Positive semidefinite quadratic determinantal representations

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1. Symmetric quadratic determinantal representations
   - Positive maps
   - Self-dual sheaves

2. Linear matrix inequalities
   - Spectrahedral cone $\leftrightarrow$ hyperbolic polynomial
   - Cubic symmetroid
   - Criteria for $P$ to be positive

3. Polynomial nonnegativity
   - PSD and SOS polynomials
   - PSD and SOS matrices
   - Šivic biquadratic form
Warm up question

Given a homogeneous nonnegative polynomial $p(x, y, z)$ of degree 6, does there exist a positive linear map $P : \text{Sym}_3 \rightarrow \text{Sym}_3$ such that

$$\det P(xx^T) = p(x, y, z) \text{ for all } x = [x, y, z]^T?$$

We will “tackle” this question from three sides:

- symmetric quadratic determinantal representations and the associated sheaves (kernels);
- semidefinite linear determinantal representations (LMI representations of hyperbolic polynomials);
- polynomial algebra (SOS and PSD polynomials and matrices).
Definition

A linear map \( P : \text{Sym}_3 \rightarrow \text{Sym}_3 \) is positive if it sends positive semidefinite matrices to positive semidefinite matrices.

Positive maps were popular in the 70s as they describe various quantum states in quantum physics. In the last decade there is again very active and fertile research in this area due to its connection to optimization.
Positive maps

**Definition**

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Positive maps were popular in the 70s as they describe various quantum states in quantum physics. In the last decade there is again very active and fertile research in this area due to its connection to optimization.
Clearly it is enough to check the positivity of $P$ on rank 1 matrices. In coordinates our question then becomes

**Question**

> Given a nonnegative plane sextic $C$ in $\mathbb{P}^2$, does there exist a symmetric quadratic determinantal representation of $C$ which is semidefinite for all $(x, y, z) \in \mathbb{P}^2$?

Indeed, $P = \begin{bmatrix} x^2 & xy & xz \\ xy & y^2 & yz \\ xz & yz & z^2 \end{bmatrix} = \begin{bmatrix} p_0 & p_1 & p_3 \\ p_1 & p_2 & p_4 \\ p_3 & p_4 & p_5 \end{bmatrix},$

where $p_i = p_{0i}x^2 + p_{1i}xy + p_{2i}x^2 + p_{3i}xz + p_{4i}yz + p_{5i}z^2.$
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Indeed, $P \begin{bmatrix} x^2 & xy & xz \\ xy & y^2 & yz \\ xz & yz & z^2 \end{bmatrix} = \begin{bmatrix} p_0 & p_1 & p_3 \\ p_1 & p_2 & p_4 \\ p_3 & p_4 & p_5 \end{bmatrix}$,

where $p_i = p_{0i}x^2 + p_{1i}xy + p_{2i}x^2 + p_{3i}xz + p_{4i}yz + p_{5i}z^2$. 
C. Scheiderer. *Hilbert’s theorem on positive ternary quartics: A refined analysis*, JAG, 2010

This is exactly the question Scheiderer asked and thoroughly answered in the case of **plane quartics**.

- Quadratic determinantal representations
  \[
  \begin{pmatrix}
  p_0 & p_1 \\
  p_1 & p_2
  \end{pmatrix}
  \]
  globally generated (i.e., non-exceptional) ACM rank 1 sheaves with selfduality \( \mathcal{F} \cong \mathcal{H}om(\mathcal{F}, \mathcal{O}_C(2)) \). The number of such representations depends only on the singularities of \( C \).

- Determine which and how many of the above determinantal representations are semidefinite.
Without the nonnegativity and semidefiniteness conditions this is a classical case of determinantal hypersurfaces

**Question**

*Given a plane curve $C$ of degree $2d$, does there exist a $d \times d$ symmetric quadratic determinantal representation of $C$?*
On an integral curve $C$, a coherent torsion-free rank 1 (arithmetically Cohen-Macaulay, ACM) sheaf $\mathcal{F}$ that is generated by its global sections and $\mathcal{F} \cong \mathcal{H}om(\mathcal{F}, \mathcal{O}_C(2d-2))$ admits a resolution

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^2}(-2)^d \xrightarrow{M} \mathcal{O}_{\mathbb{P}^2}^d \rightarrow \mathcal{F} \rightarrow 0,$$

where $M$ is a symmetric quadratic matrix with $\det M = p$.

