FUNCTIONAL IDENTITIES ON TENSOR PRODUCTS OF ALGEBRAS

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ABSTRACT. Let R and S be unital algebras. We show that if X is a d-free subset of R and S is finite dimensional, then the set $\mathfrak{X} = \{x \otimes s \mid x \in X, s \in S\}$ is a d-free subset of the algebra $R \otimes S$. The assumption that S is finite dimensional turns out to be necessary in general. However, we show that some important functional identities have only standard solutions on $\mathfrak{X}$ even when S is infinite dimensional.

1. INTRODUCTION

A functional identity is, roughly speaking, an identical relation involving arbitrary elements from a subset of a ring along with with arbitrary functions that are considered as unknowns. Basic definitions and sample results will be given below, but for details and a more clear picture we refer the reader to the book [8].

The theory of functional identities is based on the concept of a d-free set, which is briefly described in Section 2. On a d-free set one can handle quite general functional identities [8, Chapter 4], and, more importantly, one can solve a variety of problems arising in different mathematical areas [8, Chapters 6–8]. The major problem is to show that d-free sets actually exist. The fundamental theorem in this context states that under a mild (and necessary) assumption every prime ring A is a d-free subset of its maximal left ring of quotients $Q_{ml}(A)$ [8, Theorem 5.11]. Using this one can then find various d-free subsets of prime rings, such as ideals, Lie and Jordan ideals, symmetric and skew-symmetric elements if the ring is equipped with involution, etc. [8, Section 5.2]. Most of known examples of d-free sets are actually subsets of prime rings. Among other examples, we list the following:

(a) If S is an arbitrary ring, then the matrix ring $M_n(S)$ is a d-free subset of itself, as long as $n \geq d$ [8, Corollary 2.22].
(b) If a unital ring A is a d-free subset of a unital ring R, then $T_n(A)$, the ring of all upper triangular matrices over A, is a d-free subset of $T_n(R)$. This was recently established by Eremita [9].
(c) The tensor product $A \otimes S$ of a prime algebra A (satisfying the usual restrictions) and an arbitrary finite dimensional algebra S is a d-free subset of $Q_{ml}(A) \otimes S$; moreover, if A is a simple unital algebra, then S can be infinite dimensional. See [1].

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In this paper we will deal, partially for simplicity, only with algebras over a field, which we denote by $\mathbb{F}$. Within this framework, all statements (a), (b), and (c) get the same form: If an algebra A is a d-free subset of an algebra R, then $A \otimes S$ is a d-free subset of $R \otimes S$. In (a), $A = M_n(\mathbb{F})$ and S is an arbitrary unital algebra. In (b), A is an arbitrary d-free algebra and $S = T_n(\mathbb{F})$. In (c), A is a prime algebra and S is an arbitrary finite dimensional unital algebra, or an arbitrary unital algebra in case A is simple and unital ($A = M_n(\mathbb{F})$ is of course just a special case). All this indicates that a more general phenomenon might be hidden behind these results. The purpose of this paper is to explore it.

In Section 3 we restrict ourselves to the case where S is a finite dimensional (but otherwise arbitrary) unital algebra. In this context, the result that we obtain (Theorem 3.2) is definitive of its kind: If X is a d-free subset of an algebra R, then $\{x \otimes s \,|\, x \in X, s \in S\}$ is a d-free subset of the algebra $R \otimes S$. In this way we obtain a new, large family of d-free sets for which the general theory from [8] is directly applicable.

The restriction to finite dimensions is necessary in general. This is shown in Section 4. We actually give an example of a simple non-unital algebra A which is a d-free subset of a larger algebra R for every $d \geq 1$, yet $A \otimes \mathbb{F}[\xi]$ is not a 2-free subset of $R \otimes \mathbb{F}[\xi]$. The second result from (c) therefore does not hold without the assumption that A is unital.

The example just mentioned shows that, unfortunately, we have no control of general functional identities if S is infinite dimensional. However, in Section 5 we will see that in this case we can still handle some indeed quite special, but particularly important identities. First of all, this turns out to be the case for the "one-sided" identities (Theorem 5.1). The main theme of Section 5 is the identity $[F(y,z),w] + [F(w,y),z] + [F(z,w),y] = 0$, which is a prototype of a functional identity appearing in different problems (cf. [8, Section 1.4]). The main result of the section (Theorem 5.2) tells us that this identity has only standard solutions under an additional technical assumption, which is fulfilled in the case where the first algebra is prime (Corollary 5.4). Among possible applications, we discuss only the one concerning Lie isomorphisms (Corollary 5.6). The reason for this is that, as shown in [1] and [2], Lie automorphisms of the tensor product between a "nice" algebra A and an arbitrary unital algebra S naturally appear in the study of gradings of Lie algebras (see [10] for the recent survey of this theory). In [1] only the case where S is finite dimensional was treated, while in [2] the need for treating an infinite dimensional algebra S appeared. In that paper, the problem was solved by relying on special properties of the algebra A that was considered. Now it seems plausible that one could obtain similar results for considerably more general algebras A.

2. Preliminaries on d-free sets

The purpose of this preliminary section is to recall the definition of a d-free set, which is due to Beidar and Chebotar [4]. Simultaneously we will introduce some necessary notation.

Let R be a unital ring with center $Z = Z_R$, let X be a nonempty subset of R, and let m be a positive integer. For elements $x_i \in X$, $i = 1, 2, \ldots, m$, we set

$$\begin{aligned} \overline{x}_m &= (x_1, \ldots, x_m) \in X^m, \\ \overline{x}_m^i &= (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_m) \in X^{m-1}, \\ \overline{x}_m^{ij} = \overline{x}_m^{ji} &= (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{j-1}, x_{j+1} \ldots, x_m) \in X^{m-2}. \end{aligned}$$

Let I, J be subsets of $\{1, 2, \ldots, m\}$. For each $i \in I$ and $j \in J$ let

$$E_i : X^{m-1} \to R \quad \text{and} \quad F_j : X^{m-1} \to R$$

be arbitrary functions. For $m = 1$ we regard E_i and F_j as elements in R. The basic functional identities are

$$\sum_{i\in I} E_i(\overline{x}_m^i)x_i + \sum_{j\in J} x_j F_j(\overline{x}_m^j) = 0 \quad \text{for all } \overline{x}_m \in X^m, \tag{2.1}$$

$$\sum_{i\in I} E_i(\overline{x}_m^i)x_i + \sum_{j\in J} x_j F_j(\overline{x}_m^j) \in Z \quad \text{for all } \overline{x}_m \in X^m. \tag{2.2}$$

Note that (2.1) trivially implies (2.2), so one should not understand that (2.1) and (2.2) are satisfied simultaneously by the same functions E_i and F_j. Each of the two identities should be treated separately.

The *standard solution* of both functional identities (2.1) and (2.2) is defined as

$$\begin{aligned} E_i(\overline{x}_m^i) &= \sum_{\substack{j\in J,\\ j\neq i}} x_j p_{ij}(\overline{x}_m^{ij}) + \lambda_i(\overline{x}_m^i), \quad i \in I, \\ F_j(\overline{x}_m^j) &= -\sum_{\substack{i\in I,\\ i\neq j}} p_{ij}(\overline{x}_m^{ij})x_i - \lambda_j(\overline{x}_m^j), \quad j \in J, \\ &\lambda_k = 0 \quad \text{if} \quad k \notin I \cap J, \end{aligned} \tag{2.3}$$

where

$$\begin{aligned} &p_{ij} : X^{m-2} \to R, \;\; i \in I, \; j \in J, \; i \neq j, \\ &\lambda_k : X^{m-1} \to Z, \;\; k \in I \cup J, \end{aligned}$$

are arbitrary functions (for $m = 1$ one should understand this as that $p_{ij} = 0$ and λ_k is an element in Z). Note that (2.3) indeed implies (2.1), and hence also (2.2). The standard solutions can be viewed as the "obvious", or, more precisely, as unavoidable solutions that always exist, independently of the structure of the ring in question.

We remark that the cases where one of the sets I and J is empty are not excluded. We will follow the convention that the sum over $\emptyset$ is 0. Thus, if $J = \emptyset$, (2.1) reads as

$$\sum_{i\in I} E_i(\overline{x}_m^i)x_i = 0 \quad \text{for all } \overline{x}_m \in X^m,$$

and the standard solution of this functional identity is $E_i = 0$ for all $i \in I$. Similarly, the standard solution of

$$\sum_{j\in J} x_j F_j(\overline{x}_m^j) = 0 \quad \text{for all } \overline{x}_m \in X^m$$

is $F_j = 0$ for each j.

Definition 2.1. Let d be a positive integer. We say that X is *d-free subset* of R if the following two conditions hold for all $m \geq 1$ and all $I, J \subseteq \{1, 2, \dots, m\}$:

(a) If $\max\{|I|, |J|\} \leq d$, then (2.1) implies (2.3).
(b) If $\max\{|I|, |J|\} \leq d-1$, then (2.2) implies (2.3).

Note that (b) can be replaced by

(b') If $\max\{|I|, |J|\} \leq d-1$, then (2.2) implies (2.1).

Namely, (2.3) trivially implies (2.1), and, according to (a), (2.1) implies (2.3) if $\max\{|I|, |J|\} \leq d-1$.

We remark that conditions (a) and (b) are usually handled in a similar manner, but are independent in general. In applications of functional identities one usually uses both, so each of them is necessary.

Remark 2.2. Suppose that X is a d-free subset of R and (2.1) with $|I| \leq d$ and $|J| \leq d-1$ holds for some functions E_i, F_j. Assume further that $i \in I$ is such that $i \notin J$ and E_i maps into Z. Then $E_i = 0$. Namely, by definition of d-freeness we know that E_i is of the form

$$E_i(\overline{x}_m^i) = \sum_{j \in J} x_j p_{ij}(\overline{x}_m^{ij}) + \lambda_i(\overline{x}_m^i).$$

Now, since $i \notin I \cap J$ we have $\lambda_i = 0$, and since

$$\sum_{\substack{j \in J,\\ j \neq i}} x_j p_{ij}(\overline{x}_m^{ij}) \in Z$$

it follows from (b) that each $p_{ij} = 0$.

