Positive semidefinite quadratic determinantal representations

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Outline I

1. Symmetric quadratic determinantal representations
   - Positive maps
   - Self-dual sheaves

2. Linear matrix inequalities
   - Spectrahedral cone $\leftrightarrow$ hyperbolic polynomial
   - Cubic symmetroid
   - Criteria for $P$ to be positive

3. Polynomial nonnegativity
   - PSD and SOS polynomials
   - PSD and SOS matrices
   - Šivic biquadratic form
Warm up question

Given a homogeneous nonnegative polynomial \( p(x, y, z) \) of degree 6, does there exist a positive linear map \( P : \text{Sym}_3 \rightarrow \text{Sym}_3 \) such that

\[
\det P(xx^T) = p(x, y, z) \quad \text{for all} \quad x = [x, y, z]^T?
\]

We will “tackle” this question from three sides:

- symmetric quadratic determinantal representations and the associated sheaves (kernels);
- semidefinite linear determinantal representations (LMI representations of hyperbolic polynomials);
- polynomial algebra (SOS and PSD polynomials and matrices).
Definition

A linear map $P : \text{Sym}_3 \rightarrow \text{Sym}_3$ is positive if it sends positive semidefinite matrices to positive semidefinite matrices.

Positive maps were popular in the 70s as they describe various quantum states in quantum physics. In the last decade there is again very active and fertile research in this area due to its connection to optimization.
Positive maps

Definition

A linear map $P : \text{Sym}_3 \to \text{Sym}_3$ is **positive** if it sends positive semidefinite matrices to positive semidefinite matrices.

Positive maps were popular in the 70s as they describe various quantum states in quantum physics. In the last decade there is again very active and fertile research in this area due to its connection to optimization.
Clearly it is enough to check the positivity of $P$ on rank 1 matrices. In coordinates our question then becomes

**Question**

*Given a nonnegative plane sextic $\mathcal{C}$ in $\mathbb{P}^2$, does there exist a symmetric quadratic determinantal representation of $\mathcal{C}$ which is semidefinite for all $(x, y, z) \in \mathbb{P}^2$?*

Indeed, 

$$
\begin{bmatrix}
  x^2 & xy & xz \\
  xy & y^2 & yz \\
  xz & yz & z^2
\end{bmatrix} = 
\begin{bmatrix}
p_0 & p_1 & p_3 \\
p_1 & p_2 & p_4 \\
p_3 & p_4 & p_5
\end{bmatrix},
$$

where $p_i = p_{0i}x^2 + p_{1i}xy + p_{2i}x^2 + p_{3i}xz + p_{4i}yz + p_{5i}z^2$. 
Clearly it is enough to check the positivity of $P$ on rank 1 matrices. In coordinates our question then becomes

**Question**

*Given a nonnegative plane sextic $C$ in $\mathbb{P}^2$, does there exist a symmetric quadratic determinantal representation of $C$ which is semidefinite for all $(x, y, z) \in \mathbb{P}^2$?*

Indeed, $P \begin{bmatrix} x^2 & xy & xz \\ xy & y^2 & yz \\ xz & yz & z^2 \end{bmatrix} = \begin{bmatrix} p_0 & p_1 & p_3 \\ p_1 & p_2 & p_4 \\ p_3 & p_4 & p_5 \end{bmatrix},$

where $p_i = p_{0i}x^2 + p_{1i}xy + p_{2i}x^2 + p_{3i}xz + p_{4i}yz + p_{5i}z^2.$
Without the nonnegativity and semidefiniteness conditions this is a classical case of determinantal hypersurfaces

**Question**

*Given a plane curve \( C \) of degree \( 2d \), does there exist a \( d \times d \) symmetric quadratic determinantal representation of \( C \)?
On an integral curve $C$, a coherent torsion-free rank 1 (arithmetically Cohen-Macaulay, ACM) sheaf $F$ that is generated by its global sections and $F \cong \mathcal{H}om(F, \mathcal{O}_C(2d - 2))$ admits a resolution

$$0 \to \mathcal{O}_{\mathbb{P}^2}(-2)^d \xrightarrow{M} \mathcal{O}_{\mathbb{P}^2}^d \to F \to 0,$$

where $M$ is a symmetric quadratic matrix with $\det M = p$.

