# COMPUTING DEDEKIND SUMS USING THE EUCLIDEAN ALGORITHM

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**Abstract**

This paper presents an algorithm for computing Dedekind sums $\sigma_i\big(\frac{1}{r}(a_1,\ldots,a_n)\big)$ by using the Euclidean algorithm and algebraic properties of $\sigma_i$. In Theorem 2.2 an explicit relation between a polynomial $\alpha$ with rational coefficients, which is obtained by the Euclidean algorithm, and $\sigma_0,\ \sigma_1,\ldots,\sigma_{r-1}$ will be determined. The main advantage of the algorithm is that by knowing $\alpha$ all $\sigma_i$ can be calculated simultaneously. When $a_1,\ldots,a_n$ are relatively prime to $r$, the algorithm can be put in a particularly simple and compact form. Several examples are explicitly computed by using `Mathematica`.

**Definition 0.1** Fix positive integers $r$ and $a_1,\ldots,a_n$. The sums $\sigma_i$ are defined by

$$\sigma_i=\frac{1}{r}\sum_{\substack{\varepsilon\in\mu_r\\ \varepsilon^{a_i}\neq 1\,\forall i=1,\ldots,n}}\frac{\varepsilon^i}{(1-\varepsilon^{a_1})\cdots(1-\varepsilon^{a_n})},\tag{1}$$

where $\varepsilon$ runs over $r$th roots of unity. It is obvious that $\sigma_i=\sigma_{r+i}$. Therefore we only need to consider $\sigma_i$ for $i=0,1,\ldots,r-1$ and call it the $i$th *Dedekind sum.* If we want to stress that not all $a_1,\ldots,a_n$ are relatively prime to $r$, then $\sigma_i$ is called the $i$th *generalised Dedekind sum.* When $a_1,\ldots,a_n$ are relatively prime to $r$ the above sum is taken over all $r$th roots of unity. Let

$$A=\prod_{i=1}^{n}(1-t^{a_i})\qquad\text{and}\qquad B=\frac{1-t^r}{1-t}.\tag{2}$$

Then by the Euclidean algorithm there exist polynomials $\alpha,\beta\in\mathbb{Q}(t)$ such that

$$1=\alpha A+\beta\frac{B}{\mathrm{hcf}\,(A,B)}.\tag{3}$$

This way obtained $\alpha$ is called *the Inverse of* $\prod_{i=1}^{n}(1-t^{a_i})$ *modulo* $\frac{1-t^r}{1-t}=1+t+\cdots+t^{r-1}$.

**Keywords: Dedekind sums, Euclidean algorithm, computing.**

### Presenting Author's Biography

Anita Buckley. Her main research area is algebraic geometry. Currently a teaching assistant for mathematics at the University of Ljubljana. Completed PhD in Mathematics at the University of Warwick, England in 2003. Graduated in theoretical mathematics at the University of Ljubljana, Slovenia in 1998.

## 1 Introduction

There is an abundant amount of literature on Dedekind sums, appearing in many areas of mathematics such as analytic number theory [5], topology [7], combinatorial geometry [10], algorithmic complexity [8] and singularity theory [11, 4]. Dedekind sums were among many others studied also by Berndt, Carlitz, Grosswald, Knuth, Rademacher and Zagier. In a recent book [2] on this topic, the authors Beck and Robins connect the Dedekind sums to the 'coin exchange problem' and also to the number of lattice points inside a certain polyhedra. From here the discrete volume and the normal (or continuous) volume of polyhedra and polytopes can be computed.

Recent research is focused on finding properties and relations among Dedekind sums that would simplify their computation [1]. The existing methods for evaluating such sums are *factorisation*, *multisection* and third most powerful method of *partial fractions* [6].

The main objective of this paper is to describe an algorithm for computing Dedekind sums using the Euclidean algorithm. The Sequential arithmetic time and Parallel arithmetic complexity of the Euclidean algorithm are well understood [3]. In practice the fast Fourier transform (FFT) algorithm is used.