3. The finite dimensional case

We now assume that R is an algebra over a field $\mathbb{F}$, X is a nonempty subset of R, and S is a finite dimensional unital algebra over $\mathbb{F}$. Fix a basis

$$\{b_1, \dots, b_N\}$$

of S over $\mathbb{F}$. Set

$$\mathfrak{R} = R \otimes S$$

and

$$\mathfrak{X} = \{x \otimes s \,|\, x \in X, s \in S\}.$$

We will show that $\mathfrak{X}$ is a d-free subset of $\mathfrak{R}$ if X is a d-free subset of R. We identify X by $X \otimes 1 \subseteq \mathfrak{X}$. Accordingly, we will often write $x \otimes 1$ simply as x. By Z_S we denote the center of S. As is well-known, the center of $R \otimes S$ is equal to

$$\mathfrak{Z} = Z \otimes Z_S \, (= Z_R \otimes Z_S).$$

Throughout, we assume that X is a d-free subset of R. Our goal is to show that then $\mathfrak{X}$ is a d-free subset of $\mathfrak{R}$. Thus, we have to show that functions $E_i, F_j : \mathfrak{X}^{m-1} \to \mathfrak{R}$, $i \in I$, $j \in J$ satisfying either

$$\sum_{i\in I} E_i(\overline{y}^i_m)y_i + \sum_{j\in J} y_jF_j(\overline{y}^j_m) = 0 \quad \text{for all } \overline{y}_m \in \mathfrak{X}^m \tag{3.1}$$

or

$$\sum_{i\in I} E_i(\overline{y}^i_m)y_i + \sum_{j\in J} y_jF_j(\overline{y}^j_m) \in \mathfrak{Z} \quad \text{for all } \overline{y}_m \in \mathfrak{X}^m \tag{3.2}$$

are of standard form if $\max\{|I|, |J|\} \le d$ (in case of (3.1)) and $\max\{|I|, |J|\} \le d-1$ (in case of (3.2)). We remark that in (3.2) we could replace $\mathfrak{Z}$ by $Z \otimes S$, but we will not bother with this generalization.

In the first lemma we consider (3.1) and (3.2) restricted to X^{m-1} (here, X stands for $X \otimes 1$). As we will see, it is rather straightforward to derive that the functions are of the desired form (2.3), however, with λ_k mapping into $Z \otimes S$ rather than into $\mathfrak{Z} = Z \otimes Z_S$. Note that one cannot say more in this setting.

Lemma 3.1. *Let $E_i, F_j : X^{m-1} \to \mathfrak{R}$, $i \in I$, $j \in J$, and set*

$$\Phi(\overline{x}_m) = \sum_{i\in I} E_i(\overline{x}^i_m)x_i + \sum_{j\in J} x_jF_j(\overline{x}^j_m).$$

Suppose that either

(a) $\Phi(\overline{x}_m) = 0$ *for all* $\overline{x}_m \in X^m$ *and* $\max\{|I|, |J|\} \le d$, *or*
(b) $\Phi(\overline{x}_m) \in \mathfrak{Z}$ *for all* $\overline{x}_m \in X^m$ *and* $\max\{|I|, |J|\} \le d-1$.

Then there exist $p_{ij} : X^{m-2} \to \mathfrak{R}$, $i \in I$, $j \in J$, $i \ne j$, and $\lambda_k : X^{m-1} \to Z \otimes S$, $k \in I \cup J$, such that (2.3) holds.

Proof. Let us write

$$E_i(\overline{x}^i_m) = \sum_{t=1}^N e_{it}(\overline{x}^i_m) \otimes b_t,$$
$$F_j(\overline{x}^j_m) = \sum_{t=1}^N f_{jt}(\overline{x}^j_m) \otimes b_t,$$

where $e_{it}, f_{jt} : X^{m-1} \to R$. We have

$$\begin{aligned}\Phi(\overline{x}_m) &= \sum_{i\in I}\Big(\sum_{t=1}^N e_{it}(\overline{x}^i_m) \otimes b_t\Big) \cdot (x_i \otimes 1) + \sum_{j\in J}(x_j \otimes 1) \cdot \Big(\sum_{t=1}^N f_{jt}(\overline{x}^j_m) \otimes b_t\Big)\\ &= \sum_{t=1}^N \Big(\sum_{i\in I} e_{it}(\overline{x}^i_m)x_i + \sum_{j\in J} x_jf_{jt}(\overline{x}^j_m)\Big) \otimes b_t.\end{aligned}$$

Consequently, for all $\overline{x}_m \in X^m$ and all $1 \le t \le N$ we have

$$\sum_{i\in I} e_{it}(\overline{x}^i_m)x_i + \sum_{j\in J} x_jf_{jt}(\overline{x}^j_m) = 0$$

if (a) holds, and

$$\sum_{i\in I} e_{it}(\overline{x}_m^i)x_i + \sum_{j\in J} x_j f_{jt}(\overline{x}_m^j) \in Z$$

if (b) holds. Since X is a d-free subset of R, in each of the two cases we get the same conlusion, namely that for each t there exist

$$\begin{aligned} &p_{ijt} : X^{m-2} \to R,\ \ i\in I,\ j\in J,\ i\neq j,\\ &\lambda_{kt} : X^{m-1} \to Z,\ \ k\in I\cup J, \end{aligned}$$

such that

$$\begin{aligned} e_{it}(\overline{x}_m^i) &= \sum_{\substack{j\in J,\\ j\neq i}} x_j p_{ijt}(\overline{x}_m^{ij}) + \lambda_{it}(\overline{x}_m^i), \quad i\in I,\\ f_{jt}(\overline{x}_m^j) &= -\sum_{\substack{i\in I,\\ i\neq j}} p_{ijt}(\overline{x}_m^{ij})x_i - \lambda_{jt}(\overline{x}_m^j), \quad j\in J,\\ &\lambda_{kt} = 0 \quad \text{if} \quad k\notin I\cap J. \end{aligned}$$

Now define $p_{ij} : X^{m-2} \to \mathfrak{R}$, $i\in I$, $j\in J$, $i\neq j$, and $\lambda_k : X^{m-1} \to Z\otimes S$, $k\in I\cup J$, by

$$\begin{aligned} p_{ij}(\overline{x}_m^{ij}) &= \sum_{t=1}^N p_{ijt}(\overline{x}_m^{ij})\otimes b_t,\\ \lambda_k(\overline{x}_m^k) &= \sum_{t=1}^N \lambda_{kt}(\overline{x}_m^k)\otimes b_t. \end{aligned}$$

Note that

$$\begin{aligned} E_i(\overline{x}_m^i) &= \sum_{t=1}^N \Big(\sum_{\substack{j\in J,\\ j\neq i}} x_j p_{ijt}(\overline{x}_m^{ij}) + \lambda_{it}(\overline{x}_m^i)\Big)\otimes b_t\\ &= \sum_{\substack{j\in J,\\ j\neq i}} x_j p_{ij}(\overline{x}_m^{ij}) + \lambda_i(\overline{x}_m^i) \end{aligned}$$

for every $i\in I$, and similarly,

$$F_j(\overline{x}_m^j) = -\sum_{\substack{i\in I,\\ i\neq j}} p_{ij}(\overline{x}_m^{ij})x_i - \lambda_j(\overline{x}_m^j)$$

for every $j\in J$. We also have $\lambda_k = 0$ if $k\notin I\cap J$. □

Let us introduce some auxiliary terminology and notation that will be used in the proof of the main theorem. For any $1\leq s\leq m$ we write

$$I_s = I\cap\{1,\ldots,s\}, \quad J_s = J\cap\{1,\ldots,s\},$$

$$I_s' = I\setminus I_s, \quad J_s' = J\setminus J_s,$$

and

$$\begin{aligned}\overline{x}_{s,m} &= (x_s,\dots,x_m),\\ \overline{x}^i_{s,m} &= (x_s,\dots,x_{i-1},x_{i+1},\dots,x_m),\\ \overline{x}^{ij}_{s,m}=\overline{x}^{ji}_{s,m} &= (x_s,\dots,x_{i-1},x_{i+1},\dots,x_{j-1},x_{j+1}\dots,x_m).\end{aligned}$$

We also set $\overline{x}_{s,m}=\emptyset$ for $s>m$.

We now assume that E_i,F_j, $i\in I$, $j\in J$, are defined on $\mathfrak{X}^{m-1}$. Let $0\le r\le m-1$. We will say that E_i,F_j are *standard on* $\mathfrak{X}^r\times X^{m-r-1}$ if for every $i\in I_{r+1}$ we have

$$\begin{aligned}E_i(\overline{y}^i_{r+1},\overline{x}_{r+2,m}) = &\sum_{\substack{j\in J_{r+1},\\ j\neq i}} y_jp_{ij}(\overline{y}^{ij}_{r+1},\overline{x}_{r+2,m})\\ &+\sum_{j\in J'_{r+1}} x_jp_{ij}(\overline{y}^i_{r+1},\overline{x}^j_{r+2,m})+\lambda_i(\overline{y}^i_{r+1},\overline{x}_{r+2,m}),\end{aligned}\tag{3.3}$$

for every $i\in I'_{r+1}$ we have

$$\begin{aligned}E_i(\overline{y}_r,\overline{x}^i_{r+1,m}) = &\sum_{j\in J_r} y_jp_{ij}(\overline{y}^j_r,\overline{x}^i_{r+1,m})\\ &+\sum_{\substack{j\in J'_r,\\ j\neq i}} x_jp_{ij}(\overline{y}_r,\overline{x}^{ij}_{r+1,m})+\lambda_i(\overline{y}_r,\overline{x}^i_{r+1,m}),\end{aligned}\tag{3.4}$$

and, similarly, for every $j\in J_{r+1}$ we have

$$\begin{aligned}F_j(\overline{y}^j_{r+1},\overline{x}_{r+2,m}) = &-\sum_{\substack{i\in I_{r+1},\\ i\neq j}} p_{ij}(\overline{y}^{ij}_{r+1},\overline{x}_{r+2,m})y_i\\ &-\sum_{i\in I'_{r+1}} p_{ij}(\overline{y}^j_{r+1},\overline{x}^i_{r+2,m})x_i-\lambda_j(\overline{y}^j_{r+1},\overline{x}_{r+2,m}),\end{aligned}\tag{3.5}$$

and for every $j\in J'_{r+1}$ we have

$$\begin{aligned}F_j(\overline{y}_r,\overline{x}^j_{r+1,m}) = &-\sum_{i\in I_r} p_{ij}(\overline{y}^i_r,\overline{x}^j_{r+1,m})y_i\\ &-\sum_{\substack{i\in I'_r,\\ i\neq j}} p_{ij}(\overline{y}_r,\overline{x}^{ij}_{r+1,m})x_i-\lambda_j(\overline{y}_r,\overline{x}^j_{r+1,m})\end{aligned}\tag{3.6}$$

for all $x_i\in X$ and all $y_i\in\mathfrak{X}$, where p_{ij} map into $\mathfrak{R}$ and λ_k map into $\mathfrak{Z}$; moreover, $\lambda_k=0$ if $k\notin I\cap J$. In an analogous fashion we define when E_i,F_j are standard on $\mathfrak{X}^{u_1}\times X^{v_1}\times\cdots\times\mathfrak{X}^{u_l}\times X^{v_l}$ for any choice of $u_i,v_i\ge 0$ such that $\sum_{i=1}^l u_i+v_i=m-1$. Further, we will say that E_i,F_j are *r-standard* if they are standard on $\mathfrak{X}^{u_1}\times X^{v_1}\times\cdots\times\mathfrak{X}^{u_l}\times X^{v_l}$ whenever $\sum_{i=1}^l u_i=r$ (and hence $\sum_{i=1}^l v_i=m-r-1$). If all conditions in this definition are fulfilled except that λ_k map into $Z\otimes S$ rather than into $\mathfrak{Z}$, then we will say that E_i,F_j are *r-standard modulo* $\mathfrak{Z}$ (Lemma 3.1 thus states that E_i,F_j are 0-standard modulo $\mathfrak{Z}$). Note that saying that E_i and F_j are standard solutions of (3.1) and (3.2) is the same as saying that they are $(m-1)$-standard.

