COMPACTNESS CONDITIONS FOR ELEMENTARY OPERATORS

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Abstract. Various topics concerning compact elementary operators on Banach algebras are studied: their ranges, their coefficents, and the structure of algebras having nontrivial compact elementary operators. In the first part of the paper we consider separately elementary operators of certain simple types. In the second part we obtain our main results which deal with general elementary operators.

1. Introduction

Let $\mathcal{A}$ be an algebra. If $\mathcal{A}$ does not have an identity element, then we denote by $\mathcal{A}^1$ the algebra obtained from $\mathcal{A}$ by adjoining the identity element (if $\mathcal{A}$ has an identity element, then we set $\mathcal{A}^1 = \mathcal{A}$). Given $a \in \mathcal{A}^1$, we define multiplication operators $L_a, R_a : \mathcal{A} \to \mathcal{A}$ by $L_a(x) = ax$ and $R_a(x) = xa$. An operator E from an algebra $\mathcal{A}$ into itself is called an *elementary operator* on $\mathcal{A}$ if there exist $a_i, b_i \in \mathcal{A}^1$ such that $E = \sum_{i=1}^{n} L_{a_i} R_{b_i}$. The elements a_i, b_i will be called the *coefficients* of E.

In our recent paper [9] compact derivations on Banach algebras were considered. The present paper continues this line of investigations. Our main purpose is to study elementary operators on Banach algebras that are simultaneously compact operators. Occasionally we will also consider elementary operators on general algebras that have a finite rank.

In the first sections 2-6 we treat elements $a, b \in \mathcal{A}$ such that some of the operators L_a, R_a, $L_a R_a$, $L_a - R_b$ are compact or of finite rank. We shall see that these conditions have an impact on the algebraic nature of the algebra in question, so that these local properties (concerned with a single element a) determine the global structure of $\mathcal{A}$. A typical conclusion is that $\mathcal{A}$ contains a central idempotent e such that the ideal $e\mathcal{A}$ is finite dimensional (i. e. L_e has finite rank). The results from these sections have rather short proofs and they vary from elementary observations to somewhat more profound statements (Theorem 3.2, for example).

After the first "warm-up" sections we proceed to study general elementary operators (sections 7-10). Section 7 is devoted to an auxiliary algebraic result which is in subsequent sections used as a crucial technical tool. Section 8 is closely connected to the theory of radicals in Banach algebras developed in

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[25]. This theory makes it possible for us to obtain some information about the range of a compact elementary operator; for instance, it turns out that the spectrum of every element in the range is finite or countable. In section 9 we show that the existence of a nonzero compact elementary operator on a semisimple Banach algebra $\mathcal{A}$ yields a nice structural property of $\mathcal{A}$: it has a nonzero socle. Section 10 is concerned with the question whether the coefficients of a compact elementary operator must be compact elements (a is said to be a compact element if L_aR_a is a compact operator). This question has been studied before, see e. g. [13, 19, 24, 26]. However, we shall use a rather different approach, based on the (purely algebraic) concept of the extended centroid of a prime algebra. The theorem that we obtain generalizes and unifies two results from the literature, the theorem by Fong and Sourour on $\mathcal{B}(X)$ [13] and the theorem by Mathieu on C^*-algebras [19].

Throughout the paper we will combine analytic and algebraic tools and techniques. Although the main goal of the paper is the study of compactness conditions, in several results we treat algebras over arbitrary fields. Most of them are later used in the analytic setting, but they might also be of some interest in their own right.

2. Completely finite rank elements

We begin with an introductory algebraic section. We shall gather together several elementary assertions, some of which are perhaps already known. Anyhow, the proofs are short.

We assume throughout this section that $\mathcal{A}$ is an algebra over a field F. We shall say that $a \in \mathcal{A}$ is a *completely finite rank element* if both L_a and R_a are finite rank operators. In other words, both the right ideal $a\mathcal{A}$ and the left ideal $\mathcal{A}a$ are finite dimensional. The set of all completely finite rank elements of $\mathcal{A}$ will be denoted by $\mathcal{A}_{\mathrm{cf}}$. It is clear that $\mathcal{A}_{\mathrm{cf}}$ is an ideal of $\mathcal{A}$.

Lemma 2.1. *$\mathcal{A}_{\mathrm{cf}}$ is the sum of all finite dimensional ideals of $\mathcal{A}$. In particular, every element in $\mathcal{A}_{\mathrm{cf}}$ generates a finite dimensional ideal in $\mathcal{A}$.*

Proof. If a lies in some finite dimensional ideal $\mathcal{J}$ of $\mathcal{A}$, then $a\mathcal{A} \cup \mathcal{A}a \subset \mathcal{J}$ and so $a \in \mathcal{A}_{\mathrm{cf}}$. Therefore it suffices to show that every $a \in \mathcal{A}_{\mathrm{cf}}$ generates a finite dimensional ideal of $\mathcal{A}$. Since L_a is of finite rank, there exists a finite basis $\{b_1, \dots, b_n\}$ of $a\mathcal{A}$. Clearly $R_{b_1}, \dots, R_{b_n}$ have finite ranks, and so

$$\mathcal{I} = Fa + L_a(\mathcal{A}) + R_a(\mathcal{A}) + \sum_{i=1}^{n} R_{b_i}(\mathcal{A})$$

is finite dimensional. Note that $\mathcal{I}$ is the ideal of $\mathcal{A}$ generated by a. □

By a *minimal ideal* of an algebra $\mathcal{A}$ we mean of course a nonzero ideal $\mathcal{J}$ of $\mathcal{A}$ such that $0 \subset \mathcal{K} \subset \mathcal{J}$, where $\mathcal{K}$ is an ideal of $\mathcal{A}$, implies $\mathcal{K} = 0$ or $\mathcal{K} = \mathcal{J}$, and by a *minimal central idempotent* in $\mathcal{A}$ we mean a central idempotent e in $\mathcal{A}$ such that $ef = f$, where f is a central idempotent in $\mathcal{A}$, implies $f = 0$ or $f = e$. As usual, by $M_n(\mathcal{B})$ we denote the algebra of $n \times n$ matrices over the algebra $\mathcal{B}$.

Lemma 2.2. *Let $\mathcal{J}$ be a non-nilpotent finite dimensional ideal of $\mathcal{A}$.*

(i) *If $\mathcal{J}$ is minimal, then there is a minimal central idempotent e in $\mathcal{A}$ such that $\mathcal{J} = e\mathcal{A} \cong M_n(\mathcal{D})$ for some positive integer n and some finite dimensional division algebra $\mathcal{D}$.*

(ii) *If F is a perfect field, then there exists a nonzero idempotent $u \in \mathcal{J}$ and a finite dimensional nilpotent ideal $\mathcal{N}$ of $\mathcal{A}$ such that $ux - xu \in \mathcal{N}$ for all $x \in \mathcal{A}$.*

Proof. We set $\mathcal{N} = \mathrm{rad}(\mathcal{J})$, the (Jacobson) radical of the algebra $\mathcal{J}$. Since $\mathcal{N} = \mathcal{J} \cap \mathrm{rad}(\mathcal{A})$, $\mathcal{N}$ is an ideal of $\mathcal{A}$. Moreover, as the radical of a finite dimensional algebra, $\mathcal{N}$ is nilpotent.