## 2 The main theorem

In this section we will relate the **Inverse function** to **Dedekind sums**. More precisely, we will find an equality connecting the polynomials

$$\alpha \quad \text{and} \quad \sum_{i=0}^{r-1} \sigma_{r-i}\, t^i. \tag{4}$$

**Remark 2.1** Running the Euclidean algorithm on a computer gives $\alpha$ supported on monomials $1, t, \ldots, t^{r-2-\deg \mathrm{hcf}(A,B)}$. Note also that $\alpha$ with such support is unique.

We begin with two obvious equalities among Dedekind sums:

(i)

$$\begin{aligned} \sigma_i &= \\ \frac{1}{r}\sum_{\varepsilon\in\mu_r,\,\varepsilon^{a_i}\neq 1} \frac{\varepsilon^i}{(1-\varepsilon^{a_1})\cdots(1-\varepsilon^{a_n})} &= \\ \frac{1}{r}\sum_{\varepsilon\in\mu_r,\,\varepsilon^{a_i}\neq 1} \frac{\varepsilon^{-i}}{(1-\varepsilon^{-a_1})\cdots(1-\varepsilon^{-a_n})} &= \\ (-1)^n\frac{1}{r}\sum_{\varepsilon\in\mu_r,\,\varepsilon^{a_i}\neq 1} \frac{\varepsilon^{a_1+\cdots+a_n-i}}{(1-\varepsilon^{a_1})\cdots(1-\varepsilon^{a_n})} &= \\ (-1)^n\sigma_{a_1+\cdots+a_n-i} & \end{aligned}$$

and

(ii) Denote by $\sum_i \xi_i t^i$ an arbitrary multiple of the polynomial $\frac{B}{\mathrm{hcf}(A,B)}$. Then

$$\sum_i \xi_i\, \sigma_i = 0, \tag{5}$$

since $\sum_i \xi_i\ \varepsilon^i = 0$ for all $\varepsilon \in \mu_r$ such that $\varepsilon^{a_i} \neq 1\, \forall i = 1, \ldots, n$. This follows directly from the definitions of $B$ and $\sigma_i$. In particular, if applied on $B$ then $\sum_{i=0}^{r-1} \sigma_i = 0$ always holds.

We proceed by giving an alternative description of the Inverse function $\alpha$. For all $\varepsilon \in \mu_r$ define

$$A_\varepsilon = \begin{cases} \frac{1}{(1-\varepsilon^{a_1})\cdots(1-\varepsilon^{a_n})}, & \text{if } \varepsilon^{a_i} \neq 1\, \forall i = 1, \ldots, n \\ 0, & \text{if } \varepsilon^{a_i} = 1 \text{ for some } i. \end{cases}$$

The expression $A_\varepsilon$ can be written as a polynomial in $\varepsilon$ with coefficients in $\mathbb{Q}$. Moreover, in the proof of Theorem 2.2 a polynomial $P(t)$ will be constructed such that

$$P(\varepsilon) = A_\varepsilon \quad \text{for all} \quad \varepsilon \in \mu_r.$$

Observe that whenever $\varepsilon^{a_i} \neq 1$ for all $i = 1, \ldots, n$ also

$$\alpha(\varepsilon) = A_\varepsilon = P(\varepsilon) \quad \text{holds by (3).}$$

For example, when $r$ is prime one can simply substitute $\varepsilon$ with $t$ in the polynomial expression of $A_\varepsilon$ to obtain $\alpha$.