We can now prove our main result in this section.

Theorem 3.2. *Let R and S be unital algebras with S finite dimensional. If X is a d-free subset of R, then $\mathfrak{X} = \{x \otimes s \,|\, x \in X, s \in S\}$ is a d-free subset of $\mathfrak{R} = R \otimes S$.*

Proof. We have to show that conditions (a) and (b) from Definition 2.1 are fulfilled. We will deal with (a) and at the end mention what modifications are necessary to establish (b). Assume, therefore, that (3.1) holds with $\max\{|I|,|J|\} \leq d$.

The proof of the validity of (a) consists of two parts. We begin by introducing the setting needed for both. Let $0 \leq r \leq m-1$. A special case of (3.1), where $y_i \in \mathfrak{X}$ if $i \leq r+1$ and $y_i = x_i \in X$ if $i > r+1$, reads as follows:

$$\begin{aligned}(3.7)\qquad &\sum_{i\in I_{r+1}} E_i(\overline{y}^i_{r+1}, \overline{x}_{r+2,m})y_i + \sum_{i\in I'_{r+1}} E_i(\overline{y}_{r+1}, \overline{x}^i_{r+2,m})x_i\\ &+ \sum_{j\in J_{r+1}} y_jF_j(\overline{y}^j_{r+1}, \overline{x}_{r+2,m}) + \sum_{j\in J'_{r+1}} x_jF_j(\overline{y}_{r+1}, \overline{x}^j_{r+2,m}) = 0.\end{aligned}$$

Assume that E_i, F_j are standard on $\mathfrak{X}^r \times X^{m-r-1}$ modulo $\mathfrak{Z}$. Applying (3.3) and (3.5) to (3.7) we obtain

$$(3.8)\qquad \sum_{k\in I_{r+1}\cap J_{r+1}} \big[\lambda_k, y_k\big] + \sum_{i\in I'_{r+1}} G_ix_i + \sum_{j\in J'_{r+1}} x_jH_j = 0,$$

where

$$\lambda_k = \lambda_k(\overline{y}^k_{r+1}, \overline{x}_{r+2,m}),$$

$$(3.9)\qquad G_i = G_i(\overline{y}_{r+1}, \overline{x}^i_{r+2,m}) = E_i(\overline{y}_{r+1}, \overline{x}^i_{r+2,m}) - \sum_{j\in J_{r+1}} y_jp_{ij}(\overline{y}^j_{r+1}, \overline{x}^i_{r+2,m}),$$

and

$$(3.10)\qquad H_j = H_j(\overline{y}_{r+1}, \overline{x}^j_{r+2,m}) = F_j(\overline{y}_{r+1}, \overline{x}^j_{r+2,m}) + \sum_{i\in I_{r+1}} p_{ij}(\overline{y}^i_{r+1}, \overline{x}^j_{r+2,m})y_i.$$

Let us add a little comment before we start. In the first part we will deal with functions with indices from $I_{r+1} \cap J_{r+1}$, and in the second part with functions with indices from $I'_{r+1} \cap J'_{r+1}$. The only reason is that this is notationally easier.

We now proceed to the first part of the proof.

Claim 1. Let $0 \leq r \leq m-1$. If E_i, F_j are r-standard modulo $\mathfrak{Z}$, then they are r-standard.

Proof of Claim 1. We will only consider the set $\mathfrak{X}^r \times X^{m-r-1}$ and show that λ_k from (3.3) and (3.5), i.e., λ_k for $k \in I_{r+1} \cap J_{r+1}$, maps into $\mathfrak{Z}$. Other cases can be handled analogously, just the notation is (even) heavier (even the proof that λ_k from (3.4) and (3.6) maps into $\mathfrak{Z}$ is notationally slightly more involved for one also has to deal with the set $\mathfrak{X}^r \times X^{k-r-1} \times \mathfrak{X} \times X^{m-k-1}$).

By our assumption, (3.3) and (3.5) are fulfilled, and so (3.8) holds. We can write

$$\lambda_k = \sum_{t=1}^{N} \lambda_{kt} \otimes b_t,$$

where

$$\lambda_{kt} = \lambda_{kt}(\overline{y}^k_{r+1}, \overline{x}_{r+2,m}) \in Z.$$

Similarly we write

$$G_i = \sum_{w=1}^{N} G_{iw} \otimes b_w \quad \text{and} \quad H_w = \sum_{w=1}^{N} H_{jw} \otimes b_w,$$

where

$$G_{iw} = G_{iw}(\overline{y}_{r+1}, \overline{x}^i_{r+2,m}) \in R$$

and

$$H_{jw} = H_{jw}(\overline{y}_{r+1}, \overline{x}^j_{r+2,m}) \in R.$$

There is nothing to prove if $I_{r+1} \cap J_{r+1} = \emptyset$. We may therefore assume, without loss of generality, that

$$I_{r+1} \cap J_{r+1} = \{1, \dots, q\}$$

for some $q \leq r+1$. Let us set

$$y_k = x_k \otimes s_k, \quad k = 1, \dots, q.$$

We can now write (3.8) as

$$\sum_{k=1}^{q} \sum_{t=1}^{N} \lambda_{kt} x_k \otimes [b_t, s_k] + \sum_{i \in I'_{r+1}} \sum_{w=1}^{N} G_{iw} x_i \otimes b_w + \sum_{j \in J'_{r+1}} \sum_{w=1}^{N} x_j H_{jw} \otimes b_w = 0. \tag{3.11}$$

For each t and k take $\alpha_{ktw} \in \mathbb{F}$, $w = 1, \dots, N$, such that

$$[b_t, s_k] = \sum_{w=1}^{N} \alpha_{ktw} b_w.$$

Hence (3.11) becomes

$$\sum_{w=1}^{N} \left(\sum_{k=1}^{q} \Big(\sum_{t=1}^{N} \alpha_{ktw} \lambda_{kt} \Big) x_k + \sum_{i \in I'_{r+1}} G_{iw} x_i + \sum_{j \in J'_{r+1}} x_j H_{jw} \right) \otimes b_w = 0. \tag{3.12}$$

This, of course, implies

$$\sum_{k=1}^{q} \Big(\sum_{t=1}^{N} \alpha_{ktw} \lambda_{kt} \Big) x_k + \sum_{i \in I'_{r+1}} G_{iw} x_i + \sum_{j \in J'_{r+1}} x_j H_{jw} = 0. \tag{3.13}$$

for every $w = 1, \dots, N$. Note that we are now in a position to apply Remark 2.2 (with $(I_{r+1} \cap J_{r+1}) \cup I'_{r+1} \subseteq I$ playing the role of I and $J'_{r+1} \subseteq J \setminus \{1\}$ playing the role of J). Accordingly,

$$\sum_{t=1}^{N} \alpha_{ktw} \lambda_{kt} = 0$$

for every $k=1,\dots,q$ and every $w=1,\dots,N$. This yields

$$[\lambda_k, x_k\otimes s_k] = \sum_{t=1}^N \lambda_{kt}x_k\otimes [b_t,s_k] = \sum_{w=1}^N\Big(\sum_{t=1}^N \alpha_{ktw}\lambda_{kt}\Big)x_k\otimes b_w = 0$$

for every $k=1,\dots,q$, every $x_k\in X$ and every $s_k\in S$. This means that

$$\lambda_k = \lambda_k(\overline{y}^k_{r+1},\overline{x}_{r+2,m})$$

commutes with every element of the form $x_k\otimes s_k$, and hence with every element in $\mathfrak{R}$. Therefore $\lambda_k\in\mathfrak{Z}$ for all $k\in I_{r+1}\cap J_{r+1}$, which is the desired conclusion.

We proceed to the second part of the proof.

Claim 2. Let $0\le r\le m-2$. If E_i,F_j are r-standard, then they are also $(r+1)$-standard.

Proof of Claim 2. Again (3.3) and (3.5) hold by our assumption, and so (3.8) holds, too. However, since now we are assuming that λ_k map into $\mathfrak{Z}$, this identity reduces to

$$\sum_{i\in I'_{r+1}} G_i x_i + \sum_{j\in J'_{r+1}} x_j H_j = 0, \tag{3.14}$$

where G_i and H_j are given by (3.9) and (3.10), respectively. We can now use Lemma 3.1 (for any fixed y_i and x_k with $k\notin I'_{r+1}\cup J'_{r+1}$). Accordingly, for every $i\in I'_{r+1}$ we have

$$G_i = \sum_{\substack{j\in J'_{r+1},\\ j\ne i}} x_j p_{ij}(\overline{y}_{r+1},\overline{x}^{ij}_{r+2,m}) + \lambda_i(\overline{y}_{r+1},\overline{x}^{i}_{r+2,m}),$$

and for every $j\in J'_{r+1}$ we have

$$H_j = -\sum_{\substack{i\in I'_{r+1},\\ i\ne j}} p_{ij}(\overline{y}_{r+1},\overline{x}^{ij}_{r+2,m})x_i - \lambda_j(\overline{y}_{r+1},\overline{x}^{j}_{r+2,m}),$$

where p_{ij} map into $\mathfrak{R}$ (if $r=m-1$ then they are all zero) and λ_k map into $Z\otimes S$. Moreover, $\lambda_k=0$ if $k\notin I\cap J$. From (3.9) and (3.10) we now see that E_i and F_j are of the desired form on $\mathfrak{X}^{r+1}\times X^{m-r-2}$ for $i\in I'_{r+1}$ and $j\in J'_{r+1}$, except that λ_k have their ranges in $Z\otimes S$. The same (just notationally more annoying) proof shows that this is also true for $i\in I_{r+1}$ and $j\in J_{r+1}$, and for all other sets $\mathfrak{X}^{u_1}\times X^{v_1}\times\dots\times\mathfrak{X}^{u_l}\times X^{v_l}$ with $u_1+\dots+u_l=r+1$ and $v_1+\dots+v_l=m-r-2$. Accordingly, we can make use of Claim 1 telling us that λ_k map into $\mathfrak{Z}$. This completes the proof of Claim 2.

Both claims together with Lemma 3.1 imply the validity of condition (a). Namely, the lemma and Claim 1 show that the assumption of Claim 2 is fulfilled for $r=0$. Claim 2 then yields (a) by induction on r.

It remains to verify the validity of condition (b). Assume that (3.2) holds with $\max\{|I|,|J|\}\le d-1$, and just follow step by step the proof of the validity of (a). The first change that occurs is that in (3.7) and (3.8) one has to replace "$=0$" by "$\in\mathfrak{Z}$". The same replacement must therefore be made in the identities (3.11) and (3.12) from the proof of Claim 1. This implies that the expression in (3.13) lies in Z

rather then being equal to zero. However, since $\max\{|I|,|J|\} \le d-1$, this expression must be zero anyway (see the comment on condition (b') following Definition 2.1). The rest of the proof of Claim 1 is thus the same as above. At the beginning of the proof of Claim 2 one has to substitute "$\in \mathfrak{Z}$" for "$= 0$" in (3.14). However, since Lemma 3.1 also covers the central case, this change does not affect the proof. Thus, the condition (b) is fulfilled, too. □

Those d-free sets that appear in applications usually have some algebraic structure; the least one usually requires is that they are additive subgroups. Let us therefore record the following immediate corollary to Theorem 3.2.