**Remark:** The above $\mathcal{F}$ is non-exceptional. This is equivalent to $H^0(C, \mathcal{F}(-1)) = H^1(C, \mathcal{F}) = 0$. Then $h^0(C, \mathcal{F}) = d$ and its global sections yield $M$. 
Actually, any $\mathcal{F}$ that is self-dual

$$\mathcal{F} \cong \mathcal{H}om(\mathcal{F}, \mathcal{O}_C(2d - 2))$$

admits a resolution

$$0 \to \bigoplus_{i=1}^{l} \mathcal{O}_{\mathbb{P}^2}(-2 - d_i) \xrightarrow{M} \bigoplus_{i=1}^{l} \mathcal{O}_{\mathbb{P}^2}(d_i) \to \mathcal{F} \to 0,$$

where $M = [m_{ij}]$ is symmetric with $m_{ij}$ of degree $d_i + d_j - 2$.

**Remark:** We are only interested in non-exceptional $\mathcal{F}$, for which $d_i = 0$ for $i = 1, \ldots, d$. The set of such pairs $(C, \mathcal{F})$ is Zariski dense in the universal Jacobian $\mathcal{J}_{2d}^{2d(d-1)}$. 
Define the moduli space $R_{2d}$ of pairs $(C, \alpha)$, where $C$ is a smooth plane curve of degree $2d$ (over a field of char 0), and $\alpha$ is a half-period, i.e. a 2-torsion divisor on $\text{Jac}(C)$, i.e. a nontrivial line bundle on $C$ satisfying $\alpha \otimes 2 \cong \mathcal{O}_C$.

**Proposition**

For $(C, \alpha)$ general in $R_{2d}$, the half-period $\alpha$ admits a minimal resolution

$$0 \to \mathcal{O}_{P^2}(-d-1)^d \xrightarrow{M} \mathcal{O}_{P^2}(-d+1)^d \to \alpha \to 0,$$

where $M$ is a symmetric quadratic matrix with $\det M = p$.

Note, $\mathcal{F}$ is obtained from the half-period $\alpha$ by $\mathcal{F} = \alpha \otimes \mathcal{O}_C(d-1)$. 
Simple singularities

When $C/\mathbb{C}$ has only simple (this means AED) singularities, there are finitely many ACM sheaves with the following self-duality

$$\mathcal{F} \cong \mathcal{H}om(\mathcal{F}, \mathcal{O}_C(2d - 2)).$$

It is possible to explicitly count them using methods in [Piontkowski, 2007]. Their number depends on the genus of the curve and the local type of $\mathcal{F}$: $(\mathcal{F}_s)_{s \in \text{Sing } C}$ is a collection of self-dual modules $\mathcal{F}_s \cong \text{Hom}(\mathcal{F}_s, \mathcal{O}_{C,s}(2d - 2))$. For a simple singularity there are only finitely many isomorphism classes of indecomposable torsion-free modules over its local ring.

A smooth $C$ has $2^{2g}$ self-dual $\mathcal{F}$; the number decreases rapidly with the number and order of singularities $A_n, D_m, E_l$. When all $\mathcal{F}$ are exceptional, $C$ has no quadratic representations.
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\( \mathcal{F} \cong \text{Hom}(\mathcal{F}, \mathcal{O}_C(2d - 2)) \)

Minimal resolutions \( M = [m_{ij}] \) with \( \text{deg } m_{ij} = d_i + d_j + 2 \):

\( \mathcal{F} \) non-except.

quartic:
\[
\begin{bmatrix}
2 & 2 \\
2 & 2 \\
2 & 2 \\
2 & 2
\end{bmatrix}
\]

\( \mathcal{F} \) exceptional

\( \text{quadric SINGULAR} \):
\[
\begin{bmatrix}
2 & 2 & 1 \\
2 & 2 & 1 \\
1 & 1 & 0 \\
1 & 1 & 0
\end{bmatrix}
\]

\( \text{square} \):
\[
\begin{bmatrix}
2 & 2 & 1 & 1 \\
2 & 2 & 1 & 1 \\
1 & 1 & 0 & 0 \\
1 & 1 & 0 & 0
\end{bmatrix}
\]

sextic:
\[
\begin{bmatrix}
2 & 2 & 2 \\
2 & 2 & 2 \\
2 & 2 & 2 \\
2 & 2 & 2 \\
1 & 1 & 1 \\
1 & 1 & 1
\end{bmatrix}
\]

\( \text{quadric SQUARE} \):
\[
\begin{bmatrix}
2 & 2 & 2 & 1 \\
2 & 2 & 2 & 1 \\
2 & 2 & 2 & 1 \\
2 & 2 & 2 & 1 \\
1 & 1 & 1 & 0 \\
1 & 1 & 1 & 0
\end{bmatrix}
\]

\( \text{square} \):
\[
\begin{bmatrix}
2 & 2 & 2 & 1 & 1 \\
2 & 2 & 2 & 1 & 1 \\
2 & 2 & 2 & 1 & 1 \\
1 & 1 & 1 & 0 & 0 \\
1 & 1 & 1 & 0 & 0
\end{bmatrix}
\]
Terminology $\mathbb{A}^n \leftrightarrow \mathbb{P}^n$