**Remark:** The above $F$ is non-exceptional. This is equivalent to $H^0(C, F(-1)) = H^1(C, F) = 0$. Then $h^0(C, F) = d$ and its global sections yield $M$. 
Actually, any $\mathcal{F}$ that is self-dual

$$\mathcal{F} \cong \mathcal{H}om(\mathcal{F}, \mathcal{O}_C(2d - 2))$$

admits a resolution $0 \to \bigoplus_{i=1}^{l} \mathcal{O}_{\mathbb{P}^2}(-2 - d_i) \xrightarrow{M} \bigoplus_{i=1}^{l} \mathcal{O}_{\mathbb{P}^2}(d_i) \to \mathcal{F} \to 0$,

where $M = [m_{ij}]$ is symmetric with $m_{ij}$ of degree $d_i + d_i - 2$.

**Remark:** We are only interested in non-exceptional $\mathcal{F}$, for which $d_i = 0$ for $i = 1, \ldots, d$. The set of such pairs $(C, \mathcal{F})$ is Zariski dense in the universal Jacobian $\mathcal{J}_{2d}^{2d(d-1)}$. 
Define the moduli space $\mathcal{R}_{2d}$ of pairs $(C, \alpha)$, where $C$ is a smooth plane curve of degree $2d$ (over a field of char 0), and $\alpha$ is a half-period, i.e. a 2-torsion divisor on $\text{Jac}(C)$, i.e. a nontrivial line bundle on $C$ satisfying $\alpha \otimes 2 \cong \mathcal{O}_C$.

**Proposition**

For $(C, \alpha)$ general in $\mathcal{R}_{2d}$, the half-period $\alpha$ admits a minimal resolution

$$0 \to \mathcal{O}_{\mathbb{P}^2}(-d - 1)^d \xrightarrow{M} \mathcal{O}_{\mathbb{P}^2}(-d + 1)^d \to \alpha \to 0,$$

where $M$ is a symmetric quadratic matrix with $\det M = p$.

Note, $\mathcal{F}$ is obtained from the half-period $\alpha$ by $\mathcal{F} = \alpha \otimes \mathcal{O}_C(d - 1)$. 
Simple singularities

When $C/\mathbb{C}$ has only simple (this means AED) singularities, there are \textbf{finitely many} ACM sheaves with the following self-duality

$$\mathcal{F} \cong \mathcal{H}om(\mathcal{F}, \mathcal{O}_C(2d - 2)).$$

It is possible to \textbf{explicitly count} them using methods in [Piontkowski, 2007]. Their number depends on the \textit{genus} of the curve and the \textit{local type} of $\mathcal{F}$: $(\mathcal{F}_s)_{s \in \text{Sing}_C}$ is a collection of self-dual modules $\mathcal{F}_s \cong \text{Hom}(\mathcal{F}_s, \mathcal{O}_{C,s}(2d - 2))$. For a simple singularity there are only finitely many isomorphism classes of indecomposable torsion-free modules over its local ring.
Simple singularities

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Given a plane curve $C$ of degree $2d$ with only simple
singularities, does there exist a $d \times d$ symmetric quadratic
determinantal representation of $C$?

There are finitely many symmetric quadratic determinantal
representations of $C$ corresponding to non-exceptional
torsion-free rank 1 sheaves on $C$ with self-duality
$\mathcal{F} \cong \mathcal{H}om(\mathcal{F}, \mathcal{O}_C(2d - 2))$.

