *noncentral* element a. Note that an algebra satisfying (CC) can be described as a weakly commutative transitive algebra which is either commutative or *central*, i.e., its center consists of scalar multiples of unity. Restricting to such weakly commutative transitive algebras makes the classification problem more approachable. We remark that [3] primarily deals with finite rings and so has only a very small overlap with the

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present paper. Let us also mention the paper [4] which classifies finite dimensional Lie algebras (over algebraically closed fields with characteristic 0) whose nonzero elements have commutative centralizers. Finally, it should be pointed out that the concept of (weak) commutative transitivity has an origin in group theory where it has a much richer history (cf. [3, 4]).

2. Introductory remarks

For $S \subseteq A$ we write $C(S)$ for the *centralizer* of S in A. Thus,

$$C(S) = \{x \in A \mid xs = sx \ \text{ for all } s \in S\}.$$

Instead of $C(\{a\})$ we write $C(a)$. Note that $C(a) = C(F[a])$ and that $F[a] \subseteq C(a)$.

3. Algebras in which nonscalar elements have trivial centralizers

Borrowing the terminology from group theory, we will say that a subalgebra S of A is *self-centralizing* if $C(S) \subseteq S$. We will be interested in unital algebras in which every subalgebra properly containing F is self-centralizing. Obviously, it is enough to consider only algebras generated by a single element, and so our condition is equivalent to the condition

$$C(a) = F[a] \quad \text{for every } a \in A \setminus F. \tag{1}$$

An immediate example of an algebra satisfying (1) is $M_2(F)$, the algebra of 2×2 matrices over F; indeed, one easily checks that $C(a) = F + Fa$ for every $a \in M_2(F) \setminus F$. A slightly less obvious example is a finite dimensional division algebra D in which every subfield different from F is maximal. Indeed, if $a \in D \setminus F$, then $F[a]$ is a subfield of D (for a is algebraic) and hence every $d \in C(a)$ lies in $F[a]$ for otherwise the subalgebra generated by a and d would be a subfield properly containing $F[a]$. If D is commutative, then this condition can be read as that D is a finite field extension of F such that there are no intermediate fields between F and D. If D is not commutative, then it is necessarily central (i.e., its center is F). Every central division algebra D of prime degree p (i.e., of dimension p^2) has this property. Indeed, this is because the dimension of every subfield K of D divides the dimension of D, therefore all subfields different from F are of the same dimension p and hence they are automatically maximal. A simple concrete example with $p = 2$ and $F = \mathbb{R}$ is the division algebra of quaternions $\mathbb{H}$. As far as we know, however, central division algebras whose nontrivial subfields are all maximal are not yet fully understood. At any rate, they do exist and they do satisfy (1). Finally, let us point out that if A satisfies (1), then so do its (unital) subalgebras.

The following theorem shows that there are no other algebras satisfying (1) apart from those mentioned in the previous paragraph.

Theorem 3.1. *Let A be a unital algebra over a field F. Then $C(a) = F[a]$ for every $a \in A \setminus F$ if and only if one of the following statements holds:*

(i) *A can be embedded into $M_2(F)$.*
(ii) *A is a finite dimensional division algebra in which every subfield different from F is maximal.*

Proof. The proof of the "if" part is straightforward and has already been outlined above. Therefore we only prove the "only if" part. Thus, assume that A satisfies $C(a) = F[a]$ for every $a \in A \setminus F$.

