

MULTIPLICATION ALGEBRA AND MAPS DETERMINED BY ZERO PRODUCTS

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Dedicated to Professor Pjek-Hwee Lee on his 65th birthday

ABSTRACT. Let A be a finite dimensional central simple algebra. By the Skolem-Noether theorem, every automorphism of A is inner. We will give a short proof of a somewhat more general result. The concept behind this proof is the fact that every linear map on A belongs to the multiplication algebra of A . As an application we will describe linear maps $\alpha, \beta : A \rightarrow A$ such that $\alpha(x)\beta(y) = 0$ whenever $xy = 0$.

1. INTRODUCTION

Describing zero product preserving linear maps, i.e., maps α from an algebra A into itself satisfying “ $xy = 0 \implies \alpha(x)\alpha(y) = 0$ ”, is one of the most studied linear preserver problems. It has been treated in both analytic and algebraic context. For history we refer the reader to [1, 7, 8]. Let us just mention that the first algebraic result is apparently due to Wong [13] who, in particular, described bijective zero product preserving linear maps on finite dimensional simple algebras that are not division algebras. By Wedderburn’s theorem, such an algebra is isomorphic to $M_n(D)$, $n \geq 2$, the algebra of all $n \times n$ matrices over a finite dimensional division algebra D .

The study of a linear preserver problem typically begins on matrix algebras. Later, when the theory unfolds, extensions to various infinite dimensional algebras take place. In this article we will revisit the finite dimensional situation of the zero product preserving problem. We will avoid assuming the bijectivity of our maps. Moreover, we will actually treat a more general condition where a pair of linear maps α, β satisfies “ $xy = 0 \implies \alpha(x)\beta(y) = 0$ ”. To the best of our knowledge this condition has not been studied yet in the literature. Anyhow, the emphasis will be on the method of proof rather than on the complexity and originality of results. It should be clear to experts that appropriate versions of some of our results hold in more general rings and algebras. Our goal, however, is to make the paper accessible to a wider audience, and therefore we shall stick with the simplest setting.

Our starting point is the fact that every linear map on a finite dimensional central simple algebra A belongs to the multiplication algebra of A . Although this result is well-known, it is our impression that it is not always exploited to its full potential. We shall demonstrate its usefulness in the proof of a theorem describing linear maps $\varphi, \alpha, \beta : A \rightarrow A$ satisfying the functional identity $\varphi(xy) = \alpha(x)\beta(y)$ for all $x, y \in A$. As a byproduct we will obtain a simple and short proof of the classical result, an important special case of the Skolem-Noether theorem, stating that every automorphism of a finite dimensional central simple algebra is inner. The problem on maps

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satisfying “ $xy = 0 \implies \alpha(x)\beta(y) = 0$ ” will be reduced to the one concerning $\varphi(xy) = \alpha(x)\beta(y)$. In the proof of this reduction we will use the approach based on idempotents, which was initiated by Chebotar, Ke, Lee and Wong in [8], and further developed in [3, 7, 11].

2. MULTIPLICATION ALGEBRA

In this section we prove two results, which are both well-known. Anyway, the proofs are short and will be given for the sake of completeness.

Let A be an algebra over a (fixed) field F . For $a, b \in A$ we define multiplication maps $L_a, R_b : A \rightarrow A$ by $L_a(x) = ax$, $R_b(x) = xb$. It is clear that for all $a, b \in A$, $\lambda, \mu \in F$ we have

$$\begin{aligned} L_{ab} &= L_a L_b, \quad R_{ab} = R_b R_a, \\ L_a R_b &= R_b L_a, \\ L_{\lambda a + \mu b} &= \lambda L_a + \mu L_b, \quad R_{\lambda a + \mu b} = \lambda R_a + \mu R_b. \end{aligned}$$

Clearly, $L_a, R_b \in \text{End}_F(A)$, the algebra of all linear maps from A into A . The subalgebra $M(A)$ of $\text{End}_F(A)$ generated by all L_a and R_b , $a, b \in A$, is called the *multiplication algebra of A* . Elements in $M(A)$ are also called *elementary operators*. If A is unital, then every $\varphi \in M(A)$ can be written as $\varphi = \sum_{i=1}^n L_{a_i} R_{b_i}$. The elements a_i, b_i are not unique. However, at least we can always achieve that the a_i 's are linearly independent, and the b_i 's are linearly independent. For instance, this clearly holds if we require that n is minimal possible.

Recall that an algebra A is said to be *central* if its center consists of scalar multiples of unity.