$\mathbb{A}^n$: spectrahedron
RZ-polynomial
rigidly convex set
PSD
LMI

$\mathbb{P}^n$: spectrahedral cone
hyperbolic poly.
hyperbolicity cone
SD
LMI

A. Buckley
Positive semidefinite quadratic determinantal representations
Cubic curve \((x_0 - x_1)(x_0 + x_1)(x_0 - 4x_1) - x_1x_2^2 = 0\)
Cubic curve \((x_0 - x_1)(x_0 + x_1)(x_0 - 4x_1) - x_1x_2^2 = 0\)

has three symmetric determinantal representations:

- two are definite determinantal representations

\[
\begin{bmatrix}
  x_0 & -\frac{x_1}{\sqrt{2}} & \frac{x_1}{\sqrt{2}} \\
  -\frac{x_1}{\sqrt{2}} & \frac{8x_0-x_1+4x_2}{8} & -\frac{x_1}{8} \\
  \frac{x_1}{\sqrt{2}} & -\frac{x_1}{8} & \frac{8x_0-x_1-4x_2}{8}
\end{bmatrix},
\begin{bmatrix}
  x_0 & -\frac{x_1}{2\sqrt{2}} & \frac{x_1}{2\sqrt{2}} \\
  -\frac{x_1}{2\sqrt{2}} & \frac{8x_0-x_1+4x_2}{8} & -\frac{7x_1}{8} \\
  \frac{x_1}{2\sqrt{2}} & -\frac{7x_1}{8} & \frac{8x_0-x_1-4x_2}{8}
\end{bmatrix}
\]

- one is nondefinite

\[
x_0 \text{Id} + x_2 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & -1/2 \end{bmatrix} + x_1 \begin{bmatrix} 0 & \frac{i}{2\sqrt{2}} & -\frac{i}{2\sqrt{2}} \\ \frac{i}{2\sqrt{2}} & -\frac{1}{8} & \frac{2\sqrt{2}}{9} \\ -\frac{i}{2\sqrt{2}} & \frac{9}{8} & -\frac{1}{8} \end{bmatrix}
\]
In coordinates write $\mathbb{P}^5$ as a symmetric matrix

$$
\begin{bmatrix}
Z_0 & Z_1 & Z_3 \\
Z_1 & Z_2 & Z_4 \\
Z_3 & Z_4 & Z_5
\end{bmatrix},
$$

and consider the Veronese embedding

$$
\nu_2 : \mathbb{P}^2 \longrightarrow \mathbb{P}^5
$$

$$(x, y, z) \mapsto \begin{bmatrix}
x^2 & xy & xz \\
xy & y^2 & yz \\
xz & yz & z^2
\end{bmatrix}.$$
Cubic symmetroid in $\mathbb{P}^5$ is the hypersurface defined by

$$\det \begin{bmatrix} z_0 & z_1 & z_3 \\ z_1 & z_2 & z_4 \\ z_3 & z_4 & z_5 \end{bmatrix} = 0.$$ 

It is singular along the Veronese surface. Semidefinite matrices lie in the spectrahedral cone

$$\left\{ (z_0, \ldots, z_5) \in \mathbb{P}^5 : \begin{bmatrix} z_0 & z_1 & z_3 \\ z_1 & z_2 & z_4 \\ z_3 & z_4 & z_5 \end{bmatrix} \succeq 0 \right\}.$$
Cubic symmetroid as LMI
Cubic symmetroid as LMI

“Being” a spectrahedral cone is much more than a convex cone!

SD matrix ↔ symmetroid

rk 1 ↔ points on the Veronese
rk 2 ↔ ∂ spectrahedral cone
\[ \lambda a a^T + (1 - \lambda) b b^T, \lambda \in [0, 1] \]
rk ≤ 3 ↔ spectrahedral cone
\[ \alpha a a^T + \beta b b^T + \gamma c c^T, \alpha, \beta, \gamma \geq 0 \]
Hyperplane sections of symmetroid

The Veronese map $\nu_2$ is given by the complete linear system. Thus the preimage of a hyperplane $H$ in $\mathbb{P}^5$ is a conic $Q_H$ in $\mathbb{P}^2$.