Suppose that $\mathcal{J}$ is a minimal ideal. Since $\mathcal{N} \subset \mathcal{J}$ and $\mathcal{J}$ is non-nilpotent, we have $\mathcal{N} = 0$. That is, $\mathcal{J}$ is a semisimple algebra. By the Wedderburn structure theorem $\mathcal{J}$ in particular contains an identity element e, so that $\mathcal{J} = e\mathcal{A}$ and e is a central idempotent in $\mathcal{A}$. This implies that every ideal of the algebra $\mathcal{J}$ is also an ideal of $\mathcal{A}$. Therefore $\mathcal{J}$ is a simple algebra, and as such, again by the Wedderburn theorem, isomorphic to $M_n(\mathcal{D})$ for some positive integer n and a finite dimensional division algebra $\mathcal{D}$. If f is a central idempotent in $\mathcal{A}$ such that $ef = f$, then $0 \subset f\mathcal{A} \subset e\mathcal{A} = \mathcal{I}$ and so $f\mathcal{A} = 0$ or $f\mathcal{A} = e\mathcal{A}$, which implies $f = 0$ or $f = e$. Thus (i) is proved.

To prove (ii), we first apply, using the assumption that F is perfect, the Wedderburn principal theorem (see e.g. [20, Theorem 2.5.37]) to obtain that $\mathcal{J} = \mathcal{B} \oplus \mathcal{N}$ (the vector space direct sum), where $\mathcal{B}$ is a subalgebra of $\mathcal{J}$ isomorphic to $\mathcal{J}/\mathcal{N}$. Since $\mathcal{J}$ is non-nilpotent, $\mathcal{B} \neq 0$. Thus $\mathcal{B}$ is a finite dimensional semisimple algebra and so in particular it has an identity element $u \neq 0$. For any $y \in \mathcal{J}$ we have $y - uy, y - yu \in \mathcal{N}$. If $x \in \mathcal{A}$ is arbitrary then $ux, xu \in \mathcal{J}$, whence $ux - uxu, xu - uxu \in \mathcal{N}$ and therefore $ux - xu = (ux - uxu) - (xu - uxu) \in \mathcal{N}$. □

We shall say that an algebra $\mathcal{A}$ is *finitely semiprime* if $\mathcal{A}$ has no nonzero finite dimensional nilpotent ideals. Besides semiprime algebras this class of algebras for instance also includes amenable Banach algebras (see e.g. [15]). Clearly every finite dimensional minimal ideal $\mathcal{I}$ of a finitely semiprime algebra $\mathcal{A}$ is of the form $\mathcal{J} = e\mathcal{A} \cong M_n(\mathcal{D})$.

Corollary 2.3. *Let $\mathcal{A}$ be a finitely semiprime algebra such that $\mathcal{A}_{\mathrm{cf}} \neq 0$. Then $\mathcal{A}_{\mathrm{cf}}$ is the direct sum of its finite dimensional minimal ideals. Accordingly, $\mathcal{A} = \mathcal{A}_{\mathrm{cf}}$ if and only if $\mathcal{A} \cong \oplus_\alpha M_{n_\alpha}(\mathcal{D}_\alpha)$ where n_α is a positive integer and $\mathcal{D}_\alpha$ is a finite dimensional division algebra.*

Proof. Let $a \in \mathcal{A}_{\mathrm{cf}}$ and let $\mathcal{J}$ be the ideal of $\mathcal{A}$ generated by a. By Lemma 2.1, $\mathcal{J}$ is finite dimensional. Since $\mathrm{rad}(\mathcal{J})$ is a finite dimensional nilpotent ideal of $\mathcal{A}$ it follows that $\mathcal{J}$ is semisimple. Using Wedderburn theorem and arguing similarly as in the proof of Lemma 2.2 we see that $\mathcal{J} = f\mathcal{A}$ for some central idempotent f in $\mathcal{A}$, and $\mathcal{J} = \mathcal{J}_1 \oplus \mathcal{J}_2 \oplus \ldots \oplus \mathcal{J}_n$ where each $\mathcal{J}_k$ is an ideal of $\mathcal{A}$ and a finite dimensional simple algebra. Clearly, $\mathcal{J}_k$ is a finite dimensional minimal ideal of $\mathcal{A}$. This shows that $\mathcal{A}_{\mathrm{cf}}$ is the sum of its finite dimensional minimal ideals. We still have to show that this sum is direct. It is clear that $\mathcal{I}_1 \cap \mathcal{I}_2 = 0$ for any different finite dimensional minimal ideals $\mathcal{I}_1$ and $\mathcal{I}_2$. Accordingly, if $\mathcal{I}_1 = e_1\mathcal{A}$ and $\mathcal{I}_2 = e_2\mathcal{A}$ where e_1, e_2 are central idempotents, then $e_1e_2 = 0$, and so $e_1\mathcal{I}_2 = 0$. From this we infer that a finite dimensional minimal ideal $\mathcal{I} = e\mathcal{A}$ has trivial intersection with the sum of all finite dimensional minimal ideals different from $\mathcal{I}$. □

The last assertion may be viewed as a slight extension of the classical Wedderburn structure theorem. We remark that if $\mathcal{A}$ is a normed algebra, then the sum in Corollary 2.3 is a topological direct sum.

3. Completely continuous elements

Until further notice we assume that $\mathcal{A}$ is a complex Banach algebra. Let $a \in \mathcal{A}$. By $\sigma(a)$ we denote the spectrum of a, and by $r(a)$ we denote its spectral radius. If $\mathcal{A}$ does not have an identity element, then $\sigma(a)$ is taken with respect to $\mathcal{A}^1$. As above, by $\mathrm{rad}(\mathcal{A})$ we denote the Jacobson radical of $\mathcal{A}$.

We say that a is a *completely continuous element* if both L_a and R_a are compact operators. The set $\mathcal{A}_{\mathrm{cc}}$ of all completely continuous elements in $\mathcal{A}$ is clearly a closed ideal of $\mathcal{A}$, and $\mathcal{A}_{\mathrm{cf}} \subset \mathcal{A}_{\mathrm{cc}}$.

One of motivations for studying completely continuous elements is their connection to compact derivations. If $a \in \mathcal{A}_{\mathrm{cc}}$ then $\mathrm{ad}\, a = L_a - R_a$ is a compact derivation. Conversely, if D is a compact derivation, then we see from the formulas $L_{D(a)} = [D, L_a]$ and $R_{D(a)} = [D, R_a]$ that the range of D lies in $\mathcal{A}_{\mathrm{cc}}$. This observation was used in our recent paper [9] studying compact derivations, and from the arguments in that paper one can extract some conclusion about completely continuous elements. Theorem 3.2 below, however, is considerably more general. We remark that in particular it yields a more refined information concerning compact derivations than given in [9, Theorem 2.1].

Suppose that $a \in \mathcal{A}_{\mathrm{cc}}$ is such that $r(a) > 0$, i. e. a is not quasinilpotent. Pick a nonzero $\lambda \in \sigma(a)$. Since L_a is a compact operator, λ is an isolated point in $\sigma(a)$. Therefore there exists a nonzero spectral idempotent $p \in \mathcal{A}^1$ of a corresponding to λ, given by

$$p = \frac{1}{2\pi i}\int_{\Gamma} (\mu - a)^{-1} d\mu, \tag{3.1}$$

where Γ is an appropriate contour in the complex plane.