Let us first consider the case where A contains an idempotent e different from 0 and 1. Since $eAe \subseteq C(e) = F + Fe$ it clearly follows that $eAe = Fe$. Similarly, $(1-e)A(1-e) = F(1-e)$. Further, since the elements in $eA(1-e)$ commute among themselves, it is easy to see that $eA(1-e)$ is either 0 or is 1-dimensional. The same is true for $(1-e)Ae$. If both $eA(1-e)$ and $eA(1-e)$ are 0, then $A = Fe \oplus F(1-e)$ is isomorphic to $F \times F$, and hence to the subalgebra of $M_2(F)$ consisting of all diagonal matrices. Suppose that $eA(1-e) \neq 0$ and $(1-e)Ae = 0$. Taking $0 \neq n \in eA(1-e)$ we then have $A = Fe \oplus F(1-e) \oplus Fn$ with $n^2 = 0$, $en = n$, and $ne = 0$. It is obvious that A is isomorphic to the subalgebra of $M_2(F)$ consisting of all upper triangular matrices. The same is true if $eA(1-e) = 0$ and $(1-e)Ae \neq 0$. Assume, finally, that $eA(1-e) \neq 0$ and $(1-e)Ae \neq 0$. Pick $0 \neq n \in eA(1-e)$ and $0 \neq m \in (1-e)Ae$. Thus, $A = Fe \oplus F(1-e) \oplus Fn \oplus Fm$, $nm = \alpha e$ and $mn = \beta(1-e)$ for some $\alpha, \beta \in F$. Hence $\beta m = (mn)m = m(nm) = \alpha m$, implying that $\alpha = \beta$. If $\alpha = \beta = 0$, then $m \in C(n) = F + Fn$ which is easily seen to be impossible for $n \in eA(1-e)$ and $m \in (1-e)Ae$. Thus, $\alpha = \beta \neq 0$, and by replacing n by $\alpha^{-1}n$ we may in fact assume that $\alpha = \beta = 1$. It is now clear that $A \cong M_2(F)$. We have thereby shown that A satisfies (i) in case it contains a nontrivial idempotent. From now on we assume that 0 and 1 are the only idempotents in A.

Since $a \in C(a^2) = F[a^2]$ whenever $a^2 \notin F$, every element in A is algebraic over F. That is to say, the algebra $F[a]$ is finite dimensional for every $a \in A$. Note that our assumption on A implies that every commutative subalgebra of A is of the form $F[a]$ for some $a \in A$. Accordingly, every commutative subalgebra of A is finite dimensional. But then A itself is finite dimensional by the result of Laffey [5].

Let N be the radical of A, i.e., the (unique) maximal nilpotent ideal of A. As is well-known, idempotents in A/N can be lifted to idempotents in A. Since we have assumed that A has no nontrivial idempotents, the same is true for A/N. As a semisimple algebra, A/N is then necessarily a division algebra by Wedderburn's structure theorem. This means that A is a local ring. Every element in $A \setminus N$ is thus invertible (indeed, if $b + N$ is the inverse of $a + N$ in A/N, then $(ab-1)^s = (ba-1)^s = 0$, implying that a is invertible in A).

Suppose that $N \neq 0$. Let $s \geq 2$ be such that $N^s = 0$ and $N^{s-1} \neq 0$. Take $0 \neq r \in N^{s-1}$ and $b \in N$. Then $rb = 0 = br$. In particular, $b \in C(r)$ and hence $b = \lambda_0 + \lambda_1 r + \cdots + \lambda_m r^m$ for some $\lambda_i \in F$. Multiplying by r it follows that $\lambda_0 r = 0$, and hence $\lambda_0 = 0$. Now, multiplying $b = \lambda_1 r + \cdots + \lambda_m r^m$ by any element $b' \in N$ it follows that $bb' = 0$. Thus, $N^2 = 0$ (and so $s = 2$). Moreover, each $b \in N$ is a scalar multiple of r, i.e., $N = Fr$. Consequently, for every $a \in A$ there exists $\lambda_a \in F$ such that $ar = \lambda_a r$. Since elements in $A \setminus N$ are invertible and $(a - \lambda_a)r = 0$ it follows that $a - \lambda_a \in N = Fr$. Therefore $A = F \oplus Fr$ and A is isomorphic to the subalgebra of $M_2(F)$ consisting of all matrices of the form $\left[\begin{smallmatrix} \lambda & \mu \\ 0 & \lambda \end{smallmatrix}\right]$, $\lambda, \mu \in F$.

