AN ELEMENTARY APPROACH TO WEDDERBURN'S STRUCTURE THEORY

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Abstract. Wedderburn's theorem on the structure of finite dimensional (semi)simple algebras is proved by using minimal prerequisites.

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1. Introduction

One hundred years have passed since J. H. M. Wedderburn published the paper on the structure of algebras [1]. In particular he proved that a finite dimensional simple algebra is isomorphic to a matrix algebra over a division algebra. Versions of this result appear in countless graduate algebra textbooks, often on their first pages. Standard proofs are based on the concept of a module over an algebra. The module-theoretic approach is certainly elegant and efficient, and moreover it gives a basis for developing various more general theories. However, Wedderburn's theorem has a very simple formulation which does not involve modules. Therefore it seems natural to seek for more direct approaches. The goal of this article is to present a proof which uses only the most elementary tools. It is short, but so are some module-theoretic proofs. Its main advantage is the conceptual simplicity. It cannot replace standard proofs if one has a development of a more sophisticated theory in mind. But it might be more easily accessible to students. Wedderburn's theorem is a typical graduate level topic, but using this approach it could be included in an undergraduate algebra course. Students often find an introduction to algebra somewhat dry and formal, and therefore enliven it with colorful theorems might make them more interested. We wish to show that Wedderburn's beautiful and important theorem is one such option.

2. Wedderburn's theorem

Besides some standard algebraic notions, the only prerequisites needed for our proof of Wedderburn's theorem are two simple lemmas. Both of them are well known. Nevertheless, we will give the proofs to make the paper self-contained.

First we introduce the notation and terminology. By an "algebra" we shall mean an associative algebra over a fixed, but arbitrary field $\mathbb{F}$. The unity of an algebra $\mathcal{A}$ will be denote by $1_{\mathcal{A}}$; until further notice we assume that *all our algebras have a unity.* If e is an idempotent in $\mathcal{A}$, then $e\mathcal{A}e$ is a subalgebra of $\mathcal{A}$ with $1_{e\mathcal{A}e} = e$.

Let n be a positive integer. Elements $e_{ij} \in \mathcal{A}$, $i, j = 1, \dots, n$, are called *matrix units* if $e_{11} + \dots + e_{nn} = 1_{\mathcal{A}}$ and $e_{ij}e_{kl} = \delta_{jk}e_{il}$ for all i, j, k, l (here, δ_{jk} is the "Kronecker delta"). In particular, e_{ii} are idempotents such that $e_{ii}e_{jj} = 0$ if $i \neq j$. Note that each $e_{ij} \neq 0$. If $\mathcal{C}$ is any algebra, then $\mathcal{A} = M_n(\mathcal{C})$, the algebra of all $n \times n$ matrices over $\mathcal{C}$, has matrix units. Indeed, the standard (but not the only) example is the following: e_{ij} is the matrix whose (i, j)-entry is $1_{\mathcal{C}}$ and all other entries are 0. In our first lemma we show that $M_n(\mathcal{C})$ is also the only example of an algebra with matrix units.

Lemma 2.1. *If an algebra $\mathcal{A}$ contains matrix units e_{ij}, $i, j = 1, \dots n$, then $\mathcal{A} \cong M_n(e_{tt}\mathcal{A}e_{tt})$ for each $t = 1, \dots, n$.*

Proof. For every $a \in \mathcal{A}$ we set $a_{ij} = e_{ti}ae_{jt}$. We can also write $a_{ij} = e_{tt}e_{ti}ae_{jt}e_{tt}$ and so $a_{ij} \in e_{tt}\mathcal{A}e_{tt}$. Now define $\varphi : \mathcal{A} \to M_n(e_{tt}\mathcal{A}e_{tt})$ by $\varphi(a) = (a_{ij})$. We claim that φ is an algebra isomorphism. Clearly, φ is linear and preserves unities. The (i, j)-entry of $\varphi(a)\varphi(b)$ is equal to $\sum_{k=1}^n e_{ti}ae_{kt}e_{tk}be_{jt} = e_{ti}a(\sum_{k=1}^n e_{kk})be_{jt} = e_{ti}abe_{jt}$, which is the (i, j)-entry of $\varphi(ab)$. Thus, $\varphi(ab) = \varphi(a)\varphi(b)$. If $a_{ij} = 0$ for all i, j, then $e_{ii}ae_{jj} = e_{it}a_{ij}e_{tj} = 0$, and so $a = 0$ since the sum of all e_{ii} is $1_{\mathcal{A}}$. Thus φ is injective. Note that $\varphi(e_{it}ae_{tj})$ is the matrix whose (i, j)-entry is $e_{tt}ae_{tt}$ and all other entries are 0. This implies the surjectivity of φ. □