Lemma 3.1. *Let $a \in \mathcal{A}_{\mathrm{cc}}$ and let $0 \neq \lambda \in \sigma(a)$. If p is a spectral idempotent of a corresponding to λ, then $p \in \mathcal{A}_{\mathrm{cf}}$. Moreover, p lies in the ideal of $\mathcal{A}$ generated by a.*

Proof. Since $L_{(\mu-a)^{-1}} = (\mu - L_a)^{-1}$ for all μ from the resolvent set of a, L_p is a spectral projection of L_a corresponding to λ. Similarly, R_p is a spectral projection of R_a corresponding to λ. From the classical theory of compact operators applied to compact operators L_a and R_a it follows that L_p and R_p are of finite rank, i. e. $p \in \mathcal{A}^1_{\text{cf}}$, and $(L_a - \lambda)L_p$ is a quasinilpotent operator on $\mathcal{A}^1$. Since $(L_a - \lambda)L_p$ is of finite rank, $(L_a - \lambda)L_p$ is a nilpotent operator on $\mathcal{A}^1$, whence there exists a positive integer n such that $(ap - \lambda p)^n = 0$. From this we readily infer that $\lambda^n p$, and hence also p, lies in the ideal of $\mathcal{A}$ generated by a (in particular, $p \in \mathcal{A}$). □

Theorem 3.2. *Let $\mathcal{A}$ be a Banach algebra and suppose there exists $a \in \mathcal{A}_{\text{cc}} \setminus \text{rad}\,(\mathcal{A})$. Then there exist a nonzero idempotent $u \in \mathcal{A}_{\text{cf}}$ and a finite dimensional nilpotent ideal $\mathcal{N}$ of $\mathcal{A}$ such that $ux - xu \in \mathcal{N}$ for all $x \in \mathcal{A}$. Moreover, u lies in the ideal of $\mathcal{A}$ generated by a.*

Proof. Let $\mathcal{I}$ be the ideal of $\mathcal{A}$ generated by a. If $r(a) = 0$ then, since $a \notin \text{rad}\,(\mathcal{A})$, there exists $a' \in a\mathcal{A} \subset \mathcal{I}$ with $r(a') > 0$. Therefore we may assume without loss of generality that $r(a) > 0$. Now we may use Lemma 3.1 which in particular tells us that there exists a nonzero idempotent $p \in \mathcal{A}_{\text{cf}} \cap \mathcal{I}$. Let $\mathcal{J}$ be the ideal of $\mathcal{A}$ generated by p. Note that $\mathcal{J}$ is finite dimensional by Lemma 2.1, it is not nilpotent since it contains p, and $\mathcal{J} \subset \mathcal{I}$. Now Lemma 2.2 (ii) yields the desired conclusion. □

Corollary 3.3. *Let $\mathcal{A}$ be a Banach algebra such that $\mathcal{A}_{\text{cc}} \setminus \text{rad}\,(\mathcal{A}) \neq 0$. Then $\mathcal{A}_{\text{cf}} \setminus \text{rad}\,(\mathcal{A}) \neq 0$, i.e. $\mathcal{A}$ has a nonzero finite dimensional ideal which is not contained in* rad $(\mathcal{A})$.

The existence of an element in $\mathcal{A}_{\text{cc}} \setminus \text{rad}\,(\mathcal{A})$ in general does not guarantee that $\mathcal{A}_{\text{cf}}$ contains nonzero central idempotents (just take the algebra of all $n \times n$ complex matrices that have nonzero entries only in the first row).

Corollary 3.4. *Let $\mathcal{A}$ be a finitely semiprime Banach algebra and suppose there exists $a \in \mathcal{A}_{\text{cc}} \setminus \text{rad}\,(\mathcal{A})$. Then there is a nonzero central idempotent $u \in \mathcal{A}_{\text{cf}}$. Moreover, u lies in the ideal of $\mathcal{A}$ generated by a.*

Our last result in this section will be also derived from Lemma 3.1. Let us point out that $\overline{\mathcal{A}_{\text{cf}}} \subset \mathcal{A}_{\text{cc}}$ are closed ideals of $\mathcal{A}$.

Corollary 3.5. *For any Banach algebra $\mathcal{A}$, $\mathcal{A}_{\text{cc}}/\overline{\mathcal{A}_{\text{cf}}}$ is a radical Banach algebra. Accordingly, if $\mathcal{A}/\overline{\mathcal{A}_{\text{cf}}}$ is semisimple, then $\overline{\mathcal{A}_{\text{cf}}} = \mathcal{A}_{\text{cc}}$.*

Proof. We have to show that $\mathcal{A}_{\text{cc}}/\overline{\mathcal{A}_{\text{cf}}}$ consists of quasinilpotents. Suppose this is not true, that is, suppose there is $a \in \mathcal{A}_{\text{cc}}$ such that $\sigma(a + \overline{\mathcal{A}_{\text{cf}}}) \neq \{0\}$. Pick $0 \neq \lambda \in \sigma(a + \overline{\mathcal{A}_{\text{cf}}})$, and let f be the spectral idempotent of $a + \overline{\mathcal{A}_{\text{cf}}}$ corresponding to λ. Of course, λ also belongs to $\sigma(a)$. Denote by p be the spectral idempotent of a corresponding to λ. Let $\varphi : \mathcal{A}^1 \to \mathcal{A}^1/\overline{\mathcal{A}_{\text{cf}}}$ be the quotient map. Since φ is a unital epimorphism, we have $\varphi((\mu - a)^{-1}) = (\mu - \varphi(a))^{-1}$ for all μ from the resolvent set of a. Applying φ to both sides of (3.1) we see that $\varphi(p) = f$, i.e., $f = p + \overline{\mathcal{A}_{\rm cf}}$. However, $p \in \mathcal{A}_{\rm cf}$ by Lemma 3.1, so that $f = 0$ – a contradiction.

Since $\mathcal{A}_{\rm cc}/\overline{\mathcal{A}_{\rm cf}}$ is a closed ideal of $\mathcal{A}/\overline{\mathcal{A}_{\rm cf}}$, the semisimplicity of $\mathcal{A}/\overline{\mathcal{A}_{\rm cf}}$ of course implies $\overline{\mathcal{A}_{\rm cf}} = \mathcal{A}_{\rm cc}$. □

4. Left (right) completely continuous elements

We say that $a \in \mathcal{A}$ is a *left* (resp. *right*) *completely continuous* element of $\mathcal{A}$ if L_a (resp. R_a) is a compact operator on $\mathcal{A}$. The set of all left (resp. right) completely continuous elements of $\mathcal{A}$ will be denoted by $\mathcal{A}_{\rm lcc}$ (resp. $\mathcal{A}_{\rm rcc}$). It is clear that $\mathcal{A}_{\rm lcc}$ and $\mathcal{A}_{\rm rcc}$ are closed ideals of $\mathcal{A}$, and $\mathcal{A}_{\rm cc} = \mathcal{A}_{\rm lcc} \cap \mathcal{A}_{\rm rcc}$. Further, by $\mathcal{A}_{\rm lcf}$ (resp. $\mathcal{A}_{\rm rcf}$) we denote the set of all elements $a \in \mathcal{A}$ such that L_a (resp. R_a) is of finite rank. Clearly $\mathcal{A}_{\rm lcf}$ and $\mathcal{A}_{\rm rcf}$ are ideals of $\mathcal{A}$ and $\mathcal{A}_{\rm cf} = \mathcal{A}_{\rm lcf} \cap \mathcal{A}_{\rm rcf}$. Note also that $\mathcal{A}_{\rm lcf}$ (resp. $\mathcal{A}_{\rm rcf}$) is the sum of all right (resp. left) finite dimensional ideals of $\mathcal{A}$.

To avoid a tedious repetition we shall formulate the results in this section only for left completely continuous elements; of course, analogous results can be proved for right completely continuous elements.

Some of the arguments from the previous section still work in the present context. For example, an analogue of Lemma 3.1 is true: If p is a spectral idempotent of $a \in \mathcal{A}_{\rm lcc}$ corresponding to a nonzero $\lambda \in \sigma(a)$, then $p \in \mathcal{A}_{\rm lcf}$. This can be used to prove an analogue of Corollary 3.5, i. e. $\mathcal{A}_{\rm lcc}/\overline{\mathcal{A}_{\rm lcf}}$ is a radical Banach algebra. On the other hand, this shows the existence of a nonzero element p in $\mathcal{A}_{\rm lcf}$. But there is another, simpler way to establish that $\mathcal{A}_{\rm lcf} \neq \{0\}$, which we will now show.

