ON BILINEAR MAPS DETERMINED BY RANK ONE IDEMPOTENTS

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Abstract. Let M_n, $n \geq 2$, be the algebra of all $n \times n$ matrices over a field F of characteristic not 2, and let Φ be a bilinear map from $M_n \times M_n$ into an arbitrary vector space X over F. Our main result states that if $\phi(e, f) = 0$ whenever e and f are orthogonal rank one idempotents, then there exist linear maps $\Phi_1, \Phi_2 : M_n \to X$ such that $\phi(a, b) = \Phi_1(ab) + \Phi_2(ba)$ for all $a, b \in M_n$. This is applicable to some linear preserver problems.

1. Introduction

Over the last couple of years several papers characterizing bilinear maps on algebras through their action on elements whose product is zero were written. Some of them are linear algebraic [4, 6, 7], and the other ones are analytic [1, 2, 3]. Although technically completely different, the philosophy in these two series of papers is similar: certain classical problems concerning *linear* maps that preserve zero products, commutativity etc., can be sometimes effectively solved by considering *bilinear* maps that preserve certain zero product properties. Somewhat surprisingly, the more complicated setting of bilinear maps has turned out to be more suitable than the original setting of linear ones. On the other hand, we believe that these problems with bilinear maps are of some interest in their own right, and might have some other applications that are yet to be found.

In the recent paper [3] the following problem arose: Given an algebra A, a linear space X, and a bilinear map $\phi : A \times A \to X$, does it follow that ϕ must be of the form $\phi(a, b) = \Phi_1(ab) + \Phi_2(ba)$ for some linear maps $\Phi_1, \Phi_2 : A \to X$ in case $\phi(a, b) = 0$ whenever $a, b \in A$ satisfy $ab = ba = 0$? Even for the simplest algebras A this question seems to be nontrivial. In [3] we have answered it in the affirmative only for the algebra $A = M_n(\mathbb{C})$. The proof, however, is involved and uses C*-algebra techniques.

Here we shall give a direct proof of a more general result. We will consider the algebra $A = M_n(F)$, where F is any field of characteristic not 2, and

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derive the same conclusion $\phi(a,b) = \Phi_1(ab) + \Phi_2(ba)$ under the assumption that $\phi(e,f) = 0$ whenever e and f are rank one idempotents that are orthogonal, i.e., they satisfy $ef = fe = 0$. A partial motivation for treating this problem is coming from a different area: maps preserving rank one idempotents (and projections) have been studied extensively in different areas. An account on this topic is given in the recent monograph by L. Molnár [8]. At the end of the paper we will present applications of the main theorem to some linear preserver problems.

From the technical point of view there is nothing too surprising in this short article. The proofs are based on elementary calculations with matrix units. Our main goal has been to point out a new type of results that can be obtained, and to indicate their usefulness.

2. The main result

Let us fix the notation. By F we denote a field of characteristic not 2, and by $M_n = M_n(F)$ the algebra of all $n \times n$ matrices over F. The linear span of all commutators $[a,b] = ab - ba$ in M_n, i.e., the set of matrices with zero trace, will be denoted by $[M_n, M_n]$. As usual, matrix units in M_n will be denoted by e_{ij}. By $\mathbf{1}$ we denote the identity matrix. Finally, by X we denote an arbitrary linear space over F; it will play just a formal role in this paper.

We shall need the following result which is a special case of [6, Theorem 2.1]. Its proof uses elementary tools.

Proposition 2.1. *Let $\psi : M_n \times M_n \to X$, $n \geq 2$, be a skew-symmetric bilinear map. Suppose that for all $i,j,k,l \in \{1,2,\dots,n\}$ we have*

(a) $\psi(e_{ij}, e_{kl}) = 0$ *whenever* $j \neq k$ *and* $i \neq l$,
(b) $\psi(e_{ij}, e_{jl}) = \psi(e_{ik}, e_{kl})$ *whenever* $i \neq l$,
(c) $\psi(e_{ji}, e_{ij}) + \psi(e_{ik}, e_{ki}) + \psi(e_{kj}, e_{jk}) = 0$.

Then there exists a linear operator $\Psi : [M_n, M_n] \to X$ such that

$$\psi(a,b) = \Psi([a,b]) \quad (a,b \in M_n).$$

We are now in a position to prove the main theorem.