Recall that an algebra $\mathcal{A}$ is said to be *prime* if for all $a, b \in \mathcal{A}$, $a\mathcal{A}b = \{0\}$ implies $a = 0$ or $b = 0$. If $\mathcal{D}$ is a division algebra, then $M_n(\mathcal{D})$ is a prime algebra for every $n \geq 1$. Indeed, one easily checks that if $a, b \in M_n(\mathcal{D})$ are such that $ae_{ij}b = 0$ for all standard matrix units e_{ij}, then $a = 0$ or $b = 0$. Next, an algebra $\mathcal{A}$ is said to be *semiprime* if for all $a \in \mathcal{A}$, $a\mathcal{A}a = \{0\}$ implies $a = 0$. Obviosuly, prime algebras are semiprime. The direct product $\mathcal{A}_1 \times \mathcal{A}_2$ of semiprime algebras $\mathcal{A}_1$ and $\mathcal{A}_2$ is a semiprime, but not a prime algebra. The semiprimeness of $\mathcal{A}$ can be equivalently defined through the condition that $\mathcal{A}$ does not have nonzero nilpotent ideals. Similarly, $\mathcal{A}$ is prime if and only if the product of any two nonzero ideals of $\mathcal{A}$ is nonzero. But we shall not need these alternative definitions.

If $\mathcal{A}$ is a simple algebra and a, b are its nonzero elements, then $a\mathcal{A}b \neq \{0\}$ since the ideals generated by a and b are equal to $\mathcal{A}$. Thus, simple algebras are prime. The converse is not true in general. For example, the polynomial algebra $\mathbb{F}[X]$ is prime but not simple. In the finite dimensional context, however, the notions of primeness and simplicity coincide. This is well known and also follows from our version of Wedderburn's theorem. The reason for dealing with prime algebras instead of with (more common but less general) simple ones in this version is not because of seeking for a greater level of generality, but because the proofs run more smoothly in this setting. For similar reason we consider semiprime algebras instead of semisimple ones.

Before stating the next lemma we mention an illustrative example. If e_{tt} is a standard matrix unit of $M_n(\mathcal{D})$, then $e_{tt}M_n(\mathcal{D})e_{tt}$ consists of all matrices whose (t,t)-entry is an arbitrary element in $\mathcal{D}$ and all other entries are 0. Therefore $e_{tt}M_n(\mathcal{D})e_{tt} \cong \mathcal{D}$.

Lemma 2.2. *If $\mathcal{A}$ is a nonzero finite dimensional semiprime algebra, then there exists an idempotent $e \in \mathcal{A}$ such that $e\mathcal{A}e$ is is a division algebra.*

Proof. Pick a nonzero left ideal $\mathcal{L}$ of minimal dimension, i.e., $\dim_{\mathbb{F}} \mathcal{L} \leq \dim_{\mathbb{F}} \mathcal{J}$ for every nonzero left ideal $\mathcal{J}$ of $\mathcal{A}$. Obviously, $\{0\}$ is then the only left ideal that is properly contained in $\mathcal{L}$. Let $0 \neq x \in \mathcal{L}$. Since $\mathcal{A}$ is semiprime, there exists $a \in \mathcal{A}$ such that $xax \neq 0$. As $y = ax \in \mathcal{L}$, we have found $x, y \in \mathcal{L}$ with $xy \neq 0$. In particular, $\mathcal{L}y \neq \{0\}$. But $\mathcal{L}y$ is a left ideal of $\mathcal{A}$ contained in $\mathcal{L}$, and so $\mathcal{L}y = \mathcal{L}$. Accordingly, as $y \in \mathcal{L}$ we have $ey = y$ for some $e \in \mathcal{L}$. This implies that $e^2 - e$ belongs to the set $\mathcal{J} = \{z \in \mathcal{L} \,|\, zy = 0\}$. Clearly, $\mathcal{J}$ is again a left ideal of $\mathcal{A}$ contained in $\mathcal{L}$. Since $x \in \mathcal{L} \setminus \mathcal{J}$, this time we conclude that $\mathcal{J} = \{0\}$. In particular, $e^2 = e$. As $e \in \mathcal{L}$, we have $\mathcal{A}e \subseteq \mathcal{L}$, and since $0 \neq e \in \mathcal{A}e$ it follows that $\mathcal{L} = \mathcal{A}e$. Now consider the subalgebra $e\mathcal{A}e$ of $\mathcal{A}$. Let $c \in \mathcal{A}$ be such that $ece \neq 0$. The lemma will be proved by showing that ece is invertible in $e\mathcal{A}e$. We have $\{0\} \neq \mathcal{A}ece \subseteq \mathcal{A}e = \mathcal{L}$, and so $\mathcal{A}ece = \mathcal{L}$. Therefore there is $b \in \mathcal{A}$ such that $bece = e$, and hence also $(ebe)(ece) = e$. Now, ebe is again a nonzero element in $e\mathcal{A}e$, and so by the same argument there is $c' \in \mathcal{A}$ such that $(ec'e)(ebe) = e$. But then $(ece)^{-1} = ebe$. □