Theorem 2.1. *Let A be a central simple algebra. If $a_i, b_i \in A$ are such that $\sum_{i=1}^n L_{a_i} R_{b_i} = 0$ and the b_i 's are linearly independent, then each $a_i = 0$.*

Proof. Suppose $a_n \neq 0$. We are going to prove by induction on n that this is impossible. As A is simple, there exist $w_j, z_j \in A$ such that $\sum_{j=1}^m w_j a_n z_j = 1$. Consequently,

$$0 = \sum_{j=1}^m L_{w_j} \left(\sum_{i=1}^n L_{a_i} R_{b_i} \right) L_{z_j} = \sum_{i=1}^n \left(\sum_{j=1}^m L_{w_j a_i z_j} \right) R_{b_i} = \sum_{i=1}^n L_{c_i} R_{b_i}$$

where $c_i = \sum_{j=1}^m w_j a_i z_j$. In particular, $c_n = 1$. This shows that $n \neq 1$, so let $n > 1$. For every $x \in A$ we have

$$0 = \left(\sum_{i=1}^n L_{c_i} R_{b_i} \right) L_x - L_x \left(\sum_{i=1}^n L_{c_i} R_{b_i} \right) = \sum_{i=1}^{n-1} L_{c_i x - x c_i} R_{b_i}.$$

The induction assumption implies that $c_i x - x c_i = 0$, and therefore $c_i \in F$. Accordingly, $\sum_{i=1}^n L_{c_i} R_{b_i} = 0$ can be written as $R_{c_1 b_1 + \dots + c_n b_n} = 0$, which contradicts the linear independence of the b_i 's. \square

We remark that Theorem 2.1 holds for considerably more general centrally closed prime algebras (cf. [2]). The proof in this more general setting is similar, but somewhat more involved.

In our main results we will consider finite dimensional central simple algebras, i.e., algebras isomorphic to $M_n(D)$ where D is a finite dimensional central division algebra.

Corollary 2.2. *If A is a finite dimensional central simple algebra, then $M(A) = \text{End}_F(A)$.*

Proof. Let $\{b_1, \dots, b_n\}$ be a basis of A . Theorem 2.1 implies that the maps $L_{b_i} R_{b_j}$, $1 \leq i, j \leq n$, are linearly independent. Therefore the dimension of $M(A)$ is $\geq n^2$, which is the dimension of $\text{End}_F(A)$. \square

3. A SKOLEM-NOETHER TYPE THEOREM

The condition treated in the next theorem can be viewed as a special functional identity, for which, however, the general theory [4] does not directly yield any information.

Theorem 3.1. *Let A be a finite dimensional central simple algebra. If $\varphi, \alpha, \beta : A \rightarrow A$ are linear maps such that $\varphi \neq 0$ and $\varphi(xy) = \alpha(x)\beta(y)$ for all $x, y \in A$, then there exist $u, v, p \in A$, with p invertible, such that $\varphi(x) = uxv$, $\alpha(x) = uxp$, and $\beta(x) = p^{-1}xv$ for all $x \in A$.*

Proof. By Corollary 2.2 there exist $a_i, b_i \in A$ such that $\beta = \sum_{i=1}^n L_{a_i} R_{b_i}$. We may require that the b_i 's are linearly independent. Setting $c_i = \alpha(1)a_i$ we have $\varphi = L_{\alpha(1)}\beta = \sum_{i=1}^n L_{c_i} R_{b_i}$. Since $\varphi \neq 0$, we may assume that $c_1 \neq 0$. Now, noticing that our assumption can be written as $\varphi L_x = L_{\alpha(x)}\beta$ for every $x \in A$, we obtain $\sum_{i=1}^n L_{c_i x - \alpha(x)a_i} R_{b_i} = 0$. Theorem 2.1 tells us that, in particular, $c_1 x - \alpha(x)a_1 = 0$ for every $x \in A$. Consequently, $(\sum_i y_i \alpha(x_i))a_1 = \sum_i y_i c_1 x_i$ for all $x_i, y_i \in A$. As A is simple we have $\sum_i y_i c_1 x_i = 1$ for some $x_i, y_i \in A$, implying that a_1 has a left inverse. Since A is finite dimensional, this already proves that a_1 is invertible. Setting $u = c_1$ and $p = a_1^{-1}$ we thus have $\alpha(x) = uxp$, $x \in A$. Since $\varphi(x) = \alpha(x)\beta(1)$ it follows that $\varphi(x) = uxv$ where $v = p\beta(1)$. Finally, from $\varphi(xy) = \alpha(x)\beta(y)$ it now readily follows that $yv = p\beta(y)$, and hence $\beta(y) = p^{-1}yv$, $y \in A$. \square

Corollary 3.2. (Skolem-Noether) *Every automorphism of a finite dimensional central simple algebra A is inner.*

This corollary indeed follows easily from the conclusion of Theorem 3.1. On the other hand, assuming, in this theorem, that $\varphi = \alpha = \beta$, the proof can be slightly simplified. Specifically, c_1 is then equal to a_1 , and so the last step of the above proof is redundant.