**Theorem 2.2** *Fix positive integers $r$ and $a_1, \ldots, a_n$. The Inverse of $\prod_{i=1}^n (1 - t^{a_i})$ modulo $\frac{1-t^r}{1-t}$ denoted by $\alpha$ equals*

$$\sum_{i=0}^{r-1} \sigma_{r-i}\, t^i \quad \textit{modulo the polynomial} \quad \frac{1+t+\cdots+t^{r-1}}{\mathrm{hcf}\left(\prod_{i=1}^n (1-t^{a_i}), 1+\cdots+t^{r-1}\right)}.$$

PROOF From now on fix a generator $\varepsilon \in \mu_r$ and denote the polynomial

$$\sum_{j=1}^{r-1} A_{\varepsilon^j} t^j \tag{6}$$

by $p(t)$. Then

$$\begin{aligned} p(0) &= 0, \\ p(1) &= r\sigma_0, \\ &\vdots \\ p(\varepsilon^i) &= r\sigma_i, \\ &\vdots \\ p(\varepsilon^{r-1}) &= r\sigma_{r-1}. \end{aligned}$$

The above formulae can be considered as a linear system of $r$ equations in $A_{\varepsilon^j}$ and variable $\varepsilon$

$$\begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \varepsilon & \varepsilon^2 & \dots & \varepsilon^{r-1} \\ 1 & \varepsilon^2 & \varepsilon^4 & \dots & \varepsilon^{2(r-1)} \\ \dots & \dots & \dots & \dots & \\ 1 & \varepsilon^{r-1} & \dots & \dots & \varepsilon \end{pmatrix} \begin{pmatrix} 0 \\ A_\varepsilon \\ A_{\varepsilon^2} \\ \vdots \\ A_{\varepsilon^{r-1}} \end{pmatrix}$$

$$= r \begin{pmatrix} \sigma_0 \\ \sigma_1 \\ \sigma_2 \\ \vdots \\ \sigma_{r-1} \end{pmatrix}.$$

Since the Vandermonde matrix is invertible, we can solve the system and obtain

$$\begin{aligned} A_{\varepsilon^k} &= \textstyle\sum_{i=0}^{r-1} \sigma_i \varepsilon^{k(r-i)} \\ &= \textstyle\sum_{i=0}^{r-1} \sigma_{r-i} \varepsilon^{ki} \quad \text{for } k = 1, 2, \dots, r-1. \end{aligned}$$

Moreover, by (ii)

$$A_1 = A_{\varepsilon^0} = \sum_{i=0}^{r-1} \sigma_i = \sum_{i=0}^{r-1} \sigma_{r-i} = 0.$$

This proves that for all $\varepsilon \in \mu_r$ the polynomial

$$P(t) = \sum_{i=0}^{r-1} \sigma_{r-i} t^i \tag{7}$$

evaluated in $\varepsilon$ equals $A_\varepsilon$. Since we are working over algebraically closed $\mathbb{C}$, our polynomial $P(t)$ is uniquely determined by its $r$ values and support $1, t, \dots, t^{r-1}$.

On the other hand recall that for all

$$\varepsilon \in \mu_r \quad \text{such that} \quad \varepsilon^{a_i} \neq 1 \, \forall i = 1, \dots, n$$

the Inverse function $\alpha(\varepsilon)$ equals $A_\varepsilon = P(\varepsilon)$. This means that

$$\begin{aligned} \alpha(t) - P(t) &= \\ \gamma(t) \prod_{\substack{\varepsilon \in \mu_r \\ \varepsilon^{a_i} \neq 1 \, \forall i = 1, \dots, n}} (t - \varepsilon) &= \\ \gamma(t) \, \frac{\frac{1-t^r}{1-t}}{\text{hcf}\left(\prod_{i=1}^n (1-t^{a_i}), \frac{1-t^r}{1-t}\right)} & \end{aligned}$$

for some polynomial $\gamma$ with rational coefficients of degree $\text{hcf}\left(\prod_{i=1}^n (1-t^{a_i}), \frac{1-t^r}{1-t}\right)$. In other words

$$\alpha(t) \equiv P(t) \ \text{ modulo } \ \frac{\frac{1-t^r}{1-t}}{\text{hcf}\left(\prod_{i=1}^n (1-t^{a_i}), \frac{1-t^r}{1-t}\right)}.$$

□

## 3 The computing algorithm

The Euclidean algorithm and Theorem 2.2 will be used to calculate $\sigma_i\left(\frac{1}{r}(a_1, \dots, a_n)\right)$ for $i = 0, 1, \dots, r-1$. We will split the description of our method into three steps. As before denote

$$A = \prod_{i=1}^n (1 - t^{a_i}) \quad \text{and} \quad B = \frac{1-t^r}{1-t}.$$