Theorem 2.2. *Let $\phi : M_n \times M_n \to X$, $n \geq 2$, be a bilinear map such that $\phi(e,f) = 0$ whenever e and f are orthogonal rank one idempotents. Then there exist linear operators $\Phi_1, \Phi_2 : M_n \to X$ such that*

$$\phi(a,b) = \Phi_1(ab) + \Phi_2(ba) \quad (a,b \in M_n).$$

Proof. Our first goal is to prove that ϕ satisfies the condition (a) of Proposition 2.1, i.e.,

$$\phi(e_{ij}, e_{kl}) = 0 \quad \text{whenever } j \neq k \text{ and } i \neq l. \tag{1}$$

If $i = j$ and $k = l$ then (1) clearly follows from our assumption. Further, if $i = j$ but $k \neq l$, then $e_{kk} + e_{kl}$ and e_{kk} are both rank one idempotents orthogonal to e_{ii}, and so

$$\phi(e_{ii}, e_{kl}) = \phi(e_{ii}, e_{kk} + e_{kl}) - \phi(e_{ii}, e_{kk}) = 0.$$

Similarly we deal with the case where $i \neq j$ and $k = l$. So let $i \neq j$ and $k \neq l$. Assume further that $i \neq k$. Then $e_{ii} + e_{ij}$ and $e_{kk} + e_{kl}$ are orthogonal rank one idempotents, which, making use of what was already proved, readily yields (1). Similarly we consider the case where $j \neq l$. Thus it remains to consider the case where $i = k$ and $j = l$, that is, we have to show that $\phi(e_{ij}, e_{ij}) = 0$ if $i \neq j$. Noting that $e_{ii} - e_{ij}$ and $e_{jj} + e_{ij}$ are orthogonal rank one idempotents it follows that $\phi(e_{ii} - e_{ij}, e_{jj} + e_{ij}) = 0$. Since $\phi(e_{ii}, e_{jj}) = 0$, this can be rewritten as

$$\phi(e_{ii}, e_{ij}) - \phi(e_{ij}, e_{jj}) - \phi(e_{ij}, e_{ij}) = 0.$$

On the other hand, $e_{ii} + e_{ij}$ and $e_{jj} - e_{ij}$ are also orthogonal rank one idempotents, from which we infer

$$-\phi(e_{ii}, e_{ij}) + \phi(e_{ij}, e_{jj}) - \phi(e_{ij}, e_{ij}) = 0.$$

Comparing both identities we get $\phi(e_{ij}, e_{ij}) = 0$, and so the proof of (1) is now complete. On the other hand, we have also showed that

$$\phi(e_{ii}, e_{ij}) = \phi(e_{ij}, e_{jj}). \tag{2}$$

In a similar fashion, by considering $e_{ii} + e_{ji}$ and $e_{jj} - e_{ji}$, we obtain

$$\phi(e_{ii}, e_{ji}) = \phi(e_{ji}, e_{jj}). \tag{3}$$

We claim that (2) and (3) can be generalized to

$$\phi(e_{ik}, e_{kj}) = \phi(e_{il}, e_{lj}) \quad \text{whenever } i \neq j \tag{4}$$

and

$$\phi(e_{ki}, e_{jk}) = \phi(e_{li}, e_{jl}) \quad \text{whenever } i \neq j. \tag{5}$$

In view of (2), (4) will be proved by showing that $\phi(e_{ik}, e_{kj}) = \phi(e_{ij}, e_{jj})$ for every k such that $k \neq i$ and $k \neq j$. This follows from noting that $e_{ii} + e_{ik} + e_{ij}$ and $e_{jj} - e_{kj}$ are orthogonal rank one idempotents, forcing

$$\phi(e_{ii} + e_{ik} + e_{ij}, e_{jj} - e_{kj}) = 0,$$

and hence we get the desired relation by making use of (1). Similarly, by considering orthogonal rank one idempotents $e_{ii} + e_{ki} + e_{ji}$ and $e_{jj} - e_{jk}$, we establish (5).