Corollary 3.3. *Let R and S be unital algebras with S finite dimensional, and let X be a linear subspace of R. If X is a d-free subset of R, then $X \otimes S$ is a d-free subset of $R \otimes S$.*

Proof. The space $X \otimes S$ is the linear span of $\mathfrak{X} = \{x \otimes s \,|\, x \in X, s \in S\}$. The desired conclusion therefore follows from the fact that if a set is d-free, then a larger set is d-free, too [8, Corollary 3.5]. □

4. A counterexample in infinite dimensions

In this short section we give an example showing that the result of the previous section in general does not hold if S is infinite dimensional, not even if the set X is a simple algebra which is a d-free subset of some larger algebra R for any $d \ge 1$.

Example 4.1. Let R be an $\mathbb{F}$-algebra and let X be its linear subspace. Suppose there exists a sequence $(e_n)_{n=1}^{\infty}$ of elements in R which satisfies the following conditions:

(a) For each n there exists $x \in X$ such that $xe_n \neq 0$.
(b) For each $x \in X$ we have $xe_n = e_n x = 0$ for all but finitely many n.

Let us show that then $X \otimes \mathbb{F}[\xi]$ is a not a 2-free subset of $R \otimes \mathbb{F}[\xi]$. We identify $R \otimes \mathbb{F}[\xi]$ with $R[\xi]$ (and hence $X \otimes \mathbb{F}[\xi]$ with $X[\xi]$). Define $E, F : X[\xi] \to R[\xi]$ by

$$E\Big(\sum_{i=0}^{k} x_i \xi^i\Big) = \sum_{i=0}^{k}\sum_{n=1}^{\infty} x_i e_n \xi^{n+i},$$
$$F\Big(\sum_{j=0}^{l} u_j \xi^j\Big) = \sum_{j=0}^{l}\sum_{n=1}^{\infty} e_n u_j \xi^{n+j}.$$

In view of (b), each of this summations contains only finitely many nonzero terms. The definitions therefore make sense. As one immediately checks,

$$E(y)z = yF(z)$$

holds for all $y, z \in X[\xi]$. If $X[\xi]$ was a 2-free subset of $R[\xi]$, there would exist $p = \sum_{t=1}^{m} r_k \xi^k$ such that $E(y) = yp$ and $F(z) = pz$ for all $y, z \in R[\xi]$. Therefore $E(x_0)$ had degree at most m for every $x_0 \in X$. However, by (a) we can choose x_0 so that $x_0 e_{m+1} \neq 0$, and so $E(x_0)$ has degree at least $m+1$ in light of definition of E. Therefore $X[\xi]$ is not 2-free in $R[\xi]$.

The point we wish to make is that, under the above conditions, X can still be a 2-free subset of R. Consider the following concrete example. Let $X = A$ be the algebra all (countably) infinite matrices with finitely many nonzero entries. It is well-known that A is a (non-unital) simple algebra. Moreover, A does not satisfy a nontrivial polynomial identity. The general theory therefore tells us that there exist algebras R containing A such that A is their d-free subsets for every $d \geq 1$; say, the maximal left algebra of quotients of A is an example of such an algebra R [8, Corollary 5.12]. Let $e_n \in A$ be the diagonal matrix whose only nonzero term is 1 on the (n, n) position. It is immediate that the sequence $(e_n)_{n=1}^{\infty}$ satisfies conditions (a) and (b). Therefore $A \otimes \mathbb{F}[\xi]$ is a not a 2-free subset of $R \otimes \mathbb{F}[\xi]$.

5. The infinite dimensional case

In Section 3 we were assuming that S is finite dimensional. This assumption was actually used only at one place, namely in the definition of p_{ij} and λ_k in the proof of Lemma 3.1, where summations make sense only if they are finite. If $J = \emptyset$ and S is infinite dimensional, then by following the proof of this lemma (the only difference is that a basis of S is now infinite) we see that all p_{ijt} and λ_{kt} are 0, so one simply defines $p_{ij} = 0$ and $\lambda_k = 0$. With reference to the notation introduced in the previous sections, we can thus state the following theorem.

Theorem 5.1. *Let R and S be unital algebras, and let X be a d-free subset of R. Set $\mathfrak{X} = \{x \otimes s \,|\, x \in X, s \in S\}$ and let $E_i : \mathfrak{X}^{m-1} \to R \otimes S$, $i \in I$, be arbitrary functions. Suppose that either*

(a) $\sum_{i\in I} E_i(\overline{y}_m^i) y_i = 0$ *for all* $\overline{y}_m \in \mathfrak{X}^m$ *and* $|I| \leq d$, *or*
(b) $\sum_{i\in I} E_i(\overline{y}_m^i) y_i \in \mathfrak{Z}$ *for all* $\overline{y}_m \in \mathfrak{X}^m$ *and* $|I| \leq d - 1$.

Then each $E_i = 0$.

A similar theorem of course holds for functional identities

$$\sum_{j\in J} y_j F_y(\overline{y}_m^i) = 0 \quad \text{and} \quad \sum_{j\in J} y_j F_y(\overline{y}_m^i) \in \mathfrak{Z}.$$

As we saw, Theorem 3.2 does not hold for infinite dimensional algebras S. In the next theorem we will see that a special but important functional identity can be handled in infinite dimensions under the following mild technical assumptions:

($*$) There exist $x_1, x_2 \in X$ such that for all α_1, α_2 in the center Z of R,

$$\alpha_1 x_1 x_2 + \alpha_2 x_2 x_1 \in Z x_1 + Z x_2 + Z$$

implies $\alpha_1 = \alpha_2 = 0$.

($**$) For each $x \in X$ there exists $u \in X$ such that for all $\alpha \in Z$,

$$\alpha u \in Z x + Z$$

implies $\alpha = 0$.

Later we will see that these conditions are satisfied in the situation in which we are primarily interested.

In the course of the proof we will make use of the results on the so-called quasi-polynomials. We refer the reader to [8, Chapter 4] (or to the original source [4]) for a complete account on this topic. Let us give here only a brief informal introduction. Let R be a ring with center Z. A *quasi-polynomial* of degree 2 on $X \subseteq R$ is a function $F : X^2 \to R$ of the form

$$F(x_1, x_2) = \lambda_1 x_1 x_2 + \lambda_2 x_2 x_1 + \mu_1(x_1)x_2 + \mu_2(x_2)x_1 + \nu(x_1, x_2),$$

where $\lambda_1, \lambda_2 \in Z$, $\mu_1, \mu_2 : X \to Z$ and $\nu : X^2 \to Z$. We call λ_i, μ_i and ν the *coefficients* of F. The coefficient ν plays a special role; we call it the *central coefficient.* A quasi-polynomial of degree m is defined analogously, i.e., as a sum of functions of the form

$$(x_1, \dots, x_m) \mapsto \lambda(x_{i_1}, \dots, x_{i_k})x_{i_{k+1}} \dots x_{i_m}$$

where $\lambda : X^k \to Z$. If a quasi-polynomial of degree at most m is zero on X^m and X is $(m + 1)$-free, then all its coefficients are zero; moreover, under the assumption that its central coefficient is 0 it is enough to assume that X is m-free. This is the content of [8, Lemma 4.4]. This result is very useful, although fairly easy. The next one is deeper. It states that if the function

$$Q(\overline{x}_m) = \sum_{i \in I} \left[E_i(\overline{x}^i_m), x_i\right]$$

is a quasi-polynomial (of degree at most m) on X, then all E_i are quasi-polynomials, provided that X is $(m + 1)$-free, or, if the central coefficient of Q is 0, m-free. This is a special case of [8, Theorem 4.13].

We now have enough information to prove the next theorem which, roughly speaking, states that under appropriate assumptions the functional identity (5.1) below has only standard solutions.

Theorem 5.2. *Let R and S be unital algebras, and let X be a 3-free subset of R which satisfies conditions $(*)$ and $(**)$. Set $\mathfrak{X} = \{x \otimes s \mid x \in X, s \in S\}$ and $\mathfrak{R} = R \otimes S$. If $F : \mathfrak{X}^2 \to \mathfrak{R}$ satisfies*

$$F(y, z)w + F(w, y)z + F(z, w)y = wF(y, z) + zF(w, y) + yF(z, w) \tag{5.1}$$

for all $y, z, w \in \mathfrak{X}$, then F is of the form

$$F(y, z) = \varepsilon yz + \varepsilon' zy + \mu(y)z + \mu(z)y + \nu(y, z)$$

for all $y, z \in \mathfrak{X}$, where $\varepsilon, \varepsilon'$ lie in the center $\mathfrak{Z}$ of $\mathfrak{R}$, and $\mu : \mathfrak{X} \to \mathfrak{Z}$, $\nu : \mathfrak{X}^2 \to \mathfrak{Z}$.

Proof. Pick a basis $\{b_t \mid t \in T\}$ of S. We can write

$$F(y, z) = \sum_{t \in T} f_t(y, z) \otimes b_t,$$

where $f_t : \mathfrak{X}^2 \to R$ and for each pair $y, z \in \mathfrak{X}$ we have $f_t(y, z) = 0$ for all but finitely many $t \in T$.

We first consider (5.1) in the case where all elements lie in $X \otimes 1$. As above, we identify X with $X \otimes 1$ and sometimes write x for $x \otimes 1$. We have

$$\sum_{t \in T} \Big(f_t(x, u)v + f_t(v, x)u + f_t(u, v)x\Big) \otimes b_t = \sum_{t \in T} \Big(v f_t(x, u) + u f_t(v, x) + x f_t(u, v)\Big) \otimes b_t$$

for all $x,u,v\in X$. Consequently,

$$f_t(x,u)v+f_t(v,x)u+f_t(u,v)x=vf_t(x,u)+uf_t(v,x)+xf_t(u,v) \tag{5.2}$$

for each $t\in T$. Since R is 3-free, we may use [8, Theorem 4.13] to conclude that each f_t is a quasi-polynomial. This means that there exist $\epsilon_t,\epsilon_t'\in Z$ and functions $\mu_t,\mu_t':X\to Z$, $\nu_t:X^2\to Z$ such that

$$f_t(x,u)=\varepsilon_t xu+\varepsilon_t' ux+\mu_t(x)u+\mu_t'(u)x+\nu_t(x,u) \tag{5.3}$$

for all $x,u\in X$. A little more can be said, namely,

$$\mu_t=\mu_t'. \tag{5.4}$$

Indeed, using (5.3) in (5.2) we obtain

$$(\mu_t-\mu_t')(x)[u,v]+(\mu_t-\mu_t')(u)[v,x]+(\mu_t-\mu_t')(v)[x,u]=0,$$

which yields (5.4) by [8, Lemma 4.4].