**Step I)** Run the Euclidean algorithm on

$$A \quad \text{and} \quad \frac{B}{\text{hcf}\,(A, B)} \quad \text{to obtain } \alpha.$$

**Step II)** Recall that by Remark 2.1 thus obtained polynomial $\alpha(t)$ is spanned on $1, t, \dots, t^{r-2-\deg \text{hcf}(A,B)}$. By Theorem 2.2 there exists a unique polynomial $\gamma(t) \in \mathbb{Q}(t)$ of degree $\text{hcf}\,(A, B)$ such that

$$\alpha(t) = \sum_{i=0}^{r-1} \sigma_{r-i} \, t^i + \gamma(t) \, \frac{B}{\text{hcf}\,(A, B)}. \tag{8}$$

**Lemma 3.1** *Equalities (i) and (ii) uniquely determine $\gamma$.*

Proof Write

$$\alpha(t) = \sum_{i=0}^{r-1} \alpha_i \, t^i. \tag{9}$$

For the sake of simpler notation we will restrict the proof to $\text{hcf}\,(A, B) = 1$ which holds if and only if $a_1, \dots, a_n$ are relatively prime to $r$. In this case $\gamma$ is a constant. The general case can be proved in the same way by using equalities (i) and (ii) and comparing the coefficients in (8).

We will separate cases for $n$ odd or even.

When $n$ is odd compare the coefficients at $t^i$ and $t^{r - \overline{a_1 + \dots + a_n + i}}$ in (8), where $\overline{\phantom{x}}$ denotes the smallest nonnegative residue mod $r$. Recall from (i) that

$$\sigma_i = (-1)^n \sigma_{a_1 + \dots + a_n - i}. \tag{10}$$

Therefore $\alpha_i - \gamma = \sigma_{r-i}$ equals

$$-\sigma_{a_1 + \dots + a_n - r + i} = -\left(\alpha_{r - \overline{a_1 + \dots + a_n + i}} - \gamma\right)).$$

From this

$$\gamma = \frac{1}{2}\left(\alpha_i + \alpha_{r - \overline{a_1 + \dots + a_n + i}}\right) \tag{11}$$

for any and thus for all $i = 0, \dots, r-1$.

When $n$ is even we use equality (ii) to determine $\gamma$:

$$\sum_{i=0}^{r-1} \alpha_i - r\gamma = \sum_{i=0}^{r-1} \sigma_i = 0$$

implies

$$\gamma = \frac{1}{r}\sum_{i=0}^{r-1} \alpha_i. \tag{12}$$

□

**Step III)** By (8) the $i$th coefficient of the above obtained polynomial $\alpha - \gamma\frac{B}{\mathrm{hcf}(A,B)}$ is exactly $\sigma_{r-i}$.

## 4 Examples

In the following examples it will be shown how to use `Mathematica` to compute the polynomials $\alpha$ and $\gamma$. In this way all Dedekind sums $\sigma_i$ are calculated simultaneously.

Load the standard package
$<<$ **Algebra‘PolynomialExtendedGCD‘**.

**Example 4.1** We start with examples where $a_1, \ldots, a_n$ are relatively prime to $r$.

$$\bullet\ \sigma_i\big(\frac{1}{23}(4,5,12)\big):$$

In step I) $\alpha$ and $\beta$ are calculated by

$$\mathtt{In}[1] := \mathbf{PolynomialExtendedGCD}[(\mathbf{1-t^4})(\mathbf{1-t^5})(\mathbf{1-t^{12}}), \mathbf{PolynomialQuotient}[\mathbf{1-t^{23}}, \mathbf{1-t}, \mathbf{t}]]$$