Now pick and distinct i, j, k, and note that

$$e = e_{ii} + e_{ij} + e_{ik} - e_{ji} - e_{jj} - e_{jk} + e_{ki} + e_{kj} + e_{kk}$$

and

$$f = 2e_{ii} + 2e_{ij} - e_{ji} - e_{jj} - e_{ki} - e_{kj}$$

are orthogonal rank one idempotents, so that $\phi(e,f)=0$. Expanding this identity and using (1) we get

$$\begin{aligned}
& 2\phi(e_{ii},e_{ii})+2\phi(e_{ii},e_{ij})-\phi(e_{ii},e_{ji})-\phi(e_{ii},e_{ki}) \\
+ \;& 2\phi(e_{ij},e_{ii})-\phi(e_{ij},e_{ji})-\phi(e_{ij},e_{jj})-\phi(e_{ij},e_{ki}) \\
+ \;& 2\phi(e_{ik},e_{ii})-\phi(e_{ik},e_{ji})-\phi(e_{ik},e_{ki})-\phi(e_{ik},e_{kj}) \\
- \;& 2\phi(e_{ji},e_{ii})-2\phi(e_{ji},e_{ij})+\phi(e_{ji},e_{jj})+\phi(e_{ji},e_{kj}) \\
- \;& 2\phi(e_{jj},e_{ij})+\phi(e_{jj},e_{ji})+\phi(e_{jj},e_{jj})+\phi(e_{jj},e_{kj}) \\
- \;& 2\phi(e_{jk},e_{ij})+\phi(e_{jk},e_{jj})+\phi(e_{jk},e_{ki})+\phi(e_{jk},e_{kj}) \\
+ \;& 2\phi(e_{ki},e_{ii})+2\phi(e_{ki},e_{ij}) \\
- \;& \phi(e_{kj},e_{ji})-\phi(e_{kj},e_{jj}) \\
- \;& \phi(e_{kk},e_{ki})-\phi(e_{kk},e_{kj})=0.
\end{aligned}$$

Applying (4) and (5) this identity reduces to

$$\begin{aligned}
& 2\phi(e_{ii},e_{ii})-\phi(e_{ij},e_{ji})-\phi(e_{ik},e_{ki}) \\
& -2\phi(e_{ji},e_{ij})+\phi(e_{jj},e_{jj})+\phi(e_{jk},e_{kj})=0.
\end{aligned} \tag{6}$$

Further, one can check that $2e_{ii}+e_{ij}-2e_{ji}-e_{jj}$ and $-e_{ii}-e_{ij}+2e_{ji}+2e_{jj}$ are orthogonal rank one idempotents for any $i\neq j$, and so

$$\phi(2e_{ii}+e_{ij}-2e_{ji}-e_{jj},-e_{ii}-e_{ij}+2e_{ji}+2e_{jj})=0.$$

Expanding and applying (4) and (5) one obtains

$$\phi(e_{ij},e_{ji})+\phi(e_{ji},e_{ij})=\phi(e_{ii},e_{ii})+\phi(e_{jj},e_{jj}). \tag{7}$$

Using this in (6) it follows that

$$\phi(e_{ii},e_{ii})+\phi(e_{jk},e_{kj})=\phi(e_{ji},e_{ij})+\phi(e_{ik},e_{ki}).$$

Further, since

$$\phi(e_{jk},e_{kj})=\phi(e_{jj},e_{jj})+\phi(e_{kk},e_{kk})-\phi(e_{kj},e_{jk})$$

by (7), we finally obtain

$$\begin{aligned}
& \phi(e_{ii},e_{ii})+\phi(e_{jj},e_{jj})+\phi(e_{kk},e_{kk}) \\
= \;& \phi(e_{ji},e_{ij})+\phi(e_{ik},e_{ki})+\phi(e_{kj},e_{jk}).
\end{aligned} \tag{8}$$

Note that (8) holds for any i,j,k, no matter whether they are distinct or not.