We say that a nonzero $b \in \mathcal{A}$ is a *left eigenvector* of $a \in \mathcal{A}$ corresponding to an *eigenvalue* $\lambda \in \mathbb{C}$ if $(a - \lambda)b = 0$. Note that every $a \in \mathcal{A}_{\rm lcc}$ with $r(a) > 0$ has nonzero eigenvalues.

Lemma 4.1. *Let $\mathcal{A}$ be a Banach algebra, and let $a \in \mathcal{A}_{\rm lcc}$. If b is a left eigenvector of a corresponding to a nonzero eigenvalue, then $b \in \mathcal{A}_{\rm lcf}$. Moreover, if $\mathcal{A}$ is semiprime, then $b \in \mathcal{A}_{\rm cf}$.*

Proof. Let $(a - \lambda)b = 0$ with a nonzero $\lambda \in \mathbb{C}$. Then $(L_a - \lambda)(b\mathcal{A}) = 0$. Since L_a is compact, the eigenspace $\ker(L_a - \lambda)$ is finite dimensional and so $b\mathcal{A}$ must also be finite dimensional, that is, $b \in \mathcal{A}_{\rm lcf}$. If $\mathcal{A}$ is semiprime, then Smyth's lemma [23] tells us that $\dim(\mathcal{A}b) < \infty$ as well, so that $b \in \mathcal{A}_{\rm cf}$. □

Is is easy to find an example of a (non-semiprime) algebra $\mathcal{A}$ containing an idempotent e such that, say, $\dim(e\mathcal{A}) < \infty$ but $\dim(\mathcal{A}e) = \infty$. This justifies the assumption that $\mathcal{A}$ is semiprime in the last assertion. If, however, we impose the stronger assumption on a that $a \in \mathcal{A}_{\rm cc}$, then the semiprimeness is redundant. Namely, as above we have that $\dim(b\mathcal{A}) < \infty$, while for the proof of $\dim(\mathcal{A}b) < \infty$ we can make use of dual operators: noting that $(R_a^* - \lambda)R_b^* = 0$ with R_a^* compact, it follows by the same argument as above that R_b^* is of finite rank, and so R_b has finite rank too.

Corollary 4.2. *Let $\mathcal{A}$ be a Banach algebra such that $\mathcal{A}_{\mathrm{lcc}} \setminus \mathrm{rad}\,(\mathcal{A}) \neq 0$. Then $\mathcal{A}_{\mathrm{lcf}} \neq 0$. Moreover, if $\mathcal{A}$ is semiprime, then $\mathcal{A}_{\mathrm{cf}}$ contains a nonzero central idempotent.*

Proof. As in the proof of Theorem 3.2 we see that there is $a \in \mathcal{A}_{\mathrm{lcc}}$ with $r(a) > 0$. Hence a has a nonzero eigenvalue, and by Lemma 4.1 every corresponding eigenvector lies in $\mathcal{A}_{\mathrm{lcf}}$. If $\mathcal{A}$ is semiprime, then it lies in $\mathcal{A}_{\mathrm{cf}}$, and then Corollary 2.3 and Lemma 2.2 (i) yield the desired conclusion. □

Prime algebras clearly do not contain central idempotents different from 0 and 1. The conclusion is therefore more definite in this case.

Corollary 4.3. *Let $\mathcal{A}$ be a prime Banach algebra. If $\mathcal{A}_{\mathrm{lcc}} \setminus \mathrm{rad}\,(\mathcal{A}) \neq 0$, then* $\dim(\mathcal{A}) < \infty$.

5. Compact elements

We say that $a \in \mathcal{A}$ is a *compact element* if $L_a R_a$ is a compact operator. The set $\mathcal{A}_{\mathrm{c}}$ of all compact elements in $\mathcal{A}$ is only a closed multiplicative semigroup in $\mathcal{A}$ in general.

Finally, we say that a is a *finite rank element* if $L_a R_a$ is a finite rank operator. The set of all finite rank elements in $\mathcal{A}$ will be denoted by $\mathcal{A}_{\mathrm{f}}$. Of course, $\mathcal{A}_{\mathrm{f}} \subset \mathcal{A}_{\mathrm{c}}$.

Our intention is to establish results on compact elements that are analogous to those from the previous sections. However, all the necessary work has basically already been done before, so we will just refer to the literature and the present section will be very short. The following lemma follows immediately from a slightly more general result [21, Lemma 3].

Lemma 5.1. [21] *Let $\mathcal{A}$ be a Banach algebra. If $\mathcal{A}_{\mathrm{c}} \setminus \mathrm{rad}\,(\mathcal{A}) \neq 0$, then $\mathcal{A}_{\mathrm{f}}$ contains a nonzero idempotent. In particular, $\mathcal{A}_{\mathrm{f}} \setminus \mathrm{rad}\,(\mathcal{A}) \neq 0$.*

In the case when $\mathcal{A}$ is a semiprime Banach algebra, $\mathcal{A}_{\mathrm{f}}$ coincides with $\mathrm{soc}(\mathcal{A})$, the *socle* of $\mathcal{A}$. For semisimple algebras this was proved a long time ago [1] and somewhat more recently it was extended to the semiprime case [10]. Moreover, algebraic versions of this result were obtained even more recently [7, 14] (more details about [7] are given in section 9). Recall that the socle of a semiprime algebra $\mathcal{A}$ is equal to the sum of all minimal left (resp. right) ideals each of which is necessarily of the form $\mathcal{A}e$ (resp. $e\mathcal{A}$) where e is a minimal idempotent in $\mathcal{A}$, i.e. an idempotent such that $e\mathcal{A}e$ is a division algebra. In the case when $\mathcal{A}$ is a Banach algebra we necessarily have $e\mathcal{A}e = \mathbb{C}e$. If $\mathcal{A}$ has no nonzero minimal left (or equivalently, right) ideals then we define $\mathrm{soc}(\mathcal{A}) = 0$.

Corollary 5.2. *Let $\mathcal{A}$ be a semiprime Banach algebra. Then $\mathcal{A}_{\mathrm{c}} \setminus \mathrm{rad}\,(\mathcal{A}) \neq 0$ if and only if $\mathrm{soc}(\mathcal{A}) \neq 0$.*

6. Compact generalized derivations

Following [6] we say that a linear map $\Delta : \mathcal{A} \to \mathcal{A}$ is a *generalized derivation* if there exists a derivation D on $\mathcal{A}$ such that $\Delta(xy) = \Delta(x)y + xD(y)$ for all $x, y \in \mathcal{A}$. Besides derivations, the other basic examples are *inner generalized derivations*, that is maps of the form $\Delta = L_a - R_b$ for some $a, b \in \mathcal{A}$ (in this case the associated derivation is $D = \operatorname{ad} b$).

In section 3 we mentioned that completely continuous elements naturally appear when studying compact derivations. Similarly the "one-sided" completely continuous elements appear when studying compact generalized derivations. If Δ is a compact generalized derivation, then we see from $R_{D(a)} = [\Delta, R_a]$ that $D(a) \in \mathcal{A}_{\mathrm{rcc}}$ for every $a \in \mathcal{A}$. The next result therefore follows from (the "right" version of) Corollary 4.2.