Conic $Q_H$ is singular $\iff$ $H$ is tangent to the Veronese surface

More precisely,

- if $Q_H$ is a line pair, then $H$ is a tangent to the Veronese surface at a single point,
- if $Q_H$ is a double line, then $H$ is tangent to the Veronese surface along the curve that is the image of $Q_H^{\text{red}}$ under the restriction of $\nu_2$. 
Recall that $P : \text{Sym}_3 \to \text{Sym}_3$ is positive if and only if

$$
\begin{bmatrix}
\ p_0 & p_1 & p_3 \\
\ p_1 & p_2 & p_4 \\
\ p_3 & p_4 & p_5 \\
\end{bmatrix} \succeq 0 \text{ for all } \mathbf{x} = (x, y, z) \in \mathbb{P}^2,
$$

where $p_i = p_{0i}x^2 + p_{1i}xy + p_{2i}y^2 + p_{3i}xz + p_{4i}yz + p_{5i}z^2$

and $P = [p_{ij}]_{0 \leq i, j \leq 5} : \mathbb{P}^5 \to \mathbb{P}^5$ in the corresponding basis.
This way $P : \text{Sym}_3 \rightarrow \text{Sym}_3$ induces

$$\nu_P : \mathbb{P}^2 \rightarrow \mathbb{P}^5 \quad (x, y, z) \mapsto (p_0, \ldots, p_5),$$

**Lemma**

$P$ is positive if and only if the image of $\nu_P$ lies in the spectrahedral cone of the cubic symmetroid.

Therefore, classifying positive maps is the same as classifying linear maps $P : \mathbb{P}^5 \rightarrow \mathbb{P}^5$ that preserve the spectrahedral cone of the cubic symmetroid.
The convex hull of $\text{Im}(\nu_P)$

When $P$ is invertible, $\nu_P$ is given by the complete linear system. In this case the image of $\nu_P$ is the singular locus of the hypersurface

$$\det P^{-1}\begin{pmatrix} x_0 & x_1 & x_3 \\ x_1 & x_2 & x_4 \\ x_3 & x_4 & x_5 \end{pmatrix} = 0.$$

In other words, the convex hull of $\text{Im} \nu_P$ equals to the spectrahedral cone of the above hypersurface. Else, the convex hull of $\text{Im} (\nu_P)$ is a projection of a spectrahedral cone shadow (inside the spectrahedral cone of the symmetroid).
The convex hull of $\text{Im} \left( \nu_{P^{-1}} \right)$

On the other side, the hypersurface

$$
\det P \begin{pmatrix}
Z_0 & Z_1 & Z_3 \\
Z_1 & Z_2 & Z_4 \\
Z_3 & Z_4 & Z_5
\end{pmatrix} = 0
$$

contains the Veronese surface in its spectrahedral cone. When $P$ is invertible, this spectrahedral cone equals to the convex hull of the image of

$$
\nu_{P^{-1}} : \mathbb{P}^2 \rightarrow \mathbb{P}^5
$$

$$(x, y, z) \mapsto P^{-1} \begin{pmatrix}
x^2 & xy & xz \\
xy & y^2 & yz \\
xz & yz & z^2
\end{pmatrix}
$$
Spectrahedral cones of:
\[
\det P([z_{ij}]) = 0, \quad \det[z_{ij}] = 0, \quad \det P^{-1}([z_{ij}]) = 0
\]
Choi’s example

Matrix

\[
\begin{bmatrix}
x^2 + z^2 & -xy & -xz \\
-xy & x^2 + y^2 & -yz \\
-xz & -yz & y^2 + z^2
\end{bmatrix}
\]

is positive definite for all \((x, y, z) \in \mathbb{P}^2\) except at the 7 points: 
\((1, 1, 1), (-1, 1, 1), (1, -1, 1), (1, 1, -1), (1, 0, 0), (0, 1, 0), (0, 0, 1)\). The Veronese surface therefore lies inside the spectrahedral cone of

\[
\det \begin{bmatrix}
z_0 + z_5 & -z_1 & -z_3 \\
-z_1 & z_0 + z_2 & -z_4 \\
-z_3 & -z_4 & z_2 + z_5
\end{bmatrix} = 0
\]

and intersects its boundary in

\((1,1,1,1,1,1), (1,-1,1,-1,1,1), (1,-1,1,1,-1,1), (1,1,1,1,1,1), (1,0,0,0,0,0), (0,0,1,0,0,0), (0,0,0,0,0,1)\)
Choi’s example and the cubic symmetroid

\[ \det P([z_{ij}]) = 0, \quad \det[z_{ij}] = 0; \]

intersected with \( z_0 = z_1 = z_2 = 1 \), thus containing
\( (1, 1, 1, 1, 1), (1, -1, 1, -1, 1), (1, -1, 1, 1, -1), (1, 1, -1, 1, -1) \):
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Choi’s example and the cubic symmetroid

$\det P([z_{ij}]) = 0$, $\det[z_{ij}] = 0$;

intersected with $z_3 = z_4 = z_5 = 1 - z_0 - z_2$, thus containing $(1, 1, 1, 1, 1, 1), (1, 0, 0, 0, 0, 0), (0, 0, 1, 0, 0, 0)$. 