We now have enough information about ϕ. Let us define a bilinear map $\psi: M_n\times M_n\to X$ by

$$\psi(a,b)=\phi(\mathbf{1},ab)-\phi(a,b).$$

Our goal is to show that ψ satisfies the conditions of Proposition 2.1. First we have to prove that ψ is skew-symmetric. It is enough to show that $\psi(e_{ij},e_{kl})=-\psi(e_{kl},e_{ij})$ for all i,j,k,l, that is,

$$\phi(\mathbf{1},e_{ij}e_{kl}+e_{kl}e_{ij})=\phi(e_{ij},e_{kl})+\phi(e_{kl},e_{ij}). \tag{9}$$

If $j \neq k$ and $i \neq l$, then (9) follows from (1). Assume that $j = k$ and $i \neq l$. We have $\phi(\mathbf{1}, e_{ij}e_{jl} + e_{jl}e_{ij}) = \phi(\mathbf{1}, e_{il})$. Writing $\mathbf{1}$ as $e_{11} + e_{22} + \ldots + e_{nn}$ and using (1) we see that this is further equal to $\phi(e_{ii}, e_{il}) + \phi(e_{ll}, e_{il})$, which in turn is equal to $\phi(e_{ij}, e_{jl}) + \phi(e_{jl}, e_{ij})$ by (4) and (5). Thus (9) holds in this case. Similarly we consider the the case when $j \neq k$ and $i = l$. Finally, let $j = k$ and $i = l$. Noting that $\phi(\mathbf{1}, e_{ii} + e_{jj}) = \phi(e_{ii}, e_{ii}) + \phi(e_{jj}, e_{jj})$ we see that in this case (9) follows from (7). Thus (9) holds in every case, and so ψ is skew-symmetric.

Since ϕ satisfies the condition (a), i. e. (1) holds, it readily follows that ψ also satisfies (a). By (5) we see that ψ satisfies (b), and by (8) we see that ψ satisfies (c). Thus we may use Proposition 2.1, and conclude that $\psi(a, b) = \Phi_2([a, b])$ for some linear operator $\Phi_2 : [M_n, M_n] \to X$. Without loss of generality we may assume that Φ_2 is defined on M_n (just choose any $x \in X$ and extend Ψ_2 to M_n by setting $\Phi_2(e_{11}) = x$). Accordingly,

$$\phi(a, b) = \phi(\mathbf{1}, ab) - \Phi_2(ab - ba) = \Phi_1(ab) + \Phi_2(ba)$$

where $\Phi_1 : M_n \to X$ is defined by $\Phi_1(a) = \phi(\mathbf{1}, a) - \Phi_2(a)$. □

Let us add that Φ_1 and Φ_2 are not uniquely determined. This is clear from the proof. Namely, $\Phi_2(e_{11})$ can be chosen arbitrarily. However, Φ_1 and Φ_2 are uniquely determined on $[M_n, M_n]$.

An analogous results for bilinear maps preserving zero products of rank one idempotents can be easily derived from Theorem 2.2.

Corollary 2.3. *Let $\phi : M_n \times M_n \to X$ be a bilinear map such that $\phi(e, f) = 0$ whenever e and f are rank one idempotents such that $ef = 0$. Then there exists a linear operator $\Phi : M_n \to X$ such that*

$$\phi(a, b) = \Phi(ab) \quad (a, b \in M_n).$$

Proof. By Theorem 2.2 we know that $\phi(a, b) = \Phi_1(ab) + \Phi_2(ba)$, and we only have to show that $\Phi_2 = 0$. As it is clear from the above comments, there is no loss of generality in assuming that $\Phi_2(e_{11}) = 0$. Picking any $i \neq j$ we see that $e = e_{jj} + e_{ij}$ and $f = e_{ii}$ are rank one idempotents satisfying $ef = 0$, and so $\Phi_2(e_{ij}) = \Phi_2(fe) = \phi(e, f) = 0$. Similarly, for any $i \neq 1$ we consider $e' = \frac{1}{2}(e_{11} + e_{1i} + e_{i1} + e_{ii})$, $f' = e_{11} - e_{i1}$, and obtain $\Phi_2(f'e') = 0$, that is, $\Phi_2(e_{11} + e_{1i} - e_{i1} - e_{ii}) = 0$. Since $\Phi_2(e_{11}) = \Phi_2(e_{1i}) = \Phi_2(e_{i1}) = 0$ it follows that $\Phi_2(e_{ii}) = 0$. But then $\Phi_2 = 0$. □

3. Applications to linear preserver problems

In the next corollaries we shall try to illustrate the usefulness of Theorem 2.2. One might regard the results that we shall obtain just as technical improvements of known results; however, the approach based on Theorem 2.2 yields new proofs of some of them, and moreover, it makes it possible for us to cover various preserver conditions simultaneously.