Let x_1,x_2 be elements satisfying $(*)$. Since $f_t(x_1,x_2)=0$ for all but finitely many $t\in T$, it follows from (5.3) that ε_t and ε_t' are 0 for all but finitely many $t\in T$. This makes it possible for us to define

$$\varepsilon=\sum_{t\in T}\varepsilon_t\otimes b_t \quad\text{and}\quad \varepsilon'=\sum_{t\in T}\varepsilon_t'\otimes b_t.$$

Let us point out that ε,ε' are elements in $Z\otimes S$ – only later we will see that they actually lie in $\mathfrak{Z}$. Now take $x\in X$ and let u be an element satisfying $(**)$. Since the identities $f_t(x,u)=0$ and $\varepsilon_t=\varepsilon_t'=0$ hold for all but finitely many $t\in T$, it follows from (5.3) that $\mu_t(x)=0$ for all but finitely many $t\in T$. Finally, since, for each pair $x,u\in X$, the elements $\varepsilon_t,\varepsilon_t',\mu_t(x),\mu_t(u),f_t(x,u)$ are 0 for all but finitely many t, the same is true for $\nu_t(x,u)$.

Now take $x,u,v\in X$ and $s\in S$. Consider the following special case of (5.1):

$$\begin{aligned}F(x,u)(v\otimes s)+F(v\otimes s,x)u+F(u,v\otimes s)x&\\ =(v\otimes s)F(x,u)+uF(v\otimes s,x)+xF(u,v\otimes s).&\end{aligned}$$

That is,

$$\begin{aligned}&\sum_{t\in T}f_t(x,u)v\otimes b_ts+\sum_{w\in T}f_w(v\otimes s,x)u\otimes b_w+\sum_{w\in T}f_w(u,v\otimes s)x\otimes b_w\\ =&\sum_{t\in T}vf_t(x,u)\otimes sb_t+\sum_{w\in T}uf_w(v\otimes s,x)\otimes b_w+\sum_{w\in T}xf_w(u,v\otimes s)\otimes b_w.\end{aligned}$$

Let $\beta_{tw},\gamma_{tw}\in\mathbb{F}$ be such that

$$b_ts=\sum_{w\in T}\beta_{tw}b_w \quad\text{and}\quad sb_t=\sum_{w\in T}\gamma_{tw}b_w. \tag{5.5}$$

Of course, for each $t \in T$ there can be only finitely many nonzero β_{tw} and γ_{tw}. Using (5.5) in the last identity it readily follows that

$$\begin{aligned} &\sum_{t\in T}\beta_{tw}f_t(x,u)v + f_w(v\otimes s,x)u + f_w(u,v\otimes s)x \\ &= \sum_{t\in T}\gamma_{tw}vf_t(x,u) + uf_w(v\otimes s,x) + xf_w(u,v\otimes s) \end{aligned} \tag{5.6}$$

for every $w \in T$. Fix $s \in S$; since, by (5.3), the functions $\sum_{t\in T}\beta_{tw}f_t(x,u)v$ and $\sum_{t\in T}\gamma_{tw}vf_t(x,u)$ are quasi-polynomials, it follows from [8, Theorem 4.13] that the functions $f_w(v\otimes s,x)$ and $f_w(u,v\otimes s)$ are quasi-polynomials, too. Therefore we have

$$f_w(v\otimes s,x) = \lambda_w vx + \lambda'_w xv + \mu_w(v\otimes s)x + \eta_w(x)v + \nu_w(v\otimes s,x), \tag{5.7}$$

and

$$f_w(u,v\otimes s) = \delta_w uv + \delta'_w vu + \sigma_w(u)v + \mu'_w(v\otimes s)u + \nu_w(u,v\otimes s), \tag{5.8}$$

where $\lambda_w, \lambda'_w, \beta_w, \beta'_w$ are elements in Z (they depend on s) and the functions μ_w, η_w etc. map into Z (we have used the same notation μ_w, ν_w as above since these new functions are in fact extensions of the old μ_w, ν_w). Rewriting (5.6) according to (5.7), (5.8), and (5.3) we get

$$\begin{aligned} &\sum_{t\in T}\beta_{tw}\varepsilon_t xuv + \sum_{t\in T}\beta_{tw}\varepsilon'_t uxv + \sum_{t\in T}\beta_{tw}\mu_t(x)uv \\ &+ \sum_{t\in T}\beta_{tw}\mu_t(u)xv + \sum_{t\in T}\beta_{tw}\nu_t(x,u)v \\ &+ \lambda_w vxu + \lambda'_w xvu + \mu_w(v\otimes s)xu + \eta_w(x)vu + \nu_w(v\otimes s,x)u \\ &+ \delta_w uvx + \delta'_w vux + \sigma_w(u)vx + \mu'_w(v\otimes s)ux + \nu_w(u,v\otimes s)x \\ =&\sum_{t\in T}\gamma_{tw}\varepsilon_t vxu + \sum_{t\in T}\gamma_{tw}\varepsilon'_t vux + \sum_{t\in T}\gamma_{tw}\mu_t(x)vu \\ &+ \sum_{t\in T}\gamma_{tw}\mu_t(u)vx + \sum_{t\in T}\gamma_{tw}\nu_t(x,u)v \\ &+ \lambda_w uvx + \lambda'_w uxv + \mu_w(v\otimes s)ux + \eta_w(x)uv + \nu_w(v\otimes s,x)u \\ &+ \delta_w xuv + \delta'_w xvu + \sigma_w(u)xv + \mu'_w(v\otimes s)xu + \nu_w(u,v\otimes s)x. \end{aligned} \tag{5.9}$$

We can interpret (5.9) as equality of two quasi-polynomials of degree 3 with zero central coefficients. By [8, Lemma 4.4], this is possible only if these quasi-polynomials have identical coefficients. Comparing the coefficients at vxu we obtain $\lambda_w = \sum_{t\in T}\gamma_{tw}\varepsilon_t$. On the other hand, examining the coefficients at uvx and xuv we get $\delta_w = \lambda_w$ and $\delta_w = \sum_{t\in T}\beta_{tw}\varepsilon_t$. Thus,

$$\lambda_w = \delta_w = \sum_{t\in T}\beta_{tw}\varepsilon_t = \sum_{t\in T}\gamma_{tw}\varepsilon_t. \tag{5.10}$$

Similarly we see that

$$\lambda'_w = \delta'_w = \sum_{t\in T}\beta_{tw}\varepsilon'_t = \sum_{t\in T}\gamma_{tw}\varepsilon'_t, \tag{5.11}$$

and also that

$$\eta_w(x) = \sum_{t\in T}\beta_{tw}\mu_t(x) = \sum_{t\in T}\gamma_{tw}\mu_t(x),$$
$$\sigma_w(u) = \sum_{t\in T}\beta_{tw}\mu_t(u) = \sum_{t\in T}\gamma_{tw}\mu_t(u),$$
$$\mu_w(v\otimes s) = \mu'_w(v\otimes s).$$

We can now rewrite (5.7) and (5.8) as

$$\begin{aligned} f_w(v\otimes s, x) &= \Big(\sum_{t\in T}\beta_{tw}\varepsilon_t\Big)vx + \Big(\sum_{t\in T}\beta_{tw}\varepsilon'_t\Big)xv \\ &\quad + \mu_w(v\otimes s)x + \Big(\sum_{t\in T}\beta_{tw}\mu_t(x)\Big)v + \nu_w(v\otimes s, x) \end{aligned} \tag{5.12}$$

and

$$\begin{aligned} f_w(u, v\otimes s) &= \Big(\sum_{t\in T}\beta_{tw}\varepsilon_t\Big)uv + \Big(\sum_{t\in T}\beta_{tw}\varepsilon'_t\Big)vu \\ &\quad + \Big(\sum_{t\in T}\beta_{tw}\mu_t(u)\Big)v + \mu_w(v\otimes s)u + \nu_w(u, v\otimes s). \end{aligned} \tag{5.13}$$

Similarly as above we see that for every $s\in S$ we have $\sum_{t\in T}\beta_{tw}\varepsilon_t = \sum_{t\in T}\beta_{tw}\varepsilon'_t = 0$ for all but finitely many w, and consequently, for every $v\otimes s\in\mathfrak{X}$ we have $\mu_w(v\otimes s) = 0$ for all but finitely many w. This makes it possible for us to define $\mu : \mathfrak{X}\to Z\otimes S$ by

$$\mu(v\otimes s) = \sum_{w\in T}\mu_w(v\otimes s)\otimes b_w.$$

There is another important relation that we can deduce from (5.10), namely

$$\begin{aligned} [\varepsilon, v\otimes s] &= \sum_{t\in T}[\varepsilon_t\otimes b_t, v\otimes s] = \sum_{t\in T}\varepsilon_t v\otimes[b_t, s] \\ &= \sum_{t\in T}\varepsilon_t v\otimes\Big(\sum_{w\in T}(\beta_{tw}-\gamma_{tw})b_w\Big) = \sum_{w\in T}\Big(\sum_{t\in T}(\beta_{tw}-\gamma_{tw})\varepsilon_t\Big)v\otimes b_w = 0. \end{aligned}$$

Similarly we infer from (5.11) that $[\varepsilon', v\otimes s] = 0$. Accordingly, $\varepsilon, \varepsilon'\in\mathfrak{Z}$.

Now take $x, u, v\in X$ and $q, s\in S$. As a special case of (5.1)we have

$$\begin{aligned} &F(x, u\otimes q)(v\otimes s) + F(v\otimes s, x)(u\otimes q) + F(u\otimes q, v\otimes s)x \\ &= (v\otimes s)F(x, u\otimes q) + (u\otimes q)F(v\otimes s, x) + xF(u\otimes q, v\otimes s). \end{aligned}$$

That is,

$$\begin{aligned} &\sum_{t\in T}f_t(x, u\otimes q)v\otimes b_t s + \sum_{t\in T}f_t(v\otimes s, x)u\otimes b_t q \\ &+ \sum_{w\in T}f_w(u\otimes q, v\otimes s)x\otimes b_w = \sum_{t\in T}vf_t(x, u\otimes q)\otimes sb_t \\ &+ \sum_{t\in T}uf_t(v\otimes s, x)\otimes qb_t + \sum_{w\in T}xf_w(u\otimes q, v\otimes s)\otimes b_w. \end{aligned} \tag{5.14}$$

Let β_{tw}, γ_{tw} be scalars satisfying (5.5), and let ρ_{tw}, τ_{tw} be scalars satisfying

$$b_t q = \sum_{w\in T} \rho_{tw} b_w \quad \text{and} \quad q b_t = \sum_{w\in T} \tau_{tw} b_w.$$

Note that (5.14) implies that for every $w \in T$ we have

$$\begin{aligned} &\sum_{t\in T} \beta_{tw} f_t(x, u\otimes q)v + \sum_{t\in T} \rho_{tw} f_t(v\otimes s, x)u + f_w(u\otimes q, v\otimes s)x \\ &= \sum_{t\in T} \gamma_{tw} v f_t(x, u\otimes q) + \sum_{t\in T} \tau_{tw} u f_t(v\otimes s, x) + x f_w(u\otimes q, v\otimes s). \end{aligned} \tag{5.15}$$