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Choi's example and the cubic symmetroid
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intersected with \( z_3 = z_4 = z_5 = 1 - z_0 - z_2 \), thus containing
\((1, 1, 1, 1, 1, 1), (1, 0, 0, 0, 0, 0), (0, 0, 1, 0, 0, 0)\).
Convex algebraic geometry

\[ P_{n,2d} = \{ \text{non-negative (PSD) forms in } \mathbb{R}[x_0, \ldots, x_{n-1}] \text{ of degree } 2d \} \]

\[ \cup \]

\[ \Sigma_{n,2d} = \{ \text{sums of squares (SOS-polynomials)} \} \]

\[ \det \uparrow \]

\[ P^M_{n,2d} = \{ \text{positive semidefinite } d \times d \text{ matrix quadratic polynomials} \} \]

\[ \cup \]

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\[ \begin{align*}
\det & \uparrow \\
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\[ \det \uparrow \]

\[ P_{n,2d}^M = \{ \text{positive semidefinite } d \times d \text{ matrix quadratic polynomials} \} \]
\[ \cup \]
\[ \Sigma_{n,2d}^M = \{ \text{SOS-matrix quadratic polynomials} \} \]
Convex cone: \( p, q \in C \Rightarrow \lambda p + \mu q \in C \) for all \( \lambda, \mu > 0 \)

\[
P_{n,2d} = \{ \text{PSD polynomials of degree } 2d \}
\]

\[
\Sigma_{n,2d} = \{ \text{SOS polynomials of degree } 2d \}
\]

are both convex cones in \( \mathbb{R}^N \) where \( N = \binom{n+2d-1}{2d} \).

We know since Hilbert that

\[
\Sigma_{n,2d} \subset P_{n,2d};
\]

- testing if \( p \in P_{n,2d} \) is NP-hard,
- but testing if \( p \in \Sigma_{n,2d} \) is an SDP (using **Gram matrix**).
Hilbert’s 17th problem

Hilbert in 1888 showed that $\Sigma_{n,2d} = P_{n,2d}$ in the following cases:

<table>
<thead>
<tr>
<th>$n \backslash 2d$</th>
<th>2</th>
<th>4</th>
<th>6</th>
<th>8</th>
<th>$\cdots$</th>
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<tbody>
<tr>
<td>2</td>
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- $2d = 2$, quadratic polynomial forms
- $n = 2$, homogeneous polynomials in two variables
- $2d = 4$, $n = 3$, quartic forms in three variables
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Hilbert in 1888 showed that $\sum_{n,2d} = P_{n,2d}$ in the following cases:

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Hilbert’s 17th problem

Artin in 1927 showed that every PSD polynomial is an SOS of rational functions.

A constructive solution was found in 1984 by Delzell.
In 1967 Motzkin constructed the first example of a positive semidefinite polynomial, that is not a sum of squares:

\[ p(x, y, z) = x^2 y^4 + x^4 y^2 + z^6 - 3x^2 y^2 z^2 \]
Blekherman in 2012 provided a geometric explanation for the containment $\Sigma_{3,6} \subset P_{3,6}$. The difference lies in fulfillment of certain linear relations (Cayley-Bacharach relations) from Hilbert’s proof.

- Robinson’s polynomial with 10 zeros (1973):

  $$x^6 + y^6 + z^6 - x^4y^2 - x^4z^2 - y^4x^2 - y^4z^2 - z^4x^2 - z^4y^2 + 3x^2y^2z^2;$$

- lots of examples from Reznick’s construction (2007).
The geometry of $P_{3,6} \setminus \Sigma_{3,6}$ remains puzzling!

An **algebraic boundary** of a cone is the hypersurface that arises as Zariski closure of its topological boundary.

- Nie, 2011: The algebraic boundary of the cone $P_{n,2d}$ is the **discriminant** of degree $n(2d - 1)^{n-1}$.
- Blekherman, Sturmfels, et al., 2011: Discriminant is also a component in the algebraic boundary of $\Sigma_{3,6}$. Besides, $\partial \Sigma_{3,6}$ has another unique non-discriminant component of degree 83200 which consists of forms that are **sums of three squares of cubics**.

**Remark:** A sextic $C$ that is a sum of three squares of cubics coincides with an ACM rk 1 sheaf $\mathcal{F} \cong \mathcal{H}om(\mathcal{F}, \mathcal{O}_C(3))$ that is globally generated; this is exactly an effective even theta characteristic.
Definition

A symmetric polynomial matrix $P(x)$ is an **SOS-matrix** if

$$P(x) = M(x) M(x)^T$$

for a possibly non-square polynomial matrix $M(x)$.

Definition

A matrix polynomial $P(x)$ is **positive semidefinite** if $P(x)$ is positive semidefinite for all $x = (x_0, \ldots, x_{n-1}) \in \mathbb{R}^n$. 
Recall the natural $1 \to 1$ correspondence between:

- positive linear maps $P : \text{Sym}_3 \to \text{Sym}_3$;
- PSD quadratic ternary matrices $P(\mathbf{x} \mathbf{x}^T)$;
- non-negative biquadratic forms $\mathbf{u}^T P(\mathbf{x} \mathbf{x}^T) \mathbf{u}$, where $\mathbf{x} = [x, y, z]^T$ and $\mathbf{u} = [u, v, w]^T$.