$$\mathtt{Out}[1] := \{1, \{-\frac{5}{23} - \frac{3t}{23} - \frac{t^2}{23} - \frac{6t^3}{23} - \frac{2t^4}{23} + \frac{4t^5}{23} + \frac{5t^6}{23} - \frac{6t^7}{23} - \frac{13t^8}{23} + \frac{3t^{10}}{23} - \frac{11t^{11}}{23} - \frac{3t^{12}}{23} - \frac{3t^{13}}{23} + \frac{5t^{14}}{23} - \frac{9t^{15}}{23} - \frac{6t^{16}}{23} + \frac{7t^{17}}{23} - \frac{11t^{19}}{23} - \frac{10t^{20}}{23} - \frac{4t^{21}}{23}, \frac{28}{23} - \frac{25t}{23} - \frac{2t^2}{23} + \frac{5t^3}{23} - \frac{9t^4}{23} - \frac{9t^5}{23} + \frac{3t^6}{23} + \frac{8t^7}{23} + \frac{6t^8}{23} + \frac{2t^9}{23} + \frac{2t^{10}}{23} + \frac{2t^{11}}{23} - \frac{26t^{12}}{23} + \frac{4t^{13}}{23} + \frac{4t^{14}}{23} - \frac{3t^{15}}{23} + \frac{11t^{16}}{23} + \frac{11t^{17}}{23} - \frac{t^{18}}{23} - \frac{6t^{19}}{23} - \frac{4t^{20}}{23}\}\}.$$

In step II) $\gamma$ is calculated using (3) since $n = 3$ is odd. Thus

$$\gamma = \frac{1}{2}\left(\alpha_0 + \alpha_{23-\overline{4+5+12}}\right) = \frac{1}{2}\left(\alpha_0 + \alpha_2\right) = -\frac{3}{23}.$$

In step III) we obtain $\sigma_i$ as the coefficient at $t^{23-i}$ in

$$\mathtt{In}[2] := \alpha - \gamma\,\mathbf{PolynomialQuotient}[\mathbf{1-t^{23}}, \mathbf{1-t}, \mathbf{t}]$$

$$\mathtt{Out}[2] := -\frac{2}{23} + \frac{2t^2}{23} - \frac{3t^3}{23} + \frac{t^4}{23} + \frac{7t^5}{23} + \frac{8t^6}{23} - \frac{3t^7}{23} - \frac{10t^8}{23} + \frac{3t^9}{23} + \frac{6t^{10}}{23} - \frac{8t^{11}}{23} + \frac{8t^{14}}{23} - \frac{6t^{15}}{23} - \frac{3t^{16}}{23} + \frac{10t^{17}}{23} + \frac{3t^{18}}{23} - \frac{8t^{19}}{23} - \frac{7t^{20}}{23} - \frac{t^{21}}{23} + \frac{3t^{22}}{23}.$$

It is easy to verify that

$$\sigma_0 = -\sigma_{21},\ \sigma_{22} = -\sigma_{22} = 0, \ldots$$

as expected.

$$\bullet\ \sigma_i\big(\frac{1}{33}(4,5,5,7,10,14)\big):$$

Step I) computes $\alpha$ by

$$\mathtt{In}[1] := \mathbf{PolynomialExtendedGCD}[(\mathbf{1-t^4})(\mathbf{1-t^5})^{\mathbf{2}}(\mathbf{1-t^7})(\mathbf{1-t^{10}})(\mathbf{1-t^{14}}), \mathbf{PolynomialQuotient}[\mathbf{1-t^{33}}, \mathbf{1-t}, \mathbf{t}]]$$