Fix s and q. Since $f_t(x, u\otimes q)$ and $f_t(v\otimes s, x)$ are quasi-polynomials by (5.12) and (5.13), [8, Theorem 4.13] implies that $f_w(u\otimes q, v\otimes s)$ is a quasi-polynomial, too. Thus,

$$\begin{aligned} f_w(u\otimes q, v\otimes s) &= \alpha_w uv + \alpha'_w vu \\ &\quad + \zeta_w(u\otimes q)v + \zeta'_w(v\otimes s)u + \nu_w(u\otimes q, v\otimes s) \end{aligned} \tag{5.16}$$

where $\alpha_w, \alpha'_w \in Z$ and ζ_w, ζ'_w, ν_w map into Z (α_w and α'_w depend on q and s, ζ_w depends on s, and ζ'_w depends on q). The usual argument in particular shows that for each pair $u\otimes q, v\otimes s \in \mathfrak{X}$ there can be only finitely many w such that $\nu_w(u\otimes q, v\otimes s) = 0$. We can therefore define $\nu : \mathfrak{X}^2 \to Z\otimes S$ by

$$\nu(u\otimes q, v\otimes s) = \sum_{w\in T} \nu_w(u\otimes q, v\otimes s)\otimes b_w.$$

Setting (5.16), as well as (5.12) and (5.13) (with some notational adjustments) to (5.15) we obtain

$$
\begin{aligned}
&\sum_{t\in T}\sum_{z\in T}\beta_{tw}\rho_{zt}\varepsilon_z xuv+\sum_{t\in T}\sum_{z\in T}\beta_{tw}\rho_{zt}\varepsilon'_z uxv\\
&+\sum_{t\in T}\sum_{z\in T}\beta_{tw}\rho_{zt}\mu_z(x)uv+\sum_{t\in T}\beta_{tw}\mu_t(u\otimes q)xv+\sum_{t\in T}\beta_{tw}\nu_t(x,u\otimes q)v\\
&+\sum_{t\in T}\sum_{z\in T}\rho_{tw}\beta_{zt}\varepsilon_z vxu+\sum_{t\in T}\sum_{z\in T}\rho_{tw}\beta_{zt}\varepsilon'_z xvu\\
&+\sum_{t\in T}\rho_{tw}\mu_t(v\otimes s)xu+\sum_{t\in T}\sum_{z\in T}\rho_{tw}\beta_{zt}\mu_z(x)vu+\sum_{t\in T}\rho_{tw}\nu_t(v\otimes s,x)u\\
&+\alpha_w uvx+\alpha'_w vux+\zeta_w(u\otimes q)vx+\zeta'_w(v\otimes s)ux+\nu_w(u\otimes q,v\otimes s)x\\
=&\sum_{t\in T}\sum_{z\in T}\gamma_{tw}\rho_{zt}\varepsilon_z vxu+\sum_{t\in T}\sum_{z\in T}\gamma_{tw}\rho_{zt}\varepsilon'_z vux\\
&+\sum_{t\in T}\sum_{z\in T}\gamma_{tw}\rho_{zt}\mu_z(x)vu+\sum_{t\in T}\gamma_{tw}\mu_t(u\otimes q)vx+\sum_{t\in T}\gamma_{tw}\nu_t(x,u\otimes q)v\\
&+\sum_{t\in T}\sum_{z\in T}\tau_{tw}\beta_{zt}\varepsilon_z uvx+\sum_{t\in T}\sum_{z\in T}\tau_{tw}\beta_{zt}\varepsilon'_z uxv\\
&+\sum_{t\in T}\tau_{tw}\mu_t(v\otimes s)ux+\sum_{t\in T}\sum_{z\in T}\tau_{tw}\beta_{zt}\mu_z(x)uv+\sum_{t\in T}\tau_{tw}\nu_t(v\otimes s,x)u\\
&+\alpha_w xuv+\alpha'_w xvu+\zeta_w(u\otimes q)xv+\zeta'_w(v\otimes s)xu+\nu_w(u\otimes q,v\otimes s)x.
\end{aligned}
\tag{5.17}
$$

We are again at a position to apply [8, Lemma 4.4]. In particular we infer that

$$\alpha_w=\sum_{t\in T}\sum_{z\in T}\beta_{tw}\rho_{zt}\varepsilon_z.$$

This yields

$$
\begin{aligned}
&\varepsilon(u\otimes q)(v\otimes s)=\Big(\sum_{z\in T}\varepsilon_z u\otimes b_z q\Big)(v\otimes s)\\
=&\bigg(\sum_{z\in T}\varepsilon_z u\otimes\Big(\sum_{t\in T}\rho_{zt}b_t\Big)\bigg)(v\otimes s)=\sum_{t\in T}\Big(\sum_{z\in T}\rho_{zt}\varepsilon_z\Big)uv\otimes b_t s\\
=&\sum_{t\in T}\Big(\sum_{z\in T}\rho_{zt}\varepsilon_z\Big)uv\otimes\Big(\sum_{w\in T}\beta_{tw}b_w\Big)=\sum_{w\in T}\Big(\sum_{t\in T}\sum_{z\in T}\beta_{tw}\rho_{zt}\varepsilon_z\Big)uv\otimes b_w\\
=&\sum_{w\in T}\alpha_w uv\otimes b_w.
\end{aligned}
\tag{5.18}
$$

Similarly we see that

$$\alpha'_w=\sum_{t\in T}\sum_{z\in T}\rho_{tw}\beta_{zt}\varepsilon'_z$$

and hence

$$\varepsilon'(v \otimes s)(u \otimes q) = \sum_{w \in T} \alpha'_w vu \otimes b_w. \tag{5.19}$$

Next we infer from (5.17) that

$$\zeta_w(u \otimes q) = \sum_{t \in T} \beta_{tw}\mu_t(u \otimes q) = \sum_{t \in T} \gamma_{tw}\mu_t(u \otimes q).$$

On the one hand, this implies

$$\begin{aligned} &\mu(u \otimes q)(v \otimes s) = \sum_{t \in T} \mu_t(u \otimes q)v \otimes b_t s \\ &= \sum_{t \in T} \mu_t(u \otimes q)v \otimes \Big(\sum_{w \in T} \beta_{tw} b_w\Big) = \sum_{w \in T}\Big(\sum_{t \in T} \beta_{tw}\mu_t(u \otimes q)\Big)v \otimes b_w \\ &= \sum_{w \in T} \zeta_w(u \otimes q)v \otimes b_w, \end{aligned} \tag{5.20}$$

and, on the other hand, it can be easily checked that it also implies

$$\mu(u \otimes q)(v \otimes s) = (v \otimes s)\mu(u \otimes q).$$

Therefore, $\mu(u \otimes q) \in \mathfrak{Z}$.

Further, (5.17) yields

$$\zeta'_w(v \otimes s) = \sum_{t \in T} \rho_{tw}\mu_t(v \otimes s),$$

from which

$$\mu(v \otimes s)(u \otimes q) = \sum_{w \in T} \zeta'_w(v \otimes s)u \otimes b_w \tag{5.21}$$

follows.

Let us gather together all information obtained so far. From (5.18), (5.19), (5.20), (5.21), along with (5.16) we see that

$$\begin{aligned} &F(u \otimes q, v \otimes s) = \sum_{w \in T} f_w(u \otimes q, v \otimes s) \otimes b_w \\ &= \sum_{w \in T}\Big(\alpha_w uv + \alpha'_w vu + \zeta_w(u \otimes q)v + \zeta'_w(v \otimes s)u + \nu_w(u \otimes q, v \otimes s)\Big) \otimes b_w \\ &= \varepsilon(u \otimes q)(v \otimes s) + \varepsilon'(v \otimes s)(u \otimes q) + \mu(u \otimes q)(v \otimes s) \\ &\quad + \mu(v \otimes s)(u \otimes q) + \nu(u \otimes q, v \otimes s). \end{aligned}$$

The only thing that remains to be shown is that $\nu(u \otimes q, v \otimes s)$ lies in $\mathfrak{Z}$. To this end, we first observe that (5.1) now reduces to

$$\nu(y,z)w + \nu(w,y)z + \nu(z,w)y = w\nu(y,z) + z\nu(w,y) + y\nu(z,w).$$

Writing $x \otimes p$ for y, $u \otimes q$ for z, and $v \otimes s$ for w and using the definition of ν we obtain

$$\sum_{t\in T}\nu_t(x\otimes p, u\otimes q)v\otimes [b_t, s] + \sum_{t\in T}\nu_t(v\otimes s, x\otimes p)u\otimes [b_t, q] + \sum_{t\in T}\nu_t(u\otimes q, v\otimes s)x\otimes [b_t, p] = 0. \tag{5.22}$$

Take $\sigma_{tw}, \xi_{tw}, \omega_{tw} \in \mathbb{F}$ such that

$$[b_t, s] = \sum_{w\in T}\sigma_{tw}b_w, \quad [b_t, q] = \sum_{w\in T}\xi_{tw}b_w, \quad [b_t, p] = \sum_{w\in T}\omega_{tw}b_w,$$

and note that (5.22) implies that for each $w \in T$ we have

$$\Big(\sum_{t\in T}\sigma_{tw}\nu_t(x\otimes p, u\otimes q)\Big)v + \Big(\sum_{t\in T}\xi_{tw}\nu_t(v\otimes s, x\otimes p)\Big)u + \Big(\sum_{t\in T}\omega_{tw}\nu_t(u\otimes q, v\otimes s)\Big)x = 0.$$

Using [8, Lemma 4.4] (or Remark 2.2) it follows that all the coefficients of this quasi-polynomial are 0. In particular,

$$\sum_{t\in T}\omega_{tw}\nu_t(u\otimes q, v\otimes s) = 0,$$

which readily yields $[\nu(u\otimes q, v\otimes s), x\otimes p] = 0$. This shows that $\nu(u\otimes q, v\otimes s) \in \mathfrak{Z}$. $\square$

Assume now that A is a prime ring. Recall that the center C of the maximal left ring of quotients $Q_{ml}(A)$ of A is a field, called the *extended centroid* of A. Given $t \in A$, we denote by $\deg(t)$ the degree of algebraicity of t over C if t is algebraic, or ∞ if it is not algebraic. We set $\deg(A) = \sup\{\deg(t) \,|\, t \in A\}$. It is well-known that $\deg(A) \le n < \infty$ if and only if A satisfies the standard polynomial identity of degree $2n$. Equivalently, A can be embedded into the ring of $n\times n$ matrices over some field. The fundamental theorem on functional identities, which is due to Beidar [3], states that A is a d-free subset of $Q_{ml}(A)$ if and only if $\deg(A) \ge d$ [8, Corollary 5.12]. This theorem combined with Theorem 5.2 will yield the next corollary. We also need the following auxiliary result, based on basic results on polynomial identities.

Lemma 5.3. *Let A be a prime ring. The following conditions are equivalent:*

(i) *For each pair $x_1, x_2 \in A$, the elements x_1x_2, x_2x_1, x_1, x_2 and 1 are linearly dependent over the extended centroid C of A.*

(ii) $\deg(A) \le 2$.