Corollary 6.1. *Let Δ be a compact generalized derivation on a Banach algebra $\mathcal{A}$. Suppose that the associated derivation D does not map $\mathcal{A}$ into* $\operatorname{rad}(\mathcal{A})$*. Then $\mathcal{A}_{\mathrm{rcf}} \neq 0$; moreover, if $\mathcal{A}$ is semiprime, then $\mathcal{A}_{\mathrm{cf}}$ contains a nonzero central idempotent.*

If $\Delta = L_a - R_b$ is an inner generalized derivation, then Corollary 6.1 says that if Δ is compact and b does not lie in the center modulo the Jacobson radical, then $\mathcal{A}_{\mathrm{rcf}} \neq 0$. Of course, similarly one can prove that that if Δ is compact and a does not lie in the center modulo the Jacobson radical, then $\mathcal{A}_{\mathrm{lcf}} \neq 0$.

In the semisimple case the above result gets a simple form.

Corollary 6.2. *Let $\mathcal{A}$ be a semisimple Banach algebra. If there exists a nonzero compact generalized derivation on $\mathcal{A}$, then $\mathcal{A}_{\mathrm{cf}}$ contains a nonzero central idempotent.*

Proof. If the associated derivation D is not zero, then we may use Corollary 6.1. So we may assume that $D = 0$. Note that then $\Delta L_a = L_{\Delta(a)}$ holds for every $a \in \mathcal{A}$, and so the result follows from Corollary 4.2. □

Corollary 6.2 was obtained for ordinary derivations in [9, Corollary 2.6].

There exists a commutative prime radical Banach algebra $\mathcal{A}$ such that $\mathcal{A}_{\mathrm{lcc}} = \mathcal{A}$ [11]. Thus, every L_a $(= R_a)$, $a \in \mathcal{A}$, is compact and quasinilpotent. Since $\mathcal{A}$ is prime, $\mathcal{A}_{\mathrm{lcf}} = \mathcal{A}_{\mathrm{cf}} = 0$, and therefore, because of commutativity of $\mathcal{A}$, even $\mathcal{A}_{\mathrm{f}} = 0$. In light of this example, the next proposition seems to be of some interest.

Proposition 6.3. *Let $\mathcal{A}$ be a Banach algebra. If there exists a compact inner generalized derivation on $\mathcal{A}$ which is not quasinilpotent, then $\mathcal{A}_{\mathrm{f}} \neq 0$.*

Proof. Let $\Delta = L_a - R_b$ be compact and non-quasinilpotent. Pick a nonzero $\lambda \in \sigma(\Delta)$. Then $X = \ker(\Delta - \lambda)$ is a finite dimensional space which is invariant under L_a. Therefore there exists a nonzero element $x \in X$ such that $ax = \mu x$ for some $\mu \in \mathbb{C}$, which in turn implies $xb = (\mu - \lambda)x$. Accordingly,

$$\Delta L_x R_x = \lambda L_x R_x.$$

Take a spectral projection P of Δ such that $1-P$ has finite rank and $r(\Delta P) < |\lambda|$. Then

$$(\Delta P)PL_xR_x = \lambda PL_xR_x,$$

whence $PL_xR_x = 0$ and $L_xR_x = (1-P)L_xR_x$ is of finite rank. □

7. On operator near-ideals

This section is entirely algebraic. By $\mathcal{A}$ we denote an algebra over an arbitrary field. Let $\mathcal{U}$ be a subspace of the algebra of linear operators on $\mathcal{A}$. We shall say that $\mathcal{U}$ is an *operator near-ideal* on $\mathcal{A}$ if $EU, UE \in \mathcal{U}$ for every $U \in \mathcal{U}$ and every elementary operator E on $\mathcal{A}$. Basic examples of operator near-ideals are of course ideals of the algebra of linear operators on $\mathcal{A}$ (or bounded linear operators in the case when $\mathcal{A}$ is a Banach algebra). Further, we shall say that $a \in \mathcal{A}$ is a $\mathcal{U}$-*element* if the elementary operator L_aR_a belongs to $\mathcal{U}$. For example, if $\mathcal{U}$ is the space of all finite rank operators on an algebra $\mathcal{A}$, then by the very definition a is a $\mathcal{U}$-element if and only if $a \in \mathcal{A}_{\mathrm{f}}$. Similarly, if $\mathcal{U}$ is the space of all compact operators on a Banach algebra $\mathcal{A}$, then a is a $\mathcal{U}$-element if and only if $a \in \mathcal{A}_{\mathrm{c}}$.

We shall say that an elementary operator E has *length* n if $E = \sum_{i=1}^n L_{a_i}R_{b_i}$ for some $a_i, b_i \in \mathcal{A}^1$ and E cannot be represented as $\sum_{i=1}^k L_{c_i}R_{d_i}$ for some $k < n$ and $c_i, d_i \in \mathcal{A}^1$. We also define that the operator 0 has *length* 0.

Lemma 7.1. *Let $\mathcal{A}$ be an arbitrary algebra and let $\mathcal{U}$ be an operator near-ideal on $\mathcal{A}$. If $\mathcal{U}$ contains a nonzero elementary operator, then $\mathcal{A}$ contains a nonzero $\mathcal{U}$-element.*

Proof. Let $E = \sum_{i=1}^n L_{a_i}R_{b_i}$ be a nonzero elementary operator of length $n \geq 1$ belonging to $\mathcal{U}$. We proceed by induction on n.

Let $n = 1$, i. e. $E = L_{a_1}R_{b_1} \in \mathcal{U}$ and $E \neq 0$. Therefore there exists $t \in \mathcal{A}$ such that $a = a_1tb_1 = L_{a_1}R_{b_1}(t) \neq 0$. Note that $L_aR_a = EL_{tb_1}R_{at_1}$ belongs to $\mathcal{U}$ and so the lemma is proved in this case.

We may now assume that $n > 1$ and that the lemma is true whenever $\mathcal{U}$ contains a nonzero elementary operator of length $< n$. Pick $y \in \mathcal{A}$ and define $F = ER_{b_ny} - R_{yb_n}E \in \mathcal{U}$. Note that

$$F = \sum_{i=1}^{n-1} L_{a_i}R_{b_nyb_i - b_iyb_n}$$

and so the length of F is $< n$. In view of the induction assumption we may therefore assume without loss of generality that $F = 0$. That is,

$$\sum_{i=1}^{n-1} L_{a_i}R_{b_nyb_i} = \sum_{i=1}^{n-1} L_{a_i}R_{b_iyb_n}$$

for every $y \in \mathcal{A}$. Accordingly,

$$L_{E(x)}R_{b_n}(y) = E(x)yb_n = \sum_{i=1}^{n} a_i x b_i y b_n = \sum_{i=1}^{n} L_{a_i}R_{b_i y b_n}(x)$$
$$= \sum_{i=1}^{n} L_{a_i}R_{b_n y b_i}(x) = \sum_{i=1}^{n} a_i x b_n y b_i = EL_{xb_n}(y)$$

for all $x, y \in \mathcal{A}$, that is,

$$L_{E(x)}R_{b_n} = EL_{xb_n}$$

for all $x \in \mathcal{A}$. Thus $\mathcal{U}$ contains elementary operators $L_{E(x)}R_{b_n}$, $x \in \mathcal{A}$, of length ≤ 1. If for some $x \in \mathcal{A}$ the length of $L_{E(x)}R_{b_n}$ is 1 then the result follows from the induction assumption. Therefore we may assume that $L_{E(x)}R_{b_n} = 0$ for all $x \in \mathcal{A}$. In fact, the same argument shows that we may assume that for every j, $1 \leq j \leq n$, we have $L_{E(x)}R_{b_j} = 0$ for all $x \in \mathcal{A}$. Consequently, $L_{E(x)}R_{a_j x b_j} = 0$ and hence also $L_{E(x)}R_{E(x)} = 0$ for all $x \in \mathcal{A}$. In particular, $E(x)$ is a $\mathcal{U}$-element. By assumption $E(x) \neq 0$ for some $x \in \mathcal{A}$ and so the desired conclusion holds true in this case as well. □

8. On the range of a compact elementary operator

We are now ready to tackle general elementary operators that are compact. In this section we are interested in the range of such an operator. First we have to recall some notions and results from the theory of radicals of Banach algebras.