We may now assume that $N = 0$ and hence that A is a finite dimensional division algebra. Suppose that A is central, i.e., its center is F. If K is a subfield of A which contains a noncentral element a, and L is another subfield of A which contains K, then $L \subseteq C(a) = F[a]$, so that $L = K$. Thus, (ii) holds in this case. Finally, assume that A is not central, i.e., its center contains a nonscalar element c. Then A is commutative since $A = C(c) = F[c]$. Note that (ii) holds in this case, too. □

4. Finite dimensional semisimple nCT algebras

An algebra A is called *weakly commutative transitive (wCT)* if $C(a)$ is commutative for every $a \in A \setminus Z(D)$ [3]. We will consider a version of this condition, namely,

$$C(a) \text{ is commutative for every } a \in A \setminus F. \tag{2}$$

Let us call an algebra satisfying (2) *nearly commutative transitive* (nCT). Note that an nCT algebra is either commutative or is a central wCT algebra. Examples of infinite dimensional central wCT algebras are the first Weyl algebra $\mathcal{A}_1$ over $\mathbb{C}$ and every free algebra $F\langle X\rangle$ (cf. [3, Example 2]). The class of nCT algebras is thus considerably larger than the class of algebras satisfying (1). We will restrict ourselves to finite dimensional algebras over a field F, for which we will assume that it is perfect (so that Wedderburn's principal theorem can be used). Recall that fields of characteristic 0, algebraically closed fields, and finite fields are all examples of perfect fields.

Lemma 4.1. *Let D be a finite dimensional central division algebra. Then the following conditions are equivalent:*

(a) *D is nCT (or, equivalently, wCT).*
(b) *Every proper subalgebra of D is commutative.*
(c) *Every subfield of D different from F is maximal.*
(d) *$C(a) = F[a]$ for every $a \in A \setminus F$.*

Proof. (a)$\Longrightarrow$(b). Take a noncommutative subalgebra S of D. Of course, S itself is a division algebra. By the double centralizer theorem we have $[D : F] = [S : F][C(S) : F]$ and $C(C(S)) = S$. If S is a proper subalgebra of D, then $C(S)$ contains a nonscalar element a and hence $S \subseteq C(a)$. Therefore $C(a)$ is noncommutative and D is not wCT.

(b)$\Longrightarrow$(c). Let K be a subfield of D. Then K is the center of $C(K)$ (see, e.g., [6, Lemma 3.1.8]). Assuming (b) it follows that $K = C(K)$, which readily implies that K is maximal.

(c)$\Longrightarrow$(d). Since $F[a]$ is a subfield of D different from F if $a \in D \setminus F$, it must be maximal if (c) holds. But then $C(a)$ cannot contain an element outside $F[a]$ for otherwise a and such an element would generate a subfield greater than $F[a]$.

(d)$\Longrightarrow$(a). Trivial. □

It is now easy to describe all finite dimensional semisimple nCT algebras.

Proposition 4.2. *Let A be a finite dimensional semisimple algebra. Then A is* nCT *if and only if one of the following statements holds:*

(i) *A is commutative.*
(ii) *$A \cong M_2(F)$.*
(iii) *A is a central division algebra in which every proper subalgebra different from F is a maximal subfield.*

Proof. In view of Lemma 4.1 it suffices to prove the "only if" part. Thus, let A be nCT, and assume that it is not commutative. Then it is a central algebra, and so Wedderburn's structure theorem obviously implies that it is simple, and moreover, that it is isomorphic to $M_n(D)$ where D is a central division algebra. It is clear that n cannot be greater than 2 since otherwise the matrix unit e_{11} would have a noncommutative centralizer. For the same reason

D must be commutative if $n = 2$, and hence (ii) holds. If $n = 1$, then $A = D$ satisfies the conditions of Lemma 4.1, i.e., (iii) holds. □

Corollary 4.3. *A noncommutative finite dimensional semisimple algebra A is nCT if and only if $C(a) = F[a]$ for every $a \in A \setminus F$.*

5. Trivial extensions with admissible bimodules

Let M be a unital bimodule over an algebra A. Recall that the vector space $A \oplus M$ becomes a (unital) algebra, called the *trivial extension of A by M*, if we define multiplication by

$$(a, m) \cdot (a', m') = (aa', am' + ma').$$

Identifying A with $A \oplus 0$ and M with $0 \oplus M$ we see that A is a subalgebra of $A \oplus M$, M is an ideal of $A \oplus M$, and $M^2 = 0$.