A smooth $C$ has $2^{2g}$ self-dual $\mathcal{F}$; the number decreases rapidly
with the number and order of singularities $A_n$, $D_m$, $E_l$. When all
$\mathcal{F}$ are exceptional, $C$ has no quadratic representations.
\[ \mathcal{F} \cong \text{Hom}(\mathcal{F}, \mathcal{O}_C(2d - 2)) \]

Minimal resolutions \( M = [m_{ij}] \) with \( \deg m_{ij} = d_i + d_j + 2 \):

- \( \mathcal{F} \) non-except.
- \( \mathcal{F} \) exceptional

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\( \text{SINGULAR} \), \( \text{SQUARE} \), square
A Hermitian / symmetric linear matrix polynomial $M$ is a pencil

$$M_0 + x_1 M_1 + \cdots + x_n M_n, \text{ where } M_i \in H_d(\mathbb{C}) / \text{Sym}_d(\mathbb{R}).$$

Spectrahedron $S(M)$ is the set of points where $M$ is positive semidefinite

$$S(M) := \{ a \in \mathbb{R}^n : M(a) \succeq 0 \}.$$

Spectrahedra are precisely the sets on which semidefinite programming SDP can be performed.
We can always assume that 0 belongs to the interior of $S(M)$, and after conjugation with a unitary / orthogonal matrix we can take $M_0 = \text{Id}$.

Now consider

$$p = \det M.$$ 

Note that $S(M)$ can be retrieved from the polynomial $p$ only. It consists of those points $a$ for which $p$ has NO zeros between the origin and $a$. Indeed, for each $a \in \mathbb{R}^n$, the nonzero eigenvalues of $a_1 M_1 + \cdots + a_n M_n$ are in 1-1 correspondence with the zeros of the univariate polynomial $p_a(t) := p(ta)$, by the following rule $\lambda \mapsto -1/\lambda$. 

spectrahedra $\leftrightarrow$ real zero polynomials
Conversely, take a real zero polynomial or RZ-polynomial $p$,

$$p(0) = 1 \text{ and } \forall a \in \mathbb{R}^n \ p(t \ a) = 0 \Rightarrow t \in \mathbb{R}.$$ 

It is natural to ask whether the rigidly convex set defined by $p$

$$\{a \in \mathbb{R}^n : p_a(t) = p(t \ a) \text{ has no roots in } [0, 1)\}$$

is a spectrahedron of some linear matrix $M$. When $M$ exists, it is called a linear matrix inequality or LMI for $p$. 
Cubic curve $(x_0 - x_1)(x_0 + x_1)(x_0 - 4x_1) - x_1x_2^2 = 0$
Symmetric quadratic determinantal representations
Linear matrix inequalities
Polynomial nonnegativity

Spectrahedral cone $\leftrightarrow$ hyperbolic polynomial
Cubic symmetroid
Criteria for $P$ to be positive

Terminology $\mathbb{A}^n \leftrightarrow \mathbb{P}^n$

$\mathbb{A}^n$: spectrahedron  RZ-polynomial  rigidly convex set  PSD  LMI
$\mathbb{P}^n$: spectrahedral cone  hyperbolic poly.  hyperbolicity cone  SD  LMI
Cubic curve \((x_0 - x_1)(x_0 + x_1)(x_0 - 4x_1) - x_1 x_2^2 = 0\) has three symmetric determinantal representations:

- two are definite determinantal representations

\[
\begin{pmatrix}
  x_0 & -\frac{x_1}{\sqrt{2}} & \frac{x_1}{\sqrt{2}} \\
-\frac{x_1}{\sqrt{2}} & \frac{8x_0 - x_1 + 4x_2}{8} & -\frac{x_1}{8} \\
\frac{x_1}{\sqrt{2}} & -\frac{x_1}{8} & \frac{8x_0 - x_1 - 4x_2}{8}
\end{pmatrix}, \quad
\begin{pmatrix}
  x_0 & -\frac{x_1}{2\sqrt{2}} & \frac{x_1}{2\sqrt{2}} \\
-\frac{x_1}{2\sqrt{2}} & \frac{8x_0 - x_1 + 4x_2}{8} & -\frac{7x_1}{8} \\
\frac{x_1}{2\sqrt{2}} & -\frac{7x_1}{8} & \frac{8x_0 - x_1 - 4x_2}{8}
\end{pmatrix}
\]