We remark that the finite dimensionality of $\mathcal{A}$ was used only for finding a minimal left ideal, i.e., a left ideal that does not properly contain any other nonzero left ideal.

We are now in a position to prove Wedderburn's theorem. Let us first outline the concept of the proof to help the reader not to get lost in an (inevitably) tedious notation in the formal proof. By Lemma 2.1 it is enough to show that the algebra $\mathcal{A}$ in question contains matrix units e_{ij} such that $e_{tt}\mathcal{A}e_{tt}$ is a division algebra for some t. Lemma 2.2 yields the existence of an idempotent e such that $e\mathcal{A}e$ is a divison algebra. Think of e as e_{nn}. A simple argument based on the induction on $\dim_{\mathbb{F}} \mathcal{A}$ shows that the algebra $(1_{\mathcal{A}} - e)\mathcal{A}(1_{\mathcal{A}} - e)$ contains matrix units $e_{11}, e_{12}, \ldots, e_{n-1,n-1}$ with $e_{tt}\mathcal{A}e_{tt}$ being division algebras. It remains to find $e_{n1}, \ldots, e_{n,n-1}$ and $e_{1n}, \ldots, e_{n-1,n}$. Finding e_{1n} and e_{n1} is the heart of the proof; here we make use of the fact that $e_{11}\mathcal{A}e_{11}$ and $e_{nn}\mathcal{A}e_{nn}$ are division algebras. The remaining matrix units can be then just directly defined as $e_{nj} = e_{n1}e_{1j}$ and $e_{jn} = e_{j1}e_{1n}$, $j = 2, \ldots, n-1$; checking that they satisfy all desired identities is straightforward.

Theorem 2.3. (Wedderburn's theorem) *Let $\mathcal{A}$ be a finite dimensional algebra. Then $\mathcal{A}$ is prime if and only if there exist a positive integer n and a division algebra $\mathcal{D}$ such that $\mathcal{A} \cong M_n(\mathcal{D})$.*

Proof. We have already mentioned that the algebra $M_n(\mathcal{D})$ is prime. Therefore we only have to prove the "only if" part. The proof is by induction on $N = \dim_{\mathbb{F}} \mathcal{A}$.