A *hereditary topological radical* on the class of all Banach algebras is a map $\mathcal{A} \mapsto \mathcal{A}_r$ which assigns to each Banach algebra $\mathcal{A}$ its closed ideal $\mathcal{A}_r$ so that the following conditions are satisfied:

(r) $(\mathcal{A}/\mathcal{A}_r)_r = 0$.
(rr) $\mathcal{J}_r = \mathcal{J} \cap \mathcal{A}_r$ for every closed ideal $\mathcal{J}$ of $\mathcal{A}$.
(rrr) $f(\mathcal{A}_r) \subset \mathcal{B}_r$ for every continuous epimorphism $f : \mathcal{A} \to \mathcal{B}$.

The following definitions are taken from [25]. A Banach algebra $\mathcal{A}$ is called *hypocompact* if a nonzero quotient of $\mathcal{A}$ by an arbitrary closed ideal always contains a nonzero compact element, and is called *scattered* if the spectrum of every element $a \in \mathcal{A}$ is finite or countable. A closed ideal $\mathcal{J}$ of $\mathcal{A}$ is called *hypocompact* (resp. *scattered*) if $\mathcal{J}$ is hypocompact (resp. scattered) as a Banach algebra. The next result, which is of crucial importance for our results in this section, follows from [25, Theorems 4 and 6] (see also [22]).

Theorem 8.1. [25] *For every Banach algebra $\mathcal{A}$ there exist the largest hypocompact ideal $\mathcal{A}_{\mathrm{hc}}$ and the largest scattered ideal $\mathcal{A}_{\mathrm{sc}}$, and $\mathcal{A}_{\mathrm{hc}} \subset \mathcal{A}_{\mathrm{sc}}$. The maps $\mathcal{A} \mapsto \mathcal{A}_{\mathrm{hc}}$ and $\mathcal{A} \mapsto \mathcal{A}_{\mathrm{sc}}$ are hereditary topological radicals.*

Let us add a few elementary remarks.

Lemma 8.2. *If $\mathcal{A}$ is a Banach algebra then $\mathcal{A}_{\mathrm{c}} \subset \mathcal{A}_{\mathrm{hc}}$.*

Proof. Let $a \in \mathcal{A}_c$ and let $\mathcal{I}$ be the closed ideal of $\mathcal{A}$ generated by a. If $\mathcal{J}$ is a proper closed ideal of $\mathcal{I}$, then $a \notin \mathcal{J}$ and so $a + \mathcal{J} \neq 0$. But $a + \mathcal{J}$ is a compact element of $\mathcal{I}/\mathcal{J}$. This shows that $\mathcal{I}$ is hypocompact, and so $a \in \mathcal{I} \subset \mathcal{A}_{hc}$. □

Therefore $\mathcal{A}_{hc}$ contains the closed ideal of $\mathcal{A}$ generated by $\mathcal{A}_c$. In general $\mathcal{A}_{hc}$ does not coincide with this ideal.

So we now know that, for every Banach algebra $\mathcal{A}$, we have

$$\mathcal{A}_f \subset \mathcal{A}_c \subset \mathcal{A}_{hc} \subset \mathcal{A}_{sc}. \tag{8.1}$$

If $\mathcal{A}$ is semiprime, and in particular if it is semisimple, then $\mathcal{A}_f = \mathrm{soc}(\mathcal{A})$ (cf. section 5).

Lemma 8.3. *If $\mathcal{A}$ is a semisimple Banach algebra, then $\mathcal{A}_{sc} \neq 0$ if and only if* $\mathrm{soc}(\mathcal{A}) \neq 0$.

Proof. In view of (8.1) we only have to prove the "only if" part. So let $\mathcal{A}_{sc} \neq 0$. As an ideal of a semisimple algebra, $\mathcal{A}_{sc}$ is also semisimple. Therefore $\mathrm{soc}(\mathcal{A}_{sc}) \neq 0$ by Barnes' theorem [4, Theorem 2.2]. This means that there exists a minimal idempotent e of $\mathcal{A}_{sc}$. Note that

$$e\mathcal{A}_{sc}e \subset e\mathcal{A}e = e(\mathcal{A}e)e \subset e\mathcal{A}_{sc}e,$$

whence $e\mathcal{A}e = e\mathcal{A}_{sc}e = \mathbb{C}e$, i. e. e is a minimal idempotent of $\mathcal{A}$. Therefore $\mathrm{soc}(\mathcal{A}) \neq 0$. □

Having Lemma 7.1 in hand it is now easy to connect the theory just sketched with compact elementary operators.

Theorem 8.4. *If E is a compact elementary operator on a Banach algebra $\mathcal{A}$, then the range of E lies in $\mathcal{A}_{hc}$. Accordingly, $\sigma(E(y)x)$ is finite or countable for all $x, y \in \mathcal{A}^1$.*

Proof. Elementary operators clearly leave each ideal invariant. In particular, $\mathcal{A}_{hc}$ is invariant under E. Let E' be the operator on $\mathcal{A}/\mathcal{A}_{hc}$ induced by E. Obviously, E' is also a compact elementary operator. If E' was nonzero then $\mathcal{A}/\mathcal{A}_{hc}$ would contain a nonzero compact element by Lemma 7.1. However, in view of the property `(r)` and Lemma 8.2 this is impossible. Therefore $E' = 0$, that is, E maps $\mathcal{A}$ into $\mathcal{A}_{hc}$. By Theorem 8.1 the range of E lies also in $\mathcal{A}_{sc}$. Taking $\mathcal{A}^1$ instead of $\mathcal{A}$ and repeating the argument, we obtain that the range of E lies in $\mathcal{A}^1_{sc}$. Hence $\sigma(E(y)x)$ is finite or countable for every $x, y \in \mathcal{A}^1$. □

Corollary 8.5. *If $\sum_{i=1}^n L_{a_i} R_{b_i}$ is a compact operator, then $\sigma(\sum_{i=1}^n a_i b_i)$ is finite or countable.*

9. On algebras having compact elementary operators

Let $\mathcal{A}$ be a semiprime algebra over a field F. The sum of all minimal left ideals $e\mathcal{A}$, where e is a minimal idempotent in $\mathcal{A}$ such that the division algebra $e\mathcal{A}e$ is *finite dimensional* over F, is called the *lower socle* of $\mathcal{A}$ and will be denoted by $\underline{\mathrm{soc}}(\mathcal{A})$ (in the Banach algebra case we of course have $\underline{\mathrm{soc}}(\mathcal{A}) = \mathrm{soc}(\mathcal{A})$). This concept was introduced and studied in the recent paper [7]. It is easy to see that $\underline{\mathrm{soc}}(\mathcal{A})$ is an ideal of $\mathcal{A}$ (contained of course in $\mathrm{soc}(\mathcal{A})$). The important result for us is [7, Theorem 3.3] stating that $\underline{\mathrm{soc}}(\mathcal{A}) = \mathcal{A}_{\mathrm{f}}$. So in particular $\mathcal{A}_{\mathrm{f}}$ is an ideal of $\mathcal{A}$ (this is not clear from the definition). We can now easily prove the following theorem that gives an additional insight to the topic considered in [7, Section 4].