- one is nondefinite

\[
x_0 \text{Id} + x_2 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & -1/2 \end{pmatrix} + x_1 \begin{pmatrix} 0 & i & -i \\ i & 2\sqrt{2} & 2\sqrt{2} \\ -i & 2\sqrt{2} & 2\sqrt{2} \end{pmatrix}
\]
Veronese surface

In coordinates write $\mathbb{P}^5$ as a symmetric matrix

$$\begin{bmatrix} Z_0 & Z_1 & Z_3 \\ Z_1 & Z_2 & Z_4 \\ Z_3 & Z_4 & Z_5 \end{bmatrix},$$

and consider the Veronese embedding

$$\nu_2 : \mathbb{P}^2 \longrightarrow \mathbb{P}^5$$

$$(x, y, z) \mapsto \begin{bmatrix} x^2 & xy & xz \\ xy & y^2 & yz \\ xz & yz & z^2 \end{bmatrix}.$$
Cubic symmetroid as LMI

Cubic symmetroid in $\mathbb{P}^5$ is the hypersurface defined by

$$\det \begin{bmatrix} z_0 & z_1 & z_3 \\ z_1 & z_2 & z_4 \\ z_3 & z_4 & z_5 \end{bmatrix} = 0.$$ 

It is singular along the Veronese surface. Semidefinite matrices lie in the spectrahedral cone

$$\left\{ (z_0, \ldots, z_5) \in \mathbb{P}^5 : \begin{bmatrix} z_0 & z_1 & z_3 \\ z_1 & z_2 & z_4 \\ z_3 & z_4 & z_5 \end{bmatrix} \succeq 0 \right\}.$$
Cubic symmetroid as LMI
“Being” a spectrahedral cone is much more than a convex cone!

SD matrix

\[ \begin{align*}
\text{rk 1} & \iff \text{points on the Veronese} \\
\text{rk 2} & \iff \partial \text{spectrahedral cone} \\
\text{rk} \leq 3 & \iff \text{spectrahedral cone}
\end{align*} \]

\[ \lambda aa^T + (1 - \lambda)bb^T, \lambda \in [0, 1] \]

\[ \alpha aa^T + \beta bb^T + \gamma cc^T, \alpha, \beta, \gamma \geq 0 \]
The Veronese map $\nu_2$ is given by the complete linear system. Thus the preimage of a hyperplane $H$ in $\mathbb{P}^5$ is a conic $Q_H$ in $\mathbb{P}^2$.

Conic $Q_H$ is singular $\iff$ $H$ is tangent to the Veronese surface

More precisely,

- if $Q_H$ is a line pair, then $H$ is a tangent to the Veronese surface at a single point,

- if $Q_H$ is a double line, then $H$ is tangent to the Veronese surface along the curve that is the image of $Q_H^{\text{red}}$ under the restriction of $\nu_2$. 
Recall that $P : \text{Sym}_3 \to \text{Sym}_3$ is positive if and only if

$$
\begin{bmatrix}
  p_0 & p_1 & p_3 \\
  p_1 & p_2 & p_4 \\
  p_3 & p_4 & p_5
\end{bmatrix} \succeq 0 \text{ for all } \mathbf{x} = (x, y, z) \in \mathbb{P}^2,
$$

where

$$p_i = p_{0i}x^2 + p_{1i}xy + p_{2i}x^2 + p_{3i}xz + p_{4i}yz + p_{5i}z^2$$

and $P = [p_{ij}]_{0 \leq i, j \leq 5} : \mathbb{P}^5 \to \mathbb{P}^5$ in the corresponding basis.
This way $P : \text{Sym}_3 \rightarrow \text{Sym}_3$ induces

$$
\nu_P : \mathbb{P}^2 \rightarrow \mathbb{P}^5
(x, y, z) \mapsto (p_0, \ldots, p_5),
$$

**Lemma**

$P$ is positive if and only if the image of $\nu_P$ lies in the spectrahedral cone of the symmetroid.