**Lemma**

$P$ is positive $\iff$ $\mathbf{u}^T P(\mathbf{x} \mathbf{x}^T) \mathbf{u}$ is a PSD polynomial

$\iff$ $P(\mathbf{x} \mathbf{x}^T)$ is a PSD quadratic matrix.
Choi matrix (analogy to the Gram matrix for SOS polynomials)

**Choi map:** A linear map $\phi : M_3 \rightarrow M_3$ induces a linear map $\Phi : M_9 \rightarrow M_9$ by the following rule

$\Phi \left( \begin{bmatrix} X_{ij} \end{bmatrix}_{i,j=1,2,3} \right) = \begin{bmatrix} \phi(X_{ij}) \end{bmatrix}_{i,j=1,2,3}$.

**Theorem (Choi, 1974)**

*Choi matrix*

$[\phi(E_{ij})]_{i,j=1,2,3}$ is positive semidefinite if and only if the restriction $\phi : \text{Sym}_3 \rightarrow \text{Sym}_3$ induces an SOS quadratic matrix $\phi(\mathbf{x}\mathbf{x}^T)$.

This is equivalent to $\mathbf{u}^T P(\mathbf{x}\mathbf{x}^T) \mathbf{u}$ being a biquadratic SOS form.

Such $\phi$ are called completely positive, in optimization they are called SOS.
The third equivalent definition of quadratic SOS matrices is the following:

**Lemma**

*Quadratic matrix* $P(xx^T)$ *is an SOS matrix if and only if there exist* $A_j \in \mathbb{R}^{3,3}$ *such that*

$$P(x, y, z) = \sum_{j=1}^{r} A_j X A_j^T, \text{ where } X = xx^T = \begin{bmatrix} x^2 & xy & xz \\ xy & y^2 & yz \\ xz & yz & z^2 \end{bmatrix}.$$
Indeed, for the $3 \times r$ linear matrix $M = [m_1 \cdots m_r]$ write

$$P(x, y, z) = MM^T = \sum_{j=1}^{r} m_j m_j^T = \sum_{j=1}^{r} \begin{bmatrix} m_{1j} \\ m_{2j} \\ m_{3j} \end{bmatrix} \cdot \begin{bmatrix} m_{1j} \\ m_{2j} \\ m_{3j} \end{bmatrix}$$

$$= \sum_{j=1}^{r} A_j \begin{bmatrix} x \\ y \\ z \end{bmatrix} \cdot \begin{bmatrix} x & y & z \end{bmatrix} A_j^T.$$  

Here the linear forms $m_{ij}$ determine $A_j$. 

A. Buckley
Positive semidefinite quadratic determinantal representations
We need examples!

Like in the polynomial case, Hilbert, 1888 → Motzkin, 1967 → Reznick, 2007, we need lots of examples to understand the difference between the convex cones $P^M_{3,6}$ and $\Sigma^M_{3,6}$. 
We need examples!

Until recently, the only examples have been derived from **Choi’s quadratic matrix**:

\[
\begin{vmatrix}
  x^2 + z^2 & -xy & -xz \\
  -xy & x^2 + y^2 & -yz \\
  -xz & -yx & y^2 + z^2
\end{vmatrix} = x^4 y^2 + y^4 z^2 + z^4 x^2 - 3 x^2 y^2 z^2.
\]

The corresponding biquadratic form has 7 zeros:

\((1, 1, 1; 1, 1, 1), (-1, 1, 1;-1, 1, 1), (1,-1, 1; 1,-1, 1), (1, 1,-1; 1, 1,-1), (1, 0, 0; 0, 0, 1), (0, 1, 0; 1, 0, 0), (0, 0, 1; 0, 1, 0).\)
Nonnegative biquadratic form with 10 zeros (max!)