Theorem 9.1. *Let $\mathcal{A}$ be a semiprime algebra. If there exists a nonzero finite rank elementary operator on $\mathcal{A}$, then $\underline{\mathrm{soc}}(\mathcal{A}) \neq 0$.*

Proof. Using Lemma 7.1 with $\mathcal{U}$ being the space of all finite rank operators on $\mathcal{A}$ we see that $\mathcal{A}$ contains a nonzero $\mathcal{U}$-element, that is to say, $\underline{\mathrm{soc}}(\mathcal{A}) = \mathcal{A}_{\mathrm{f}} \neq 0$. □

In a similar fashion we obtain an analytic version of Theorem 9.1.

Theorem 9.2. *Let $\mathcal{A}$ be a semisimple Banach algebra. If there exists a nonzero compact elementary operator on $\mathcal{A}$, then $\mathrm{soc}(\mathcal{A}) \neq 0$.*

Proof. We now apply Lemma 7.1 for the case where $\mathcal{U}$ is the space of compact linear operators on $\mathcal{A}$. Hence it follows that $\mathcal{A}$ has a nonzero compact element, and so Corollary 5.2 gives $\mathrm{soc}(\mathcal{A}) \neq 0$. □

Note that Theorem 9.2 also follows from Theorem 8.1, Lemma 8.3 and Theorem 8.4.

There are no nonzero compact elementary operators on the Calkin algebra of operators on a separable Hilbert space. This result was conjectured in [13] and proved in [2] (and later also in [16] and [19]). The following corollary is its generalization.

Corollary 9.3. *Let $\mathcal{A}$ be a simple unital Banach algebra. If there exists a nonzero compact elementary operator on $\mathcal{A}$, then $\mathcal{A} \cong M_n(\mathbb{C})$ for some $n \geq 1$.*

Proof. By Theorem 9.2 we have that $\mathrm{soc}(\mathcal{A}) \neq 0$. Since $\mathcal{A}$ is simple and unital it follows that $1 \in \mathrm{soc}(\mathcal{A}) = \mathcal{A}_{\mathrm{f}}$. Thus $\mathcal{A} = L_1 R_1(\mathcal{A})$ is finite dimensional, and hence $\mathcal{A} \cong M_n(\mathbb{C})$. □

We shall say that a Banach algebra is *bicompact* if $L_a R_b$ is a compact operator for all $a, b \in \mathcal{A}$. This is of course a generalization of the concept of a compact Banach algebra [1].

Proposition 9.4. *Let $\mathcal{A}$ be a topologically simple Banach algebra. If there exists a nonzero compact elementary operator on $\mathcal{A}$, then $\mathcal{A}$ is a bicompact algebra.*

Proof. By Lemma 7.1 $\mathcal{A}$ contains a nonzero compact element a. Note that $L_{xay} R_{zaw} = L_x R_w (L_a R_a) L_y R_z$ is compact for all $x, y, z, w \in \mathcal{A}$. Consequently, $L_u R_v$ is compact for all u, v from the ideal of $\mathcal{A}$ generated by a. Since this ideal is dense in $\mathcal{A}$, it follows that $\mathcal{A}$ is a bicompact algebra. □

If we add the further assumption to Proposition 9.4 that $\mathcal{A}$ is semisimple, then it follows from Theorem 9.2 that $\mathcal{A}$ is equal to the closure of $\mathrm{soc}(\mathcal{A})$; note that this conclusion is stronger than the one given in the proposition. However, it is not known whether a topologically simple Banach algebra is automatically semisimple, i. e. whether there exists a topologically simple radical Banach algebra.

10. On the coefficients of a compact elementary operator

If the algebra $\mathcal{A}$ is not prime then $L_a R_b$ can be zero although both a and b are nonzero. Therefore it does not seem to be easy to get some interesting information about the coefficients of an elementary operator in non-prime algebras. We shall therefore confine ourselves to prime algebras.

Lemma 7.1 will be indirectly used also in this section (via Theorem 9.2). But we need other tools too; a disadvantage of the approach based on this lemma is that we lose a track of the coefficients of an elementary operator in question. We shall now present an alternative approach based on an algebraic result by Erickson, Martindale and Osborn [12, Theorem 3.1] (see also generalizations [5, Theorem 2.3.3] and [8, Theorem 1.2]) which is stated (in an equivalent form) below as Theorem 10.1. This result involves the concept of the *extended centroid* of a prime algebra and we refer the reader to the book [5] for an account on this theory. Let us just mention here that the extended centroid of a prime algebra $\mathcal{A}$ over a field F is a certain field containing F (and in fact also containing the center of $\mathcal{A}$). In the case when the extended centroid coincides with F we say that $\mathcal{A}$ is *centrally closed* over F. By a centrally closed prime Banach algebra we shall of course mean a prime Banach algebra that is centrally closed over $\mathbb{C}$. For example, primitive Banach algebras are centrally closed prime Banach algebras (this follows easily from [5, Corollary 4.1.2]). On the other hand, commutative prime algebras (i.e. commutative domains) of dimension more than 1 over a field F are not centrally closed over F and for such algebras the next theorem clearly does not hold.

Theorem 10.1. [12] *Let $\mathcal{A}$ be a centrally closed prime algebra over a field F. If $b_1, b_2 \dots, b_n \in \mathcal{A}^1$ are such that b_1 does not lie in the linear span of $b_2, \dots, b_n$, then there exists an elementary operator E on $\mathcal{A}$ such that $E(b_1) \neq 0$ and $E(b_2) = \dots = E(b_n) = 0$.*

Let us record a simple corollary to this theorem which indicates both the similarities and the differences between the present approach and the one based on Lemma 7.1.

Corollary 10.2. *Let $\mathcal{A}$ be a centrally closed prime algebra over a field F and let $E = \sum_{i=1}^{n} L_{a_i} R_{b_i}$ be an elementary operator on $\mathcal{A}$ such that b_1 does not lie in the linear span of $b_2, \dots, b_n$. If E belongs to an operator near-ideal $\mathcal{U}$ on $\mathcal{A}$, then there is a nonzero $b \in \mathcal{A}$ such that $L_{a_1} R_b \in \mathcal{U}$.*

Proof. By Theorem 10.1 there is an elementary operator $G = \sum_{j=1}^{m} L_{c_j} R_{d_j}$ such that $G(b_1) = b \neq 0$ and $G(b_i) = 0$, $i = 2, \dots, n$. Note that

$$L_{a_1} R_b = \sum_{j=1}^{m} R_{d_j} E R_{c_j} \in \mathcal{U}.$$

□

We now return to the analytic setting. Let us first recall a theorem by Vala [26] which says that if $\mathcal{A} = \mathcal{B}(X)$, the algebra of all bounded linear operators on a Banach space X, then for any $a, b \in \mathcal{A}$ we have that $L_a R_b$ is a compact operator on $\mathcal{A}$ if any only if a and b are compact operators on X.