Therefore, classifying positive maps is the same as classifying linear maps that preserve the spectrahedral cone of the symmetroid.
The convex hull of $\text{Im}(\nu_P)$

When $P$ is invertible, $\nu_P$ is given by the complete linear system. In this case the image of $\nu_P$ is the singular locus of the hypersurface

$$\det P^{-1}\begin{pmatrix} x_0 & x_1 & x_3 \\ x_1 & x_2 & x_4 \\ x_3 & x_4 & x_5 \end{pmatrix} = 0.$$

In other words, the convex hull of $\text{Im} \nu_P$ equals to the spectrahedral cone of the above hypersurface. Else, the convex hull of $\text{Im}(\nu_P)$ is a projection of a spectrahedral cone shadow (inside the spectrahedral cone of the symmetroid).
The convex hull of $\text{Im}(\nu_{P^{-1}})$

On the other side, the hypersurface

$$\det P \begin{pmatrix} z_0 & z_1 & z_3 \\ z_1 & z_2 & z_4 \\ z_3 & z_4 & z_5 \end{pmatrix} = 0$$

contains the Veronese surface in its spectrahedral cone. When $P$ is invertible, this spectrahedral cone equals to the convex hull of the image of

$$\nu_{P^{-1}} : \mathbb{P}^2 \longrightarrow \mathbb{P}^5$$

$$(x, y, z) \mapsto P^{-1}\begin{pmatrix} x^2 & xy & xz \\ xy & y^2 & yz \\ xz & yz & z^2 \end{pmatrix}$$
Spectrahedral cones of:

\[ \det P([z_{ij}]) = 0, \quad \det[z_{ij}] = 0, \quad \det P^{-1}([z_{ij}]) = 0 \]
Convex algebraic geometry

\[ P_{n,2d} = \{ \text{non-negative (PSD) forms in } \mathbb{R}[x_0, \ldots, x_{n-1}] \text{ of degree } 2d \} \]

\[ \cup \]

\[ \Sigma_{n,2d} = \{ \text{sums of squares (SOS-polynomials)} \} \]

\[ \det \uparrow \]

\[ P_{n,2d}^M = \{ \text{positive semidefinite } d \times d \text{ matrix quadratic polynomials} \} \]

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$P_{n,2d}^M = \{ \text{positive semidefinite } d \times d \text{ matrix quadratic polynomials} \}$

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$\Sigma_{n,2d}^M = \{ \text{SOS-matrix quadratic polynomials} \}$
Given a form \( p \in \mathbb{R}_{2\ell}[x_0, \ldots, x_{n-1}] \), does there exist \( x \in \mathbb{R}^n \) such that \( p(x) < 0 \)?

- if not, \( p \) is called positive semidefinite or PSD

\[
p(x) \geq 0 \text{ for all } x \in \mathbb{R}^n
\]

- the problem is NP-hard, but decidable.

To verify the answer \textbf{yes} is easy: find \( x \) such that \( p(x) < 0 \);

answer \textbf{no} is hard: we need a certificate, that is proof that there is no feasible point.
If there exist polynomials $g_1, \ldots, g_r$ such that

$$p = \sum_{i=1}^{r} g_i^2,$$

then $p$ is called a **sum-of-squares (SOS)** polynomial.

Clearly, an SOS polynomial is PSD.
If there exist polynomials $g_1, \ldots, g_r$ such that

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then $p$ is called a sum-of-squares (SOS) polynomial.

Clearly, an SOS polynomial is PSD.
Lemma

A homogeneous polynomial \( p \in \mathbb{R}[x_0, \ldots, x_{n-1}] \) of degree \( 2d \) is an SOS polynomial if and only if there exists a positive semidefinite Gram matrix

\[
Q \succeq 0 \text{ such that } p = z^T Q z,
\]

where \( z \) denotes a vector of all monomials of degree \( d \).