$$\mathtt{Out}[1] := \{1, \{\frac{113}{27} + \frac{40t}{11} + \frac{37t^2}{11} + \frac{568t^3}{297} + \frac{7t^4}{11} + \frac{47t^5}{11} + \frac{1135t^6}{297} + \frac{30t^7}{11} + \frac{15t^8}{11} + \frac{541t^9}{297} + 4t^{10} + 4t^{11} + \frac{541t^{12}}{297} + \frac{15t^{13}}{11} + \frac{30t^{14}}{11} + \frac{1135t^{15}}{297} + \frac{47t^{16}}{11} + \frac{7t^{17}}{11} + \frac{568t^{18}}{297} + \frac{37t^{19}}{11} + \frac{40t^{20}}{11} + \frac{113t^{21}}{27} + \frac{29t^{23}}{11} + \frac{1054t^{24}}{297} + \frac{41t^{25}}{11} + \frac{40t^{26}}{11} - \frac{161t^{27}}{297} + \frac{40t^{28}}{11} + \frac{41t^{29}}{11} + \frac{1054t^{30}}{297} + \frac{29t^{31}}{11}, -\frac{86}{27} - \frac{134t}{297} + \frac{3t^2}{11} + \cdots - \frac{53t^{42}}{297} - \frac{271t^{43}}{297} - \frac{29t^{44}}{11}\}\}.$$

In step II) $\gamma$ is calculated using (4) since $n = 6$ is even. Thus

$$\gamma = \frac{1}{33}\sum_{i=0}^{31}\alpha_i = \frac{1}{33}\,\frac{2431}{27} = \frac{221}{81}.$$

Step III) yields $\sigma_i$ as the coefficient at $t^{33-i}$ in

$$\mathtt{In}[2] := \alpha - \gamma\,\mathbf{PolynomialQuotient}[\mathbf{1-t^{33}}, \mathbf{1-t}, \mathbf{t}]$$

$$\mathtt{Out}[2] := \frac{118}{81} + \frac{809t}{891} + \frac{566t^2}{891} - \frac{727t^3}{891} - \frac{1864t^4}{891} + \frac{1376t^5}{891} + \frac{974t^6}{891} - \frac{t^7}{891} - \frac{1216t^8}{891} - \frac{808t^9}{891} + \frac{103t^{10}}{81} + \frac{103t^{11}}{81} - \frac{808t^{12}}{891} - \frac{1216t^{13}}{891} - \frac{t^{14}}{891} + \frac{974t^{15}}{891} + \frac{1376t^{16}}{891} - \frac{1864t^{17}}{891} - \frac{727t^{18}}{891} + \frac{566t^{19}}{891} + \frac{809t^{20}}{891} + \frac{118t^{21}}{81} - \frac{221t^{22}}{81} - \frac{82t^{23}}{891} + \frac{731t^{24}}{891} + \frac{890t^{25}}{891} + \frac{809t^{26}}{891} - \frac{2914t^{27}}{891} + \frac{809t^{28}}{891} + \frac{890t^{29}}{891} + \frac{731t^{30}}{891} - \frac{82t^{31}}{891} - \frac{221t^{32}}{81}.$$

**Example 4.2** In the following examples we consider two generalised Dedekind sums, where not all $a_1, \ldots, a_n$ are relatively prime to $r$. In this case $\gamma(t)$ is a polynomial of degree $\mathrm{hcf}\,(A,B)$ which will be computed by generalising the proof of Lemma 3.1.

$$\bullet\ \sigma_i\big(\frac{1}{12}(1,2,3)\big):$$

In this case $\mathrm{hcf}\,(A,B) = (1+t)(1+t+t^2)$. Thus in step I) $\alpha$ is obtained by running the Euclidean algorithm on

$$\mathtt{In}[1] := \mathbf{PolynomialExtendedGCD}[(\mathbf{1-t})(\mathbf{1-t^2})(\mathbf{1-t^3}), \mathbf{PolynomialQuotient}[\mathbf{1-t^{12}}, (\mathbf{1-t})(\mathbf{1+t})(\mathbf{1+t+t^2}), \mathbf{t}]]$$

$$\mathtt{Out}[1] := \{1, \{\frac{7}{12} + \frac{5t}{12} - \frac{t^2}{4} + \frac{t^4}{4} + \frac{t^5}{2} - \frac{t^6}{3} + \frac{11t^7}{12}, \frac{19}{12} + \frac{7t}{12} - \frac{11t^2}{12} - \frac{4t^3}{3} - \frac{t^4}{3} + \frac{11t^5}{12}\}\}.$$