Remark 3.3. The same method easily yields an analogous theorem stating that every derivation δ of a finite dimensional central simple algebra A is inner. Indeed, we may write $\delta = \sum_{i=1}^n L_{a_i} R_{b_i}$ with b_i 's linearly independent and $b_1 = 1$. Using this expression in $L_{\delta(x)} = \delta L_x - L_x \delta$ it follows immediately from Theorem 2.1 that $\delta(x) = [a_1, x]$.

4. ZERO PRODUCT DETERMINED ALGEBRAS

We say that an algebra A is *zero product determined* if for every bilinear map $B : A \times A \rightarrow X$, where X is an arbitrary vector space, the following condition is fulfilled: If for all $x, y \in A$, $xy = 0$ implies $B(x, y) = 0$, then there is a linear map $\varphi : A \rightarrow X$ such that $B(x, y) = \varphi(xy)$ for all $x, y \in A$. Motivated by results from [1] and [6] this concept was introduced in [5], and studied further in [9, 10, 12].

The next two results can be extracted from the arguments in [8] and [3]. Nevertheless, we will give complete proofs.

Theorem 4.1. *If a unital algebra A is generated by its idempotents, then A is zero product determined.*

Proof. Let $B : A \times A \rightarrow X$ be such that $xy = 0$ implies $B(x, y) = 0$. If $e = e^2 \in A$, then $xe \cdot (1 - e)y = x(1 - e) \cdot ey = 0$. Therefore $B(xe, y - ey) = B(x - xe, ey) = 0$. That is, $B(xe, y) = B(xe, ey)$ and $B(x, ey) = B(xe, ey)$. Comparing we get $B(xe, y) = B(x, ey)$. This means that the set $R = \{z \in A \mid B(xz, y) = B(x, zy) \text{ for all } x, y \in A\}$ contains all idempotents. As R is readily a subalgebra of A , it follows that $R = A$. Therefore $B(x, z) = B(xz, 1) = \varphi(xz)$, where $\varphi(w) = B(w, 1)$. \square

Corollary 4.2. *Let A be a finite dimensional simple algebra. If A is not a division algebra, then A is zero product determined.*

Proof. In view of Wedderburn's theorem we may assume that $A = M_n(D)$, where D is a division algebra. Note that $n \geq 2$ according to our assumption.

Let e_{ij} denote standard matrix units in A . By ae_{ij} we denote the matrix whose (i, j) entry is $a \in D$ and other entries are 0. Every matrix in A is a sum of such matrices. If $i \neq j$, then ae_{ij} can be written as a difference of two idempotents, $ae_{ij} = (e_{ii} + ae_{ij}) - e_{ii}$. From $ae_{ii} = ae_{ij} \cdot e_{ji}$ with $i \neq j$ we thus see that ae_{ii} also lies in the subalgebra generated by idempotents. Therefore A is generated by idempotents. \square

Incidentally, the exclusion of division algebras is certainly necessary. Namely, the condition " $xy = 0 \implies B(x, y) = 0$ " is automatically fulfilled for every bilinear map on a division algebra.

5. A PAIR OF MAPS DETERMINED BY ZERO PRODUCTS

The results from the previous two sections now immediately yield the following theorem.

Theorem 5.1. *Let A be a finite dimensional central simple algebra which is not a division algebra. If linear maps $\alpha, \beta : A \rightarrow A$ are such that $xy = 0$ implies $\alpha(x)\beta(y) = 0$, then either $\alpha(A)\beta(A) = 0$ or there exist $u, v, p \in A$, with p invertible, such that $\alpha(x) = uxp$ and $\beta(x) = p^{-1}xv$, $x \in A$.*

Proof. Define $B : A \times A \rightarrow A$ by $B(x, y) = \alpha(x)\beta(y)$. According to our assumption, $xy = 0$ implies $B(x, y) = 0$. Corollary 4.2 tells us that there is a linear map $\varphi : A \rightarrow A$ such that $B(x, y) = \varphi(xy)$, $x, y \in A$. Now apply Theorem 3.1. \square

In the classical situation where $\alpha = \beta$ we easily get the following corollary.

Corollary 5.2. *Let A be a finite dimensional central simple algebra which is not a division algebra. If a linear map $\alpha : A \rightarrow A$ is such that $xy = 0$ implies $\alpha(x)\alpha(y) = 0$, then either $\alpha(A)^2 = 0$ or there exist an invertible $p \in A$ and $\lambda \in F$ such that $\alpha(x) = \lambda p^{-1}xp$, $x \in A$.*

Corollary 5.2 thus states that a zero product preserving linear map is either of a standard form, i.e., a scalar multiple of an automorphism, or its range has trivial multiplication. This is analogous to [6, Theorem 4.1] saying that a zero Lie product preserving (=commutativity preserving) linear map on a finite dimensional central simple algebra (of dimension $\neq 4$) has either a standard form of such a preserver, or its range is commutative. This suggests that there might be a more general phenomena hidden behind these results.

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