- this is an SDP problem in standard primal form;
- the number of components of \( z \) is \( \binom{n+d-1}{d} \).
Proof

Factorize $Q = V V^T$ and write $V = [v_1 \cdots v_r]$ so that

$$p = z^T V V^T z = \| V^T z \|^2 = \sum_{i=1}^r (v_i^T z)^2.$$
Convex cone: \( p, q \in C \Rightarrow \lambda p + \mu q \in C \) for all \( \lambda, \mu > 0 \)

\[
P_{n,2d} = \{ \text{PSD polynomials of degree } 2d \}
\]
\[
\Sigma_{n,2d} = \{ \text{SOS polynomials of degree } 2d \}
\]

are both convex cones in \( \mathbb{R}^N \) where \( N = \binom{n+2d-1}{2d} \).

We know since Hilbert that

\[ \Sigma_{n,2d} \subset P_{n,2d}; \]

- testing if \( p \in P_{n,2d} \) is NP-hard,
- but testing if \( p \in \Sigma_{n,2d} \) is an SDP.
Hilbert’s 17th problem

Hilbert in 1888 showed that $\Sigma_{n,2d} = P_{n,2d}$ in the following cases:

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- $2d = 2$, quadratic polynomial forms
- $n = 2$, homogeneous polynomials in two variables
- $2d = 4$, $n = 3$, quartic forms in three variables
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Hilbert’s 17th problem

Artin in 1927 showed that every PSD polynomial is an SOS of rational functions. A constructive solution was found in 1984 by Delzell.
Hilbert’s 17th problem

In 1967 Motzkin constructed the first example of a positive semidefinite polynomial, that is not a sum of squares:

\[ p(x, y, z) = x^2 y^4 + x^4 y^2 + z^6 - 3x^2 y^2 z^2 \]
Blekherman in 2012 provided a geometric explanation for the containment $\Sigma_{3,6} \subset P_{3,6}$. The difference lies in fulfillment of certain linear relations (Cayley-Bacharach relations) from Hilbert’s proof.

- Robinson’s polynomial with 10 zeros (1973):
  \[ x^6 + y^6 + z^6 - x^4 y^2 - x^4 z^2 - y^4 x^2 - y^4 z^2 - z^4 x^2 - z^4 y^2 + 3x^2 y^2 z^2; \]

- lots of examples from Reznick’s construction (2007).
An algebraic boundary of a cone is the hypersurface that arises as Zariski closure of its topological boundary.

- Nie, 2011: The algebraic boundary of the cone \( P_{n,2d} \) is the discriminant of degree \( n(2d - 1)^{n-1} \).
- Blekherman, Sturmfels, et al., 2011: Discriminant is also a component in the algebraic boundary of \( \Sigma_{3,6} \). Besides, \( \partial \Sigma_{3,6} \) has another unique non-discriminant component of degree 83200 which consists of forms that are sums of three squares of cubics.

Remark: A sextic \( C \) that is a sum of three squares of cubics coincides with an ACM \( \text{rk} \, 1 \) sheaf \( F \cong \text{Hom}(F, \mathcal{O}_C(3)) \) that is globally generated; this is exactly an effective even theta characteristic.
Definition

A symmetric polynomial matrix $P(x)$ is an **SOS-matrix** if

$$P(x) = M(x) M(x)^T$$

for a possibly non-square polynomial matrix $M(x)$.

Definition

A matrix polynomial $P(x)$ is **positive semidefinite** if $P(x)$ is positive semidefinite for all $x = (x_0, \ldots, x_{n-1}) \in \mathbb{R}^n$. 

Recall that positive linear maps $P : \text{Sym}_3 \rightarrow \text{Sym}_3$ are in one-to-one correspondence with PSD quadratic ternary matrices $P(\mathbf{x}\mathbf{x}^T)$. Moreover, they are in one-to-one correspondence with non-negative biquadratic forms $\mathbf{u}^TP(\mathbf{x}\mathbf{x}^T)\mathbf{u}$, where $\mathbf{x} = [x, y, z]^T$ and $\mathbf{u} = [u, v, w]^T$.