If $N = 1$, then $\mathcal{A} = \mathbb{F}1_{\mathcal{A}}$ is a field and the result trivially holds (with $n = 1$ and $\mathcal{D} = \mathbb{F}$). We may therefore assume that $N > 1$. By Lemma 2.2 there exists an idempotent $e \in \mathcal{A}$ such that $e\mathcal{A}e$ is a division algebra. If $e = 1_{\mathcal{A}}$, then the desired result holds (with $n = 1$). Assume therefore that e is a nontrivial idempotent, and set $\widehat{\mathcal{A}} = (1_{\mathcal{A}} - e)\mathcal{A}(1_{\mathcal{A}} - e)$. Note that $\widehat{\mathcal{A}}$ is a prime algebra with unity $1_{\mathcal{A}} - e$. Further, we have $e\widehat{\mathcal{A}} = \widehat{\mathcal{A}}e = \{0\}$, and so $e \notin \widehat{\mathcal{A}}$. Therefore $\dim_{\mathbb{F}} \widehat{\mathcal{A}} < N$. Using the induction assumption it follows that $\widehat{\mathcal{A}}$ contains matrix units e_{ij}, $i, j = 1, \dots, m$, for some $m \geq 1$, such that $e_{ii}\widehat{\mathcal{A}}e_{ii}$ is a division algebra for each i. Since $e_{ii} = (1_{\mathcal{A}} - e)e_{ii} = e_{ii}(1_{\mathcal{A}} - e)$, we actually have $e_{ii}\widehat{\mathcal{A}}e_{ii} = e_{ii}\mathcal{A}e_{ii}$. Our goal is to extend these matrix units of $\widehat{\mathcal{A}}$ to matrix units of $\mathcal{A}$. We begin by setting $n = m + 1$ and $e_{nn} = e$. Then $e_{11} + \ldots + e_{n-1,n-1} + e_{nn} = (1_{\mathcal{A}} - e) + e = 1_{\mathcal{A}}$. Using the definition of primeness twice we see that $e_{11}ae_{nn}a'e_{11} \neq 0$ for some $a, a' \in \mathcal{A}$. As $e_{11}\mathcal{A}e_{11}$ is a division algebra with unity e_{11}, it follows that $(e_{11}ae_{nn}a'e_{11})(e_{11}a''e_{11}) = e_{11}$ for some $a'' \in \mathcal{A}$. Thus, $e_{11}ae_{nn}be_{11} = e_{11}$ where $b = a'e_{11}a''$. Let us set $e_{1n} = e_{11}ae_{nn}$ and $e_{n1} = e_{nn}be_{11}$, so that $e_{1n}e_{n1} = e_{11}$. Since $e_{n1} \in e_{nn}\mathcal{A}e_{11}$, we have $e_{n1} = e_{nn}e_{n1}$ and $e_{n1} = e_{n1}e_{11} = e_{n1}e_{1n}e_{n1}$. Comparing both relations we get $(e_{nn} - e_{n1}e_{1n})e_{n1} = 0$. The element $e_{nn} - e_{n1}e_{1n}$ lies in the division algebra $e_{nn}\mathcal{A}e_{nn}$. If it is nonzero, then we can multiply the last identity from the left-hand side by its inverse, which gives $e_{nn}e_{n1} = 0$, and hence $e_{n1} = 0$ - a contradiction. Therefore $e_{nn} = e_{n1}e_{1n}$. Finally we set $e_{nj} = e_{n1}e_{1j}$ and $e_{jn} = e_{j1}e_{1n}$ for $j = 2, \dots, n-1$. Note that $e_{ij} = e_{i1}e_{1j}$ then holds for all $i, j = 1, \dots, n$. Consequently, for all $i, j, k, l = 1, \dots, n$ we have $e_{ij}e_{kl} = e_{i1}e_{1j}e_{k1}e_{1l} = \delta_{jk}e_{i1}e_{11}e_{1l} = \delta_{jk}e_{i1}e_{1l} = \delta_{jk}e_{il}$. Thus e_{ij}, $i, j = 1, \dots, n$, are indeed matrix units of $\mathcal{A}$. Therefore Lemma 2.1 tells us that $\mathcal{A} \cong M_n(\mathcal{D})$ where $\mathcal{D} = e_{tt}\mathcal{A}e_{tt}$ (for any t). As we know, $\mathcal{D}$ is a division algebra. □

Theorem 2.3 can be extended in myriad ways. In appendices we will consider two important improvements. The first one is that assuming the existence of $1_{\mathcal{A}}$ is actually redundant, and the second one is a generalization to semiprime algebras. The proofs will be, just as the above one, by induction on the dimension of $\mathcal{A}$. We will give them in a somewhat loose manner, but filling in the details is supposed to be an easy task.

From now on we *do not assume anymore that our algebras should have a unity.* The definitions of prime and semiprime algebras remain unchanged in this more general setting. Also, Lemma 2.2 still holds as we can see from its proof.