Proof. (i) $\Rightarrow$ (ii). Let c_5 be the 5th Capelli polynomial, i.e.,

$$c_5 = c_5(\xi_1,\xi_2,\xi_3,\xi_4,\xi_5,,\eta_1,\eta_2,\eta_3,\eta_4) = \sum_{\sigma\in S_5}\operatorname{sgn}(\sigma)\xi_{\sigma(1)}\eta_1\xi_{\sigma(2)}\eta_2\xi_{\sigma(3)}\eta_3\xi_{\sigma(4)}\eta_4\xi_{\sigma(5)};$$

here, ξ_i and η_j are noncommuting indeterminates. The condition (i) implies that

$$f = f(\xi_1, \xi_2, \eta_1, \eta_2, \eta_3, \eta_4) = c_5(\xi_1\xi_2, \xi_2\xi_1, \xi_1, \xi_2, 1, \eta_1, \eta_2, \eta_3, \eta_4)$$

is a polynomial identity of A (see, e.g., [6, Section 6.3]). Since A satisfies a nontrivial polynomial identity, its center Z is nonzero and C is the field of quotients of Z [6, Corollary 7.57]. Multiplying

$$c_5(x_1x_2, x_2x_1, x_1, x_2, 1, y_1, y_2, y_3, y_4) = 0$$

by $\lambda_1^{-3}\lambda_2^{-3}\mu_1^{-1}\mu_2^{-1}\mu_3^{-1}\mu_4^{-1}$ where λ_i, μ_j are nonzero elements in Z, it thus follows that f is a polynomial identity of the ring of central quotients $Q_Z(A)$ of A, which is a finite dimensional central simple algebra over C [6, Theorem 7.58], and hence, by Wedderburn's structure theorem [6, Corollary 2.62], isomorphic to the matrix algebra $M_n(D)$ for some $n \geq 1$ and a finite dimensional central division algebra D over C. If C is finite, then D is a field by Wedderburn's theorem on finite division rings [6, Theorem 1.38]. Assume that C is infinite. Then f is a stable identity for $Q_Z(A)$ [6, Theorem 6.29], so that f is also a polynomial identity of $\overline{C} \otimes_C Q_Z(A)$ where $\overline{C}$ is the algebraic closure of C. Since the algebra $\overline{C} \otimes_C Q_Z(A)$ is isomorphic to $M_n(\overline{C})$ for some $n \geq 1$ [6, Theorem 4.39], in both cases we have arrived at the same conclusion: there exist $n \geq 1$ and a field K such that A can be embedded into $M_n(K)$ and f is a polynomial identity of $M_n(K)$. Suppose that $n \geq 3$. Take the matrix unit e_{12} for x_1 and the matrix unit e_{21} for x_2. Then $x_1x_2 = e_{11}$, $x_2x_1 = e_{22}$, x_1, x_2 and $1 = \sum_{i=1}^n e_{ii}$ are linearly independent matrices. Consequently, by [6, Theorem 7.45] there exist $y_1, y_2, y_3, y_4 \in M_n(K)$ such that

$$f(x_1, x_2, y_1, y_2, y_3, y_4) = c_5(x_1x_2, x_2x_1, x_1, x_2, 1, y_1, y_2, y_3, y_4) \neq 0.$$

This contradiction shows that $n \geq 2$, and hence that $\deg(A) \leq 2$.

(ii) $\Rightarrow$ (i). Using the Cayley-Hamilton theorem one can show that $\deg(A) \leq 2$ implies that there exist an additive map $\tau : A \to C$ and a biadditive map $\delta : A^2 \to C$ such that $x^2 + \tau(x)x + \delta(x, x) = 0$ for every $x \in A$ [8, Theorem C.2]. The linearization of this identity gives

$$x_1x_2 + x_2x_1 + \tau(x_1)x_2 + \tau(x_2)x_1 + \delta(x_1, x_2) + \delta(x_2, x_1) = 0,$$

which implies (i). □

We remark that in Lemma 5.3 we did not assume that A is unital. The unity 1 from (i) may belong to $Q_Z(A) \setminus A$.

We now have enough information to prove the following corollary.

Corollary 5.4. *Let A be a prime algebra with $\deg(A) \geq 3$ and let S be an arbitrary unital algebra. Set $\mathfrak{A} = A \otimes S$ and $\mathfrak{Z} = C \otimes Z_S$ where C is the extended centroid of A and Z_S is the center of S. If a bilinear map $F : \mathfrak{A}^2 \to \mathfrak{A}$ satisfies*

$$F(y, z)w + F(w, y)z + F(z, w)y = wF(y, z) + zF(w, y) + yF(z, w)$$

for all $y, z, w \in \mathfrak{A}$, then F is of the form

$$F(y, z) = \varepsilon yz + \varepsilon' zy + \mu(y)z + \mu(z)y + \nu(y, z) \tag{5.23}$$

for all $y, z \in \mathfrak{A}$, where $\varepsilon, \varepsilon' \in \mathfrak{Z}$, $\mu : \mathfrak{A} \to \mathfrak{Z}$ is a linear and $\nu : \mathfrak{A}^2 \to \mathfrak{Z}$ is a bilinear map.

Proof. Lemma 5.3 of course shows that A satisfies the condition $(*)$ (with C playing the role of Z). Since A is noncommutative (i.e., $\deg(A) \neq 1$), it trivially satisfies $(**)$. In view of [8, Corollary 5.12], we may now use Theorem 5.2 (with $Q_{ml}(A)$ playing the role of R) to conclude that F is of the desired form on $\mathfrak{X} = \{x \otimes s \mid x \in A, s \in S\}$. Thus, there exist $\varepsilon, \varepsilon' \in \mathfrak{Z}$, $\mu : \mathfrak{X} \to \mathfrak{Z}$, and $\nu : \mathfrak{X}^2 \to \mathfrak{Z}$ such that (5.23) holds for all $y, z \in \mathfrak{X}$. By assumption, $F(\lambda y, z) = \lambda F(y, z)$ for every $\lambda \in \mathbb{F}$. Consequently

$$\big(\mu(\lambda y) - \lambda\mu(y)\big)z = \lambda\nu(y, z) - \nu(\lambda y, z) \in \mathfrak{Z}. \tag{5.24}$$

We claim that this implies that $a = \mu(\lambda y) - \lambda\mu(y) \in \mathfrak{Z}$ is 0. Indeed, since $a(x \otimes 1) \in \mathfrak{Z}$, we have $[a(x \otimes 1), y \otimes 1] = 0$ for all $x, y \in A$. Writing $a = \sum_t \alpha_t \otimes b_t$ where $\alpha_t \in C$ and $b_t \in S$ are linearly independent, it follows that $\sum_t \alpha_t[x, y] \otimes b_t = 0$. Therefore $\alpha_t[x, y] = 0$ for all $x, y \in A$ and all t, implying that $\alpha_t = 0$; consequently, $a = 0$. We have thus proved that $\mu(\lambda y) = \lambda\mu(y)$ for all $\lambda \in \mathbb{F}$ and all $y \in \mathfrak{X}$, and hence, by (5.24), also $\nu(\lambda y, z) = \lambda\nu(y, z)$ for all $\lambda \in \mathbb{F}$ and all $y, z \in \mathfrak{X}$. Similarly, $\nu(y, \lambda z) = \lambda\nu(y, z)$.

We have to extend μ from $\mathfrak{X}$ to $\mathfrak{A} = \operatorname{span} \mathfrak{X}$. We do this in the obvious way, i.e.,

$$\mu\Big(\sum_i y_i\Big) = \sum_i \mu(y_i),$$

however, we have to show that this is well-defined. Suppose that $\sum_i y_i = 0$. Then

$$\sum_i F(y_i, z) = F\Big(\sum_i y_i, z\Big) = 0,$$

which in light of (5.23) yields

$$\Big(\sum_i \mu(y_i)\Big)z = -\sum_i \nu_i(y_i, z) \in \mathfrak{Z}.$$

As we saw in the previous paragraph, this implies $\sum_i \mu(y_i) = 0$. Thus, μ is well-defined. We have also shown that $\sum_i \nu(y_i, z) = 0$ follows from $\sum_i y_i = 0$. Similarly we see that $\sum_j z_j = 0$ implies $\sum_j \nu(y, z_j) = 0$. Accordingly, we can extend ν from $\mathfrak{X}^2$ to $\mathfrak{A}^2$ by

$$\nu\Big(\sum_i y_i, \sum_j z_j\Big) = \sum_i \sum_j \nu(y_i, z_j).$$

Note that μ is linear, ν is bilinear, and that (5.23) is fulfilled for all y and z in $\mathfrak{A}$. $\square$

Remark 5.5. Let $B : \mathfrak{A}^2 \to \mathfrak{A}$ be a bilinear map. The map $q : A \to A$, $q(x) = B(x, x)$, is called the trace of B. Recall that q is said to be *commuting* if $[q(y), y] = 0$ for all $y \in \mathfrak{A}$. Linearizing this identity we obtain that $F(y, z) = B(y, z) + B(z, y)$ satisfies the functional identity of Corollary 5.4. Under the additional assumption that the characteristic of $\mathbb{F}$ is not 2, the conclusion of the corollary implies that q is of standard form, i.e., $q(y) = \lambda y^2 + \mu(y)y + \delta(y, y)$ (where $\lambda = \frac{1}{2}(\varepsilon + \varepsilon')$ and $\delta(y, z) = \frac{1}{2}\nu(y, z)$). Neglecting some small differences in assumptions, Corollary 5.4 thus generalizes [5, Theorem 1] (see also [8, Theorem 5.32]) from prime algebras to tensor products of prime algebras with arbitrary unital algebras.

Recall that a linear map φ from an algebra B onto an algebra A is called a *Lie homomorphism* if

$$\varphi\big([y,z]\big) = [\varphi(y), \varphi(z)] \quad \text{for all } y, z \in B.$$

Examples include homomorphisms, the negatives of antihomomorphisms, and their direct sums. By the latter we mean that there exists a central idempotent ε in A such that $x \mapsto \varepsilon\varphi(x)$ is a homomorphism and $x \mapsto (1-\varepsilon)\varphi(x)$ is the negative of an antihomomorphism. Further, if φ is a Lie homomorphism and τ is a linear map into the center of A that vanishes on commutators, then $\varphi + \tau$ is again a Lie homomorphism.

Lie isomorphisms between prime rings were described in [5] (see also [8, Corollaries 6.4 and 6.5] for slightly more general results). The conclusion involves the extended centroid, and this is unavoidable (see [8, Example 6.10]). We will now extend this description to the tensor products of prime algebras with arbitrary unital algebras. We remark that in the prime case there are no nontrivial central idempotents, so direct sums do not appear. They obviously cannot be avoided when tensoring with an arbitrary algebra, so the next result is necessarily more complicated. The concept of the proof, however, is the same. We will actually follow closely the proof of [8, Theorem 6.1], and make some necessary adjustments at a few points.