**Lemma**

$P$ is positive $\iff \mathbf{u}^TP(\mathbf{x}\mathbf{x}^T)\mathbf{u}$ is a PSD polynomial $\iff P(\mathbf{x}\mathbf{x}^T)$ is a PSD quadratic matrix.
Choi map: A linear map $\phi : M_3 \to M_3$ induces a linear map $\Phi : M_9 \to M_9$ by the following rule
$$\Phi \left( [X_{ij}]_{i,j=1,2,3} \right) = \left[ \phi(X_{ij}) \right]_{i,j=1,2,3}.$$ 

Theorem (Choi, 1974)

*Choi matrix*

$$[\phi(E_{ij})]_{i,j=1,2,3} \text{ is positive semidefinite}$$

if and only if the restriction $\phi : \text{Sym}_3 \to \text{Sym}_3$ induces an SOS quadratic matrix $\phi(\mathbf{x}\mathbf{x}^T)$.

This is equivalent to $\mathbf{u}^T P(\mathbf{x}\mathbf{x}^T) \mathbf{u}$ being a biquadratic SOS form.

Such $\phi$ are called completely positive, in optimization they are called SOS.
The third equivalent definition of quadratic SOS matrices is the following:

**Lemma**

*Quadratic matrix* \( P(\mathbf{x} \mathbf{x}^T) \) *is an SOS matrix if and only if there exist* \( A_j \in \mathbb{R}^{3,3} \) *such that*

\[
P(x, y, z) = \sum_{j=1}^{r} A_j X A_j^T, \text{ where } X = \mathbf{x} \mathbf{x}^T = \begin{bmatrix} x^2 & xy & xz \\ xy & y^2 & yz \\ xz & yz & z^2 \end{bmatrix}.
\]
Indeed, for the $3 \times r$ linear matrix $M = [m_1 \cdots m_r]$ write

$$P(x, y, z) = MM^T = \sum_{j=1}^{r} m_j m_j^T = \sum_{j=1}^{r} \begin{bmatrix} m_{1j} \\ m_{2j} \\ m_{3j} \end{bmatrix} \cdot [m_{1j} m_{2j} m_{3j}]$$

$$= \sum_{j=1}^{r} A_j \begin{bmatrix} x \\ y \\ z \end{bmatrix} \cdot [x y z] A_j^T.$$

Here the linear forms $m_{ij}$ determine $A_j$. 
We need examples!

Like in the polynomial case (Hilbert, 1888 → Motzkin, 1967 → Reznick, 2007) we need lots of examples to understand the difference between the convex cones $P_{3,6}^M$ and $\Sigma_{3,6}^M$. Until recently, the only examples have been derived from Choi’s quadratic matrix:

$$\det \begin{bmatrix} x^2 + z^2 & -xy & -xz \\ -xy & x^2 + y^2 & -yz \\ -xz & -yx & y^2 + z^2 \end{bmatrix} = x^4 y^2 + y^4 z^2 + z^4 x^2 - 3 x^2 y^2 z^2.$$
Theorem (Šivic)

The map \( P_t : \text{Sym}_3 \rightarrow \text{Sym}_3 \) defined by

\[
\begin{bmatrix}
z_0 & z_1 & z_3 \\
z_1 & z_2 & z_4 \\
z_3 & z_4 & z_5
\end{bmatrix}
\mapsto
\begin{bmatrix}
(t^2 - 1)^2 z_0 + z_2 + t^4 z_5 & -(t^4 - t^2 + 1)z_1 & -(t^4 - t^2 + 1)z_3 \\
-(t^4 - t^2 + 1)z_1 & t^4 z_0 + (t^2 - 1)^2 z_1 + z_5 & -(t^4 - t^2 + 1)z_4 \\
-(t^4 - t^2 + 1)z_3 & -(t^4 - t^2 + 1)z_4 & z_0 + t^4 z_2 + (t^2 - 1)^2 z_5
\end{bmatrix}
\]

is positive for all \( t \in \mathbb{R} \). When \( t \notin \{1, 0, -1\} \), the associated biquadratic form \( u^T P_t(xx^T)u \) has 10 zeros:

\[
\{[1,1,1; 1,1,1], [-1,1,1; -1,1,1], [1,-1,1; 1,-1,1], [1,1,-1; 1,1,-1], [1, \pm t; 0, \pm t, 1, 0], [0, 1, \pm t; 0, \pm t, 1], [\pm t, 0, 1; 1, 0, \pm t]\}.
\]
Nonnegative biquadratic form with 10 zeros ($\text{max}$!)