Appendix A. Let $\mathcal{A}$ be a nonzero finite dimensional prime algebra. We will show that $\mathcal{A}$ *necessarily has a unity* (and hence $\mathcal{A} \cong M_n(\mathcal{D})$ by Theorem 2.3). We proceed by induction on $N = \dim_{\mathbb{F}} \mathcal{A}$. The $N = 1$ case is trivial, so let $N > 1$. Lemma 2.2 implies that $\mathcal{A}$ has a nonzero idempotent e. We may assume that $e \neq 1_{\mathcal{A}}$. Because of the absence of $1_{\mathcal{A}}$ now we cannot introduce the algebra $(1_{\mathcal{A}} - e)\mathcal{A}(1_{\mathcal{A}} - e)$; but we can simulate it. Let us write $\widehat{a} = a - ea - ae + eae$ for each $a \in \mathcal{A}$, and set $\widehat{\mathcal{A}} = \{\widehat{a} \,|\, a \in \mathcal{A}\}$. Clearly, $e\widehat{\mathcal{A}} = \widehat{\mathcal{A}}e = \{0\}$. This readily implies that $\widehat{\mathcal{A}}$ is a prime algebra with $e \notin \widehat{\mathcal{A}}$. If $\widehat{\mathcal{A}}$ was $\{0\}$, it would follow that $(a - ae)b(c - ec) = \widehat{abc} = 0$ for all $a, b, c \in \mathcal{A}$. Using the primeness of $\mathcal{A}$ it is easy to see that this contradicts $e \neq 1_{\mathcal{A}}$. Thus $\widehat{\mathcal{A}} \neq \{0\}$. By induction assumption we may now conclude that $\widehat{\mathcal{A}}$ has a unity. Set $f = 1_{\widehat{\mathcal{A}}}$. Then $ef = fe = 0$ and hence $faf = f\widehat{a}f = \widehat{a}f = (a - ea)f$. Thus, $((e+f)a - a)f = 0$ for all $a \in \mathcal{A}$. Replacing a by ab we get $((e+f)a - a)bf = 0$, and therefore $(e + f)a = a$. Similarly we derive $a(e + f) = a$. Accordingly, $e + f = 1_{\mathcal{A}}$.

Appendix B. The semiprime (or, equivalently, semisimple) version of Wedderburn's theorem reads as follows. A nonzero finite dimensional algebra $\mathcal{A}$ *is semiprime if and only if there exist positive integers* $n_1, \ldots, n_r$ *and division algebras* $\mathcal{D}_1, \ldots, \mathcal{D}_r$ *such that* $\mathcal{A} \cong M_{n_1}(\mathcal{D}_1) \times \ldots \times M_{n_r}(\mathcal{D}_r)$. (Incidentally, this implies that semiprime algebras also have a unity). The "if" part is trivial, while the "only if" can be easily derived from Theorem 2.3, as we will show in the next paragraph. Our argument will also demonstrate the advantage of not assuming a priori that algebras must have a unity.

Let $\mathcal{A}$ be a nonzero finite dimensional semiprime algebra. Again we will derive the desired conclusion by induction on $N = \dim_{\mathbb{F}} \mathcal{A}$. In view of Theorem 2.3 and Appendix A we may assume that $\mathcal{A}$ is not prime. Therefore there exists $0 \neq a \in \mathcal{A}$ such that $\mathcal{I} = \{x \in \mathcal{A} \,|\, a\mathcal{A}x = \{0\}\}$ is not $\{0\}$. Note that $\mathcal{I}$ is an ideal of $\mathcal{A}$, and hence $\mathcal{I}$ is also a semiprime algebra. Since $a \notin \mathcal{I}$, we may use the induction assumption and conclude that $\mathcal{I}$ is a direct product of matrix algebras over division algebras. In particular, $\mathcal{I}$ has a unity $e = 1_{\mathcal{I}}$. Consequently, e is a central idempotent in $\mathcal{A}$, and $\mathcal{A} \cong \mathcal{I} \times \mathcal{J}$ where $\mathcal{J}$ is the ideal consisting of elements $a - ea$, $a \in \mathcal{A}$. We may now use the induction assumption also on $\mathcal{J}$ and the result follows.

Concluding remarks. Our aim was to write an expository article presenting a shortcut from elementary definitions to a substantial piece of mathematics. The proof of Wedderburn's theorem given is certainly pretty direct. But how original is it? To be honest, we do not know. We did not find such a proof when searching the literature. But on the other hand, it is not based on some revolutionary new idea. So many mathematicians have known this theory for so many years that one hardly imagines that something essentially new can be invented. After a closer look at [1] we have realized that a few details in our construction of matrix units are somewhat similar to those used by Wedderburn himself. Thus, some of these ideas have been around for a hundred years. Or maybe even more. We conclude this article by quoting Wedderburn [1, page 78]: "Most of the results contained in the present paper have already been given, chiefly by Cartan and Frobenius, for algebras whose coefficients lie in the field of rational numbers; and it is probable that many of the methods used by these authors are capable of direct generalisation to any field. It is hoped, however, that the methods of the present paper are, in themselves and apart from the novelty of the results, sufficiently interesting to justify its publication."

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