Lemma 10.3. *Let $\mathcal{A}$ be a primitive Banach algebra with $\mathrm{soc}(\mathcal{A}) \neq 0$ and let π be the regular representation of $\mathcal{A}^1$ on the Banach space $X = \mathcal{A}e$ where e is a minimal idempotent of $\mathcal{A}$. Further, let $E = \sum_{i=1}^{n} L_{a_i} R_{b_i}$ be a compact elementary operator on $\mathcal{A}$ and assume that b_1 does not lie in the linear span of $b_2, \dots, b_n$. Then $\pi(a_1)$ is a compact operator on X. Moreover, if π^{-1} is continuous then a_1 is a compact element.*

Proof. Corollary 10.2 tells us that $L_{a_1} R_b$ is compact for some $b \neq 0$. Since $\mathcal{A}$ is a prime algebra, $e\mathcal{A}b\mathcal{A}e \neq 0$. Moreover, since $e\mathcal{A}e = \mathbb{C}e$ we have that $esbte = e$ for some $s, t \in \mathcal{A}$. Therefore $L_{a_1} R_e = R_{te}(L_{a_1} R_b) R_{es}$ is a compact operator. Its restriction to $\mathcal{A}e = X$ is then compact too, meaning that $\pi(a_1)$ (defined by $\pi(a_1)xe = a_1 xe$) is a compact operator on X. By Vala's theorem, $L_{\pi(a_1)} R_{\pi(a_1)}$ is a compact operator on $\mathcal{B}(X)$. Therefore the operator $L_{a_1} R_{a_1} = \pi^{-1} L_{\pi(a_1)} R_{\pi(a_1)} \pi$ is also compact provided that π^{-1} is continuous. □

In certain classes of algebras π^{-1} is automatically continuous. For example, this is true in the so-called ultraprime algebras introduced by Mathieu [17]: a normed algebra $\mathcal{A}$ is said to be *ultraprime* if there exists a constant $k_{\mathcal{A}} > 0$ such that $\|L_a R_b\| \geq k_{\mathcal{A}} \|a\| \|b\|$ for all $a, b \in \mathcal{A}$. If $\mathcal{A}$ is ultraprime, then so is $\mathcal{A}^1$ [17, Proposition 3.7]. Since

$$\|\pi(a)\| \|e\| \geq \|\pi(a)\| \sup_{\|x\|=1} \|xe\| \geq \sup_{\|x\|=1} \|\pi(a)xe\| = \sup_{\|x\|=1} \|axe\| = \|L_a R_e\|$$

it follows that in an ultraprime algebra $\mathcal{A}$ we have $\|\pi(a)\| \geq k_{\mathcal{A}^1} \|a\|$ for every $a \in \mathcal{A}^1$, so that π^{-1} is continuous. The class of ultraprime normed algebras contains prime C^*-algebras [18, Proposition 2.3], $\mathcal{B}(X)$ where X is any normed space (one can check this easily and, moreover, $k_{\mathcal{B}(\mathcal{X})} = 1$), and prime group algebras $l^1(G)$ where G is a discrete group [27]. Further, ideals of ultraprime normed algebras are again ultraprime [17, Proposition 3.6]. On the other hand, quotients of ultraprime algebras by their closed ideals may not be ultraprime, and there exist primitive Banach algebras with nonzero socle that are not ultraprime [3].

A subset $\mathcal{M}$ of a Banach algebra $\mathcal{A}$ is called a *bicompact subset* of $\mathcal{A}$ if $L_a R_b$ is a compact operator for all $a, b \in \mathcal{M}$. For example, Vala's theorem shows that the set of compact operators on a Banach space X is a bicompact subset of $\mathcal{B}(X)$. The next result generalizes this.

Corollary 10.4. *If $\mathcal{A}$ is a semisimple ultraprime Banach algebra, then $\mathcal{A}_{\rm c}$ is a bicompact subset and a closed ideal of $\mathcal{A}$.*

Proof. We may assume that $\mathcal{A}_{\rm c} \neq 0$. Thus $L_a R_a$ is a compact operator for some nonzero $a \in \mathcal{A}$, and so $\operatorname{soc}(\mathcal{A}) \neq 0$ by Theorem 9.2. A prime algebra with nonzero socle is primitive since the minimal left ideal $\mathcal{A}e$ is a faithful simple left $\mathcal{A}$-module for every minimal idempotent e. Let π be as in Lemma 10.3. The lemma in particular implies that $\pi(a)$ is a compact operator if $a \in \mathcal{A}_{\rm c}$. So, if $a, b \in \mathcal{A}_{\rm c}$, then $\pi(a)$ and $\pi(b)$ are compact operators, and hence $L_{\pi(a)} R_{\pi(b)}$ is compact by Vala's theorem. Since π^{-1} is continuous, it follows (as in the proof of Lemma 10.3) that $L_a R_b$ is compact on $\mathcal{A}$. Thus $\mathcal{A}_{\rm c}$ is a bicompact subset of $\mathcal{A}$. It is clearly a closed set, and in order to prove that it is an ideal we only have to show that it is closed under addition. Now, if $a, b \in \mathcal{A}_{\rm c}$, then

$$L_{a+b} R_{a+b} = L_a R_a + L_a R_b + L_b R_a + L_b R_b$$

is, as a sum of four compact operators (by what we just showed), a compact operator itself. Therefore $a + b \in \mathcal{A}_{\rm c}$. □

The following is the main result of this section.

Theorem 10.5. *Let $\mathcal{A}$ be a semisimple ultraprime Banach algebra and let $E = \sum_{i=1}^n L_{a_i} R_{b_i}$ be an elementary operator on $\mathcal{A}$ such that b_1 does not lie in the linear span of $b_2, \ldots, b_n$. If E is a compact operator then a_1 is a compact element.*

Proof. Ultraprime normed algebras are centrally closed [17, Corollary 4.7]. Our assumption that b_1 does not lie in the linear span of $b_2, \ldots, b_n$ therefore in particular implies that $a_1 = 0$ if $E = 0$ – this follows from Corollary 10.2 by choosing $\mathcal{U} = 0$. We may therefore assume that $E \neq 0$. By Theorem 9.2 $\mathcal{A}$ has a nonzero socle. Thererefore Lemma 10.3 implies that a_1 is a compact element. □

Theorem 10.5 was known before in two special cases: in the case where $\mathcal{A} = \mathcal{B}(X)$ [13, Theorem 2] and in the case where $\mathcal{A}$ is a prime C^*-algebra [19, Theorem 3.8]. It would be interesting to find other classes of prime Banach algebras for which this theorem holds (or does not hold). Let us point out that we used the ultraprimeness condition only to guarantee that $\mathcal{A}$ is centrally closed and π^{-1} is continuous, so under certain technical conditions a more general result could be stated.

We remark that by making some rather obvious modifications in the proof of Theorem 10.5 one can prove an analogous statement: If a_1 does not lie in the linear span of $a_2, \ldots, a_n$ and E is a compact operator, then b_1 is a compact element. Accordingly, if both sets $\{a_1, \ldots, a_n\}$ and $\{b_1, \ldots, b_n\}$ are linearly independent and the operator $\sum_{i=1}^n L_{a_i} R_{b_i}$ is compact, then all elements a_i, b_i are compact.

Corollary 10.6. *Let $\mathcal{A}$ be a semisimple ultraprime Banach algebra. If E is a compact elementary operator on $\mathcal{A}$, then there exist compact elements $a_1, \ldots, a_n, b_1, \ldots, b_n \in \mathcal{A}^1$ such that $E = \sum_{i=1}^n L_{a_i} R_{b_i}$.*

Proof. We may assume that $E \neq 0$. Therefore E has length $n \geq 1$. If E is represented as $E = \sum_{i=1}^n L_{a_i} R_{b_i}$, then clearly the sets $\{a_1, \ldots, a_n\}$ and $\{b_1, \ldots, b_n\}$ are linearly independent. As mentioned above, this implies that all a_i, b_i are compact. □

A result of this type appears also in [13] (for $\mathcal{B}(X)$) and in [19] (for prime C^*-algebras), and was recently also proved for general C^*-algebras [24].

Note that Corollaries 10.4 and 10.6 show that every compact elementary operator on a semisimple ultraprime Banach algebra is a sum of compact elementary operators of length 1.

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