In particular, for

$$P_t(\mathbf{x}\mathbf{x}^T) = \begin{bmatrix}
(t^2-1)x^2+y^2+t^4z^2 & -(t^4-t^2+1)xy & -(t^4-t^2+1)xz \\
-(t^4-t^2+1)xy & t^4x^2+(t^2-1)^2y^2+z^2 & -(t^4-t^2+1)yz \\
-(t^4-t^2+1)xz & -(t^4-t^2+1)yz & x^2+t^4y^2+(t^2-1)^2z^2
\end{bmatrix}$$

$\det P_t(\mathbf{x}\mathbf{x}^T)$ is nonnegative with 10 singularities of type $A_1$. 
The polynomial
\[
p(z_0, \ldots, z_5) = \det P_t([z_{ij}])
\]
is hyperbolic with respect to $\text{Id}_3 \equiv (1, 0, 1, 0, 0, 1)$.

It is straightforward to verify that the univariate polynomial $p(t \text{Id} + xx^T)$ has no zero that is strictly positive.
"TO DO LIST"

- Find examples of non-negative polynomials that have NO positive semidefinite quadratic determinantal representation.
- This proves that \( \det : P^M_{3,6} \rightarrow P_{3,6} \) is not surjective.
- We believe that Robinson’s polynomial is such, due to its particular configuration of 10 zeros

\[ \{[1,1,1], [-1,1,1], [1,-1,1], [1,1,-1], [1, \pm 1, 0], [0, 1, \pm 1], [\pm 1, 0, 1]\}. \]
Find geometric explanation for the containment
\[ \Sigma^M_{3,6} \subset P^M_{3,6}. \]

Follow Blekherman’s explanation of the difference between the two cones in the polynomial case. The proof of Hilbert’s 17th theorem for matrices is more constructive than for polynomials (because of the Cayley-Hamilton theorem).

What are the Cayley-Bacharach relations for matrix polynomials?
Find algebraic boundaries $\partial P_{3,6}^M$ and $\partial \Sigma_{3,6}^M$.

We proved that $\partial P_{3,6}^M$ is the discriminant for biquadratic ternary forms. It is an irreducible hypersurface in $\mathbb{P}^{35}$ of degree 1328.

Recall that the non-discriminant boundary for $\Sigma_{3,6}$ consists of polynomials that are sums of three squares. Our “guess” is that the non-discriminant boundary $\partial \Sigma_{3,6}^M = \left\{ \sum_{j=1}^{5} A_j X A_j^T \right\}$: Take $P \in \Sigma_{3,6}^M$ that is a sum of 4 squares. This means that $P = M M^T$ for a linear $3 \times 4$ matrix $M$. By the Cauchy-Binet formula $\det P = \det M_{123}^2 + \det M_{124}^2 + \det M_{134}^2 + \det M_{234}^2$. Therefore the set of real zeros equals to the determinantal variety $\text{rank} M \leq 2$ which consists of 6 points.
“TO DO LIST”

- Prove that that \( \det : P^M_{3,6} \rightarrow P_{3,6} \) is not a convex map.

Clearly, determinant of an SOS quadratic matrix is an SOS sextic polynomial. On the other hand, Quarez’s example

\[
\begin{bmatrix}
 x^2 + z^2 & 0 & -xz \\
 0 & x^2 + y^2 & -yz \\
 -xz & -yx & y^2 + z^2
\end{bmatrix}
\]

is a positive semidefinite quadratic matrix that is not SOS, but its determinant is an SOS sextic polynomial.
Find an example of a ternary quartic whose Hessian is positive semidefinite but not an SOS matrix.

This is another problem from optimization. Namely, a multivariate polynomial is convex if and only if its Hessian matrix of second partial derivatives is positive semidefinite. Ahmadi and Parrilo (2013) described the difference between convexity and SOS-convexity of polynomials.