In step II) we determine $\gamma = \gamma_0 + \gamma_1 t + \gamma_2 t^2 + \gamma_3 t^3$. From (i) it follows

$$\begin{array}{ccc} \sigma_0 = -\sigma_6 & \sigma_1 = -\sigma_5 & \sigma_2 = -\sigma_4 \\ & \sigma_7 = -\sigma_{11} & \sigma_8 = -\sigma_{10} \\ & \sigma_3 = -\sigma_3 = 0 & \sigma_9 = -\sigma_9 = 0. \end{array}$$

Moreover, (ii) and

$$\frac{1-t^{12}}{(1-t)(1+t)(1+t+t^2)} = 1 - t + t^2 + t^6 - t^7 + t^8$$

imply

$$\sigma_0 - \sigma_1 + \sigma_2 + \sigma_6 - \sigma_7 + \sigma_8 = 0.$$

By comparing the coefficients in

$$\sum_{i=0}^{11} \sigma_{12-i}\, t^i \ \text{ and } \ \alpha - \gamma\, \frac{1-t^{12}}{(1-t)(1+t)(1+t+t^2)}$$

the above translates into the following system of equations

$$-\tfrac{7}{12} - \gamma_0 = \tfrac{1}{3} + \gamma_0 \qquad \tfrac{11}{12} + \gamma_0 - \gamma_1 = \gamma_3$$

$$-\gamma_0 + \gamma_1 - \gamma_2 = \gamma_2 - \gamma_3 \qquad -\gamma_1 + \gamma_2 - \gamma_3 = 0$$

and

$$-\frac{7}{12} - \gamma_0 + \gamma_3 - \gamma_2 + \gamma_3 - \frac{1}{3} - \gamma_0 - \frac{1}{2} + \gamma_3 + \frac{1}{4} - \gamma_2 + \gamma_3 = 0.$$

Its solution is

$$\gamma = -\frac{11}{24} + \frac{t}{6} + \frac{11\,t^2}{24} + \frac{7\,t^3}{24}.$$

In step III) we obtain $\sigma_i$ as the coefficient at $t^{12-i}$ in

$$\texttt{In}[2] := \quad \alpha - \gamma\, \mathbf{PolynomialQuotient}[\mathbf{1 - t^{12}}, \mathbf{(1-t)(1+t)(1+t+t^2)}, \mathbf{t}]$$

$$\texttt{Out}[2] := \quad -\tfrac{1}{8} - \tfrac{5\,t}{24} - \tfrac{t^2}{12} + \tfrac{t^4}{12} + \tfrac{5\,t^5}{24} + \tfrac{t^6}{8} + \tfrac{7\,t^7}{24} + \tfrac{t^8}{6} - \tfrac{t^{10}}{6} - \tfrac{7\,t^{11}}{24}.$$

- $\sigma_i\big(\frac{1}{12}(1,2,3,4)\big)$ :

In this case

$$\begin{aligned} \mathrm{hcf}\,(A,B) &= (1+t)(1+t^2)(1+t+t^2) \\ &= (1+t+t^2)(1+t+t^2+t^3). \end{aligned}$$

Step I) calculates $\alpha$ by running the Euclidean algorithm on

$$\texttt{In}[1] := \quad \mathbf{PolynomialExtendedGCD}[\mathbf{(1-t)(1-t^2)(1-t^3)(1-t^4)}, \mathbf{PolynomialQuotient}[\mathbf{1-t^{12}}, \mathbf{(1-t)(1+t)(1+t^2)(1+t+t^2)}, \mathbf{t}]]$$

$$\texttt{Out}[1] := \{1, \{-\tfrac{7}{12} - \tfrac{t}{12} + \tfrac{t^2}{4} - \tfrac{5\,t^3}{12} - \tfrac{5\,t^4}{12} + \tfrac{5\,t^5}{12}, \tfrac{19}{12} + \tfrac{13\,t}{12} + \tfrac{t^2}{6} - \tfrac{5\,t^3}{6} - \tfrac{5\,t