Theorem (Šivic)

The map $P_t : \text{Sym}_3 \rightarrow \text{Sym}_3$ defined by

$$
\begin{bmatrix}
z_0 & z_1 & z_3 \\
z_1 & z_2 & z_4 \\
z_3 & z_4 & z_5
\end{bmatrix} \mapsto
\begin{bmatrix}
(t^2-1)^2 z_0 + z_2 + t^4 z_5 & -(t^4 - t^2 + 1) z_1 & -(t^4 - t^2 + 1) z_3 \\
-(t^4 - t^2 + 1) z_1 & t^4 z_0 + (t^2 - 1)^2 z_1 + z_5 & -(t^4 - t^2 + 1) z_4 \\
-(t^4 - t^2 + 1) z_3 & -(t^4 - t^2 + 1) z_4 & z_0 + t^4 z_2 + (t^2 - 1)^2 z_5
\end{bmatrix}
$$

is positive for all $t \in \mathbb{R}$. When $t \notin \{1, 0, -1\}$, the associated biquadratic form $\mathbf{u}^T P_t(\mathbf{x} \mathbf{x}^T) \mathbf{u}$ has 10 zeros:

$$
\{[1, 1, 1; 1, 1, 1], [-1, 1, 1; -1, 1, 1], [1, -1, 1; 1, -1, 1], [1, 1, -1; 1, 1, -1], \\
[1, \pm t, 0; \pm t, 1, 0], [0, 1, \pm t; 0, \pm t, 1], [\pm t, 0, 1; 1, 0, \pm t]\}.
$$
Nonnegative biquadratic form with 10 zeros (max!)

In particular, for

\[
P_t(\mathbf{x} \mathbf{x}^T) = \begin{bmatrix}
(t^2 - 1)^2 x^2 + y^2 + t^4 z^2 & -(t^4 - t^2 + 1)xy & -(t^4 - t^2 + 1)xz \\
-(t^4 - t^2 + 1)xy & t^4 x^2 + (t^2 - 1)^2 y^2 + z^2 & -(t^4 - t^2 + 1)yz \\
-(t^4 - t^2 + 1)xz & -(t^4 - t^2 + 1)yz & x^2 + t^4 y^2 + (t^2 - 1)^2 z^2
\end{bmatrix}
\]

\[
\det P_t(\mathbf{x} \mathbf{x}^T)/(t^2 - 1)^2 = t^4(x^6 + y^6 + z^6) + \\
(t^8 - 2t^6)(x^4 y^2 + y^4 z^2 + z^4 x^2) + (1 - 2t^6)(x^2 y^4 + y^2 z^4 + z^2 x^4) - 3(t^8 - 2t^6 + t^4 - 2t^2 + 1)x^2 y^2 z^2
\]

is the generalized Robinson’s polynomial with 10 singularities of type $A_1$. 
Extremal nonnegative biquadratic forms

Our example is a parametrization of the **extremal** PSD quadratic matrices in the family:

\[
P_{a,b,c}(x x^T) = \begin{bmatrix}
(-1+a)x^2 + by^2 + cz^2 & -xy & -xz \\
-xy & cx^2 + (-1+a)y^2 + bz^2 & -yz \\
-xz & -yz & bx^2 + cy^2 + (-1+a)z^2
\end{bmatrix}.
\]

Cho, Kye and Lee (Generalized Choi maps, LAA 1992) proved that \( P_{a,b,c} \) is positive if and only if:

\[
a \geq 1,
\]

\[
a + b + c \geq 3,
\]

\[
bc \geq (2 - a)^2 \text{ if } 1 \leq a \leq 2.
\]
The family of biquadratic forms with 8 zeros:

\[(1, 1, 1; 1, 1, 1), (-1, 1, 1; -1, 1, 1), (1, -1, 1; 1, -1, 1), (1, 1, -1; 1, 1, -1),
   (1, 0, 0; 0, 0, 1), (0, 1, 0; 1, 0, 0), (0, 0, 1; 0, 1, \mu), (0, \nu, 1; 0, 1, 0)\]

is given by a linear combination of

\[
\begin{bmatrix}
  x^2 + z^2 & -xy & -xz \\
  -xy & x^2 & 0 \\
  -xz & 0 & y^2
\end{bmatrix}
\begin{bmatrix}
  (\mu + \nu)^2 x^2 \\
  \mu (\mu + \nu) x (-y + \nu z) \\
  -\nu (\mu + \nu) x (\mu y + z)
\end{bmatrix}
\begin{bmatrix}
  \mu (\mu + \nu) x (-y + \nu z) \\
  \mu^2 (y - \nu z)^2 \\
  -\nu (\mu + \nu) x (\mu y + z)
\end{bmatrix}
\begin{bmatrix}
  -\nu (\mu + \nu) x (\mu y + z) \\
  \mu \nu (\mu y + z) (y - \nu z) \\
  \nu^2 (\mu y + z)^2
\end{bmatrix}
\]
Nonnegative biquadratic form with 8 zeros

\[ \text{sph}[\theta_1, \phi_1, \theta_2, \phi_2] := \{ \cos[\theta_1] \cos[\phi_1], \cos[\theta_1] \sin[\phi_1], \sin[\theta_1], \cos[\theta_2] \cos[\phi_2], \cos[\theta_2] \sin[\phi_2], \sin[\theta_2] \} \]

\[ \text{biq8pt}[x, y, z, u, v, w] := \{u, v, w\} \cdot P_{a, \mu, \nu}[x, y, z] \cdot \{u, v, w\} \geq 0 \]