For reasons that will become clear soon, we will be interested in a (unital) bimodule $M \neq 0$ satisfying the following condition: For all $a \in A$ and $m \in M$, $am = ma$ implies $a \in F$ or $m = 0$. Let us call such a bimodule M an *admissible bimodule*.

Example 5.1. Let A be a field extension of F and take an automorphism φ of A such that $\varphi(a) \neq a$ whenever $a \in A \backslash F$. Endow the space $M = A$ with the bimodule structure $a \cdot m = am$, $m \cdot a = m\varphi(a)$. Then M is admissible.

Example 5.2. Let M be a field extension of F, and let K, L be intermediate fields such that $K \cap L = F$. Set $A = K \times L$. Note that by defining $(x, y) \cdot m = xm$ and $m \cdot (x, y) = my$, M becomes an admissible A-bimodule.

Example 5.3. The direct sum of admissible bimodules is admissible.

Lemma 5.4. *If A is an nCT algebra and M is a admissible A-bimodule, then the trivial extension of A by M is a central nCT algebra.*

Proof. One can easily check that $A \oplus M$ is central. Pick a nonscalar element $(b, u) \in A \oplus M$. We must show that its centralizer in $A \oplus M$ is commutative. Thus, assume that $[(x, m), (b, u)] = [(y, n), (b, u)] = 0$. Our goal is to prove that $[(x, m), (y, n)] = 0$. That is, we want to establish the following:

$$[x, b] = [y, b] = 0, [x, u] = [b, m], [y, u] = [b, n] \Longrightarrow [x, y] = 0, [x, n] = [y, m] \tag{3}$$

(note that we are now using the notation $[\,\cdot\,,\,\cdot\,]$ also for commutators of elements from A and M).

If $b \in F$, then $u \neq 0$ and $[x, u] = [y, u] = 0$, yielding $x, y \in F$. Thus, (3) holds in this case. Assume therefore that $b \notin F$. Since A is nCT, $[x, b] = [y, b] = 0$ implies $[x, y] = 0$. Further, using the Jacobi identity we have

$$[[x, n], b] = [x, [n, b]] + [n, [b, x]] = [x, [n, b]] = [x, [u, y]],$$

and

$$[[y, m], b] = [m, [b, y]] + [y, [m, b]] = [y, [u, x]] = [[y, u], x] + [[x, y], u] = [x, [u, y]].$$

Thus, $[[x, n] - [y, m], b] = 0$. Since M is admissible, the desired conclusion $[x, n] = [y, m]$ follows. □

Example 5.2 shows that an algebra with an admissible module may contain a nontrivial idempotent. However, we have

Lemma 5.5. *If an algebra A has an admissible bimodule M, then A does not contain three pairwise ortogonal nonzero idempotents whose sum is* 1*.*

Proof. Suppose e_1, e_2, e_3 were pairwise orthogonal idempotents in A with $e_1 + e_2 + e_3 = 1$ and $e_i \neq 0$. Given any $m \in M$, $e_i m e_j$ must be zero for $(e_i m e_j)e_k = e_k(e_i m e_j)$ where $k \notin \{i, j\}$. But then $m = (e_1 + e_2 + e_3)m(e_1 + e_2 + e_3) = 0$ for every $m \in M$ − a contradiction. □

The next lemma in paricular shows that finite dimensional central simple algebras different from F do not have admissible bimodules.