Corollary 5.6. *Let A be a prime algebra with $\deg(A) \geq 3$ and let S be an arbitrary unital algebra. Set $\mathfrak{A} = A \otimes S$, $\mathfrak{A}' = (A + C) \otimes S$, and $\mathfrak{Z} = C \otimes Z_S$, where C is the extended centroid of A and Z_S is the center of S. If φ is a Lie isomorphism from an arbitrary algebra $\mathfrak{B}$ onto $\mathfrak{A}$, then $\varphi = \theta + \tau$ where $\theta : \mathfrak{B} \to \mathfrak{A}'$ is the direct sum of a homomorphism and the negative of an antihomomorphism, and $\tau : \mathfrak{B} \to \mathfrak{Z}$ is a linear map which vanishes on commutators.*

Proof. Our starting point is the identity $[ab, c] + [ca, b] + [bc, a] = 0$ which obviously holds for any elements in an arbitrary algebra. Applying φ to this we get

$$[\varphi(ab), \varphi(c)] + [\varphi(ca), \varphi(b)] + [\varphi(bc), \varphi(a)] = 0 \tag{5.25}$$

for all $a, b, c \in \mathfrak{B}$. Write y for $\varphi(a)$, z for $\varphi(b)$, and w for $\varphi(c)$. Then (5.25) reads as

$$[\varphi\big(\varphi^{-1}(y)\varphi^{-1}(z)\big), w] + [\varphi\big(\varphi^{-1}(w)\varphi^{-1}(y)\big), z] + [\varphi\big(\varphi^{-1}(z)\varphi^{-1}(w)\big), y] = 0$$

and so Corollary 5.4 is applicable. Therefore there exist $\varepsilon, \varepsilon' \in \mathfrak{Z}$, a linear map $\mu : \mathfrak{A} \to \mathfrak{Z}$ and a bilinear map $\nu : \mathfrak{A}^2 \to \mathfrak{Z}$ such that

$$\varphi\big(\varphi^{-1}(y)\varphi^{-1}(z)\big) = \varepsilon yz + \varepsilon' zy + \mu(y)z + \mu(z)y + \nu(y, z)$$

for all $y, z \in \mathfrak{A}$. That is,

$$\varphi(ab) = \varepsilon\varphi(a)\varphi(b) + \varepsilon'\varphi(b)\varphi(a) + \overline{\mu}(a)\varphi(b) + \overline{\mu}(b)\varphi(a) + \overline{\nu}(a, b) \tag{5.26}$$

for all $a, b \in \mathfrak{B}$ where $\overline{\mu}(a) = \mu(\varphi(a))$ and $\overline{\nu}(a, b) = \nu(\varphi(a), \varphi(b))$.

We continue by computing $\varphi(abc)$ in two ways. First we have

$$\begin{aligned}\varphi((ab)c) =&\varepsilon\varphi(ab)\varphi(c) + \varepsilon'\varphi(c)\varphi(ab) + \overline{\mu}(ab)\varphi(c) + \overline{\mu}(c)\varphi(ab) + \overline{\nu}(ab,c)\\ =&\varepsilon^2\varphi(a)\varphi(b)\varphi(c) + \varepsilon\varepsilon'\varphi(b)\varphi(a)\varphi(c) + \varepsilon\overline{\mu}(a)\varphi(b)\varphi(c) + \varepsilon\overline{\mu}(b)\varphi(a)\varphi(c)\\ &+ \varepsilon\overline{\nu}(a,b)\varphi(c) + \varepsilon\varepsilon'\varphi(c)\varphi(a)\varphi(b) + \varepsilon'^2\varphi(c)\varphi(b)\varphi(a) + \varepsilon'\overline{\mu}(a)\varphi(c)\varphi(b)\\ &+ \varepsilon'\overline{\mu}(b)\varphi(c)\varphi(a) + \varepsilon'\overline{\nu}(a,b)\varphi(c) + \overline{\mu}(ab)\varphi(c) + \varepsilon\overline{\mu}(c)\varphi(a)\varphi(b)\\ &+ \varepsilon'\overline{\mu}(c)\varphi(b)\varphi(a) + \overline{\mu}(a)\overline{\mu}(c)\varphi(b) + \overline{\mu}(b)\overline{\mu}(c)\varphi(a) + \overline{\mu}(c)\overline{\nu}(a,b) + \overline{\nu}(ab,c).\end{aligned}$$

On the other hand,

$$\begin{aligned}\varphi(a(bc)) =&\varepsilon\varphi(a)\varphi(bc) + \varepsilon'\varphi(bc)\varphi(a) + \overline{\mu}(a)\varphi(bc) + \overline{\mu}(bc)\varphi(a) + \overline{\nu}(a,bc)\\ =&\varepsilon^2\varphi(a)\varphi(b)\varphi(c) + \varepsilon\varepsilon'\varphi(a)\varphi(c)\varphi(b) + \varepsilon\overline{\mu}(b)\varphi(a)\varphi(c) + \varepsilon\overline{\mu}(c)\varphi(a)\varphi(b)\\ &+ \varepsilon\overline{\nu}(b,c)\varphi(a) + \varepsilon\varepsilon'\varphi(b)\varphi(c)\varphi(a) + \varepsilon'^2\varphi(c)\varphi(b)\varphi(a) + \varepsilon'\overline{\mu}(b)\varphi(c)\varphi(a)\\ &+ \varepsilon'\overline{\mu}(c)\varphi(b)\varphi(a) + \varepsilon'\overline{\nu}(b,c)\varphi(a) + \varepsilon\overline{\mu}(a)\varphi(b)\varphi(c) + \varepsilon'\overline{\mu}(a)\varphi(c)\varphi(b)\\ &+ \overline{\mu}(a)\overline{\mu}(b)\varphi(c) + \overline{\mu}(a)\overline{\mu}(c)\varphi(b) + \overline{\mu}(a)\overline{\nu}(b,c) + \overline{\mu}(bc)\varphi(a) + \overline{\nu}(a,bc).\end{aligned}$$

Comparing these two identities we obtain

$$\varepsilon\varepsilon'[\varphi(b),[\varphi(a),\varphi(c)]] + \overline{\omega}(a,b)\varphi(c) - \overline{\omega}(b,c)\varphi(a) \in \mathfrak{Z} \tag{5.27}$$

for some $\overline{\omega} : \mathfrak{B}^2 \to \mathfrak{Z}$. We can rewrite this as

$$\varepsilon\varepsilon'[z,[y,w]] + \omega(y,z)w - \omega(z,w)y \in \mathfrak{Z} \tag{5.28}$$

where $\omega : \mathfrak{A}^2 \to \mathfrak{Z}$. Our goal now is to show that (5.28) implies $\varepsilon\varepsilon' = 0$. Let us write $\varepsilon\varepsilon' = \sum_{t\in T}\epsilon_t \otimes b_t$ and $\omega(y,z) = \sum_{t\in T}\omega_{t\in T}(y,z)\otimes b_t$ where $\epsilon_t, \omega_t(y,z) \in C$ and $\{b_t \,|\, t \in T\}$ is a basis of S. Taking $x\otimes 1$ for y, $u\otimes 1$ for z, and $v \otimes 1$ for w it follows from (5.28) that

$$\epsilon_t[u,[x,v]] + \omega_t(x,u)v - \omega_t(u,v)x \in C \tag{5.29}$$

for every $t \in T$ (here, $\omega_t(x,w)$ stands for $\omega_t(x\otimes 1, w\otimes 1)$). Suppose $\epsilon_t \neq 0$ for some t. Fix a noncentral $u \in A$ and set $u_1 = \epsilon_t u$. Then $u_1 \notin C$. We can write (5.29) as

$$E_1(x)v + E_2(v)x + vF_1(x) + xF_2(v) = 0,$$

where $E_1(x) = u_1x + \omega_t(x,u)$, $E_2(v) = -u_1v - \omega_t(u,v)$, $F_1(x) = xu_1$, $F_2(v) = -vu_1$. Since A is a 3-free subset of $Q_{ml}(A)$ [8, Corollary 5.12] it follows that there are $p \in Q_{ml}(A)$ and $\lambda : A \to C$ such that $E_1(x) = xp+\lambda(x)$. Consequently, $u_1x-xp \in C$. However, this is impossible for $u_1 \notin C$ and A is a 3-free. Thus, each $\epsilon_t = 0$ and hence $\varepsilon\varepsilon' = 0$.

Our next goal is to show that $\varepsilon' = \varepsilon - 1$. By (5.26) we have

$$\varphi(ab) - \varphi(ba) = (\varepsilon - \varepsilon')[\varphi(a),\varphi(b)] + \overline{\nu}(a,b) - \overline{\nu}(b,a).$$

Since φ is a Lie homomorphism, this is also equal to $[\varphi(a),\varphi(b)]$. Consequently,

$$(1-\varepsilon+\varepsilon')[\varphi(a),\varphi(b)] \in C.$$

Similarly as in the previous paragraph we see that this yields $1 - \varepsilon + \varepsilon' = 0$, as desired. Note that this together with $\varepsilon\varepsilon' = 0$ implies that ε is an idempotent.

Define $\theta : \mathfrak{B} \to \mathfrak{A}'$ by

$$\theta(a) = \varphi(a) - (1 - 2\varepsilon)\overline{\mu}(a).$$

Using (5.26) one easily derives that $\rho(a, b) = \varepsilon\theta(ab) - \varepsilon\theta(a)\theta(b)$ lies in $\mathfrak{Z}$. Consequntly, computing, similarly as above, $\varepsilon\theta((ab)c) = \varepsilon\theta(a(bc))$ in two ways results in $\rho(a, b)\theta(c) - \rho(b, c)\theta(a) \in \mathfrak{Z}$, which clearly yields $\rho(a, b)\varphi(c) - \rho(b, c)\varphi(a) \in \mathfrak{Z}$. This is similar to (5.27), but simpler. Using the same approach as above one easily shows that $\rho(a, b) = 0$ for all $a, b \in \mathfrak{B}$. Thus, $\varepsilon\theta(ab) = \varepsilon\theta(a)\theta(b)$, i.e., $a \mapsto \varepsilon\theta(a)$ is a homomorphism. Similarly we see that $a \mapsto (1 - \varepsilon)\theta(a)$ is the negative of an antihomomorphism.

Finally we define $\tau : \mathfrak{B} \to \mathfrak{Z}$ by $\tau(a) = (1 - 2\varepsilon)\overline{\mu}(a)$, so that $\varphi = \theta + \tau$. Since both φ and θ are Lie homomorphisms it follows immediately that $\tau([a, b]) = 0$ for all $a, b \in \mathfrak{B}$. □

Remark 5.7. A similar theorem can be obtained for Lie derivations, cf. [8, Theorem 6.6]. Also, Jordan derivations can be handled under similar assumptions, only 3-freeness must be replaced by 4-freeness. Actually, Jordan derivations are much easier to deal with than Lie derivations – one can show that they are ordinary derivations as an immediate corollary to Theorem 5.1. See [8, p. 177]. Let us finally remark that inspired by the (preliminary) results of the present paper, the author has simultaneously written a paper on Jordan derivations and some related maps on the tensor product between an arbitrary (not necessarily d-free) algebra and a commutative algebra [7].

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