RegionPlot3D[ Apply[And, Map[biq8pt, Map[sph, RandomReal[2\[Pi], \{9000, 4\}]])], \{\mu, -2, 2\}, \{\nu, -2, 2\}, \{a, 0, 1/2\} ]
It is easy to check that for $\mu = -1/3$ and $\nu = 1/2$ the extremal PSD quadratic matrix is obtained at $a = 1/18$:

$$
\begin{bmatrix}
3/2x^2+z^2 & -1/2xz & 1/2x(y-5z) \\
-1/2xz & x^2+1/2(z-2y)^2 & 1/2(y-3z)(2y-z) \\
1/2x(y-5z) & 1/2(y-3z)(2y-z) & y^2+1/2(y-3z)^2
\end{bmatrix}.
$$

The associated nonnegative biquadratic form is also extremal with zeros:

$$(1,1,1;1,1,1), (-1,1,1;-1,1,1), (1,-1,1;1,-1,1), (1,1,-1;1,1,-1), (1,0,0;0,0,1), (0,1,0;1,0,0), (0,0,1;0,1,-1/3), (0,1/2,1;0,1,0)$$
Nonnegative biquadratic form with 9 zeros

Positive map $P =$

$$
\begin{bmatrix}
((3+2\sqrt{2})z_0+(3-2\sqrt{2})z_2+2z_5)/4 & -z_1 & -z_3 \\
-z_1 & (z_1+z_2)/2 & 0 \\
-z_3 & 0 & ((3-2\sqrt{2})z_0+(-1+2\sqrt{2})z_2+2z_5)/4
\end{bmatrix}
$$

induces an extremal nonnegative biquadratic form $u^T P (xx^T) u$

with zeros:

$(1, 1, 1; 1, 1, 1)$, $(-1, 1, 1; -1, 1, 1)$, $(1, -1, 1; 1, -1, 1)$, $(1, 1, -1; 1, 1, -1)$, $(1, 0, -1/\sqrt{2}; 1, 1, 0)$, $(1, 0, 1/\sqrt{2}; 1, -1, 0)$, $(1-\sqrt{2}, 1, 0; 1, 1, 0)$, $(\sqrt{2}-1, 1, 0; 1, 1, 0)$, $(0, 0, 1; 0, 1, 0)$. 
"TO DO LIST"

Find examples of non-negative polynomials that have **no**
PSD quadratic determinantal representation.
This would prove that $\det : P_{3,6}^M \rightarrow P_{3,6}$ is not surjective.

We believe that Robinson’s polynomial is such, due to the
particular configuration of its 10 zeros
$\{[1,1,1], [-1,1,1], [1,-1,1], [1,1,-1], [1,\pm 1, 0], [0,1,\pm 1], [\pm 1, 0, 1]\}$.

What about Motzkin polynomial?
Understand the map \( \text{det} : P_{3,6}^M \rightarrow P_{3,6} \).

Clearly, determinant of an SOS quadratic matrix is an SOS sextic polynomial. On the other hand, Quarez’s example

\[
\begin{bmatrix}
  x^2 + z^2 & 0 & -xz \\
  0 & x^2 + y^2 & -yz \\
  -xz & -yx & y^2 + z^2
\end{bmatrix}
\]

is a positive semidefinite quadratic matrix that is not SOS, but its determinant is an SOS sextic polynomial.
“TO DO LIST”

- Find geometric explanation for the containment 
  $\Sigma^M_{3,6} \subset P^M_{3,6}$.

Follow Blekherman’s explanation of the difference between the two cones in the polynomial case. The proof of Hilbert’s 17th theorem for matrices is more constructive than for polynomials (because of the Cayley-Hamilton theorem).

What are the Cayley-Bacharach relations for matrix polynomials?
Find algebraic boundaries $\partial P_{3,6}^M$ and $\partial \Sigma_{3,6}^M$.

We proved that $\partial P_{3,6}^M$ is the discriminant for biquadratic ternary forms. It is an irreducible hypersurface in $\mathbb{P}^{35}$ of degree 1328.

Recall that the non-discriminant boundary for $\Sigma_{3,6}$ consists of polynomials that are sums of three squares. Our “guess” is that the non-discriminant boundary $\partial \Sigma_{3,6}^M = \left\{ \sum_{j=1}^{5} A_j X A_j^T \right\}$: Take $P \in \Sigma_{3,6}^M$ that is a sum of 4 squares. This means that $P = M M^T$ for a linear $3 \times 4$ matrix $M$. By the Cauchy-Binet formula $\det P = \det M_{123}^2 + \det M_{124}^2 + \det M_{134}^2 + \det M_{234}^2$. Therefore the set of real zeros equals to the determinantal variety $\text{rank} M \leq 2$ which consists of 6 points.