Lemma 5.6. *Let A be a finite dimensional central simple algebra, and let M be a nonzero unital A-bimodule. Then for every $a \in A$ there exists $0 \neq m \in M$ such that $am = ma$.*

Proof. We can turn M into a left $A \otimes A^o$-module by setting $(a \otimes b)m = amb$. Recall that $A \otimes A^o \cong \mathrm{End}_F(A)$ via $a \otimes b \mapsto L_a R_b$ where $L_a(x) = ax$ and $R_b(x) = xb$. Of course, we can consider M also as an $\mathrm{End}_F(A)$-module. Since $\mathrm{End}_F(A)$ is simple, M is equal to the (direct) sum of a family of simple submodules. Let M_1 be a simple submodule. Then M_1 is isomorphic to A. Given any $a \in A$, the endomorphism $L_a - R_a$ has a nonzero kernel and hence there exists $0 \neq m \in M_1$ such that $(L_a - R_a)m = 0$. That is, $(a \otimes 1 - 1 \otimes a)m = 0$, yielding $am = ma$. □

6. Classifying finite dimensional nCT algebras

We now have enough information to classify finite dimensional nCT algebras over perfect fields.

Theorem 6.1. *A finite dimensional algebra over a perfect field F is nCT if and only if one of the following statements holds:*

(i) *A is commutative.*
(ii) *$A \cong M_2(F)$.*
(iii) *A is a central division algebra in which every proper subalgebra different from F is a maximal subfield.*
(iv) *A is the trivial extension of an extension field K of F by an admissible K-bimodule.*
(v) *A is the trivial extension of the direct product $K \times L$ of two extension fields K, L of F by an admissible $K \times L$-bimodule.*

Proof. The "if" part follows from Theorem 3.1 and Lemma 5.4. To prove the "only if" part, assume that A is nCT and is not commutative (so that A is central).

If A is semisimple, then it satisfies (ii) or (iii) by Proposition 4.2. We may thus assume that the radical N of A is not 0. Let $s \geq 2$ be such that $N^s = 0$ and $N^{s-1} \neq 0$. Since $N \subseteq C(r)$ for every $r \in N^{s-1}$ it follows that N is commutative. Given any $x \in A$ and $n, m \in N$ we thus have $m(xn) = (xn)m$ and, on the other hand, $(mx)n = nmx$. Hence nm lies in the center of A and is thus a scalar. But then $nm = 0$. That is, $N^2 = 0$.

Wedderburn's principal theorem states that A is the vector space direct sum of N and a subalgebra B of A such that $B \cong A/N$. Since N is an B-bimodule and $N^2 = 0$, we may regard A as the trivial extension of B by N. Moreover, if B is commutative, then N is admissible. Namely, if, in this case, $bm = mb$ holds for some $b \in B$ and $0 \neq m \in M$, then it follows that

$[b, N] = 0$ since $N \subseteq C(m)$. But then b commutes with every element in A and is thus a scalar.

By Wedderburn's structure theorem we know that $B \cong B_1 \times \cdots \times B_r$ for some simple algebras B_i. Suppose first that $r = 1$. If $B = B_1$ is not commutative then its center may consist only of scalars for A is nCT. Thus, B is a central simple algebra and, as a subalgebra of A, it is nCT. Hence B is either a division algebra or is isomorphic to $M_2(F)$ by Proposition 4.2. Pick $b \in B \setminus F$. By Lemma 5.6 there exists $0 \neq m \in N$ such that $bm = mb$. Since m commutes with every element in N it follows that $[b, N] = 0$. Thus, $b(xm) = (xm)b = x(mb) = xbm$ for every $x \in B$. That is, $[b, x]m = 0$. If B is a division algebra, then $[b, x]$ is invertible whenever it is not zero, leading to a contradiction $m = 0$. Similarly, if $B \cong M_2(F)$ then it has matrix units e_{ij}, $1 \leq i, j \leq 2$, and by setting e_{12}, e_{21}, and $e_{12} + e_{21}$ for x one easily infers that $[b, x]$ is invertible for some x, again leading to a contradiction. Therefore B is commutative. Since it is also simple, it is a field extension of F. As observed in the preceding paragraph, N is an admissible B-bimodule, i.e., (iv) holds. Assume now that $r \geq 2$. Then each B_i must be commutative since it is contained in the centralizer of each element from the other B_j's. Thus, each B_i is a field extension of F. Moreover, B is commutative and N is therefore an admissible B-bimodule. Lemma 5.5 implies that $r = 2$. That is, (v) holds. □

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