Corollary 3.1. *Let $T : M_n \to M_n$, $n \geq 3$, be a linear map such that $[T(e), T(f)] = 0$ whenever e and f are orthogonal rank one idempotents.*

Then either the range of T is commutative or T is of the form $T(a) = \lambda\Phi(a) + f(a)\mathbf{1}$ for all $a \in M_n$, where λ is a non-zero scalar, Φ is an automorphism or an antiautomorphism of the algebra M_n, and f is a linear functional on M_n.

Proof. Define $\phi : M_n \times M_n \to M_n$ by $\phi(a,b) = [T(a),T(b)]$ for all $a,b \in A$. Obviously ϕ satisfies the conditions of Theorem 2.2. Thus there exist linear maps $\Phi_1, \Phi_2 : M_n \to M_n$ such that $\phi(a,b) = \Phi_1(ab) + \Phi_2(ba)$. However, since ϕ is skew-symmetric it is easy to see that $\Phi_2 = -\Phi_1$. So we have $[T(a),T(b)] = \Phi_1([a,b])$ for all $a,b \in A$. In particular, $[T(a),T(a^2)] = 0$ for each $a \in A$. The result now follows immediately from [6, Theorem 4.11]. □

The condition treated in Corollary 3.1 covers three conditions that have been treated thoroughly in the literature: the condition that T preserves commutativity, the condition that T preserves zero products, and the condition that T preserves idempotents (see e.g. [5]). Perhaps this is not entirely obvious for the last condition. But in fact it is very easy to see this: If T preservers idempotents, then for every orthogonal idempotents e and f we have that $e+f$ is also an idempotent, hence $T(e)$, $T(f)$, and $T(e+f) = T(e)+T(f)$ are idempotents, which is possible only when $T(e)T(f) = 0 = T(f)T(e)$.

We remark that Corollary 3.1 holds only for maps T from M_n into itself, in general it does not hold for maps from M_n into, say, M_m with $m > n$. Under a stronger assumption we can obtain a conclusion also for maps from M_n into any other algebra B.

Corollary 3.2. *If $T : M_n \to B$ is a linear map such that $T(e)T(f) = 0$ whenever e and f are orthogonal rank one idempotents, then $T(\mathbf{1})$ commutes with every $T(a)$, $a \in M_n$, and T satisfies*

$$T(\mathbf{1})T(ab + ba) = T(a)T(b) + T(b)T(a) \quad (a, b \in M_n).$$

Proof. Define $\phi : M_n \times M_n \to B$ by $\phi(a,b) = T(a)T(b)$ for all $a,b \in M_n$. Using Theorem 2.2 it follows that $T(a)T(b) = \Phi_1(ab) + \Phi_2(ba)$ for all $a,b \in M_n$, where $\Phi_1, \Phi_2 : M_n \to B$ are linear operators. Letting first $a = \mathbf{1}$, and then $b = \mathbf{1}$, we see that $T(\mathbf{1})$ indeed commutes with every $T(a)$, and that $T(\mathbf{1})T(a) = (\Phi_1 + \Phi_2)(a)$. Accordingly,

$$T(a)T(b)+T(b)T(a) = \Phi_1(ab)+\Phi_2(ba)+\Phi_1(ba)+\Phi_2(ab) = T(\mathbf{1})T(ab+ba).$$

□

If $c = T(1)$ is invertible in B, then then we can represent T as $T(a) = c\Phi(a)$ where Φ is a Jordan homomorphism. In general, however, c may not be invertible, and T cannot be represented through Jordan homomorphisms. For example, any map T such that $T(a)T(b) = 0$ for all $a,b \in M_n$ trivially satisfies the conditions of Corollary 3.2.

We remark that Corollary 3.2 extends [5, Theorem 2.1] which states that a linear map from M_n into B that preserves idempotents is necessarily a Jordan homomorphism.

Arguing in a similar way as in the proof of Corollary 3.2, only that we use Corollary 2.3 instead of Theorem 2.2, we get the following result.

Corollary 3.3. *If $T : M_n \to B$ is a linear map such that $T(e)T(f) = 0$ whenever e and f are rank one idempotents satisfying $ef = 0$, then $T(\mathbf{1})$ commutes with every $T(a)$, $a \in M_n$, and T satisfies*

$$T(\mathbf{1})T(ab) = T(a)T(b) \quad (a, b \in M_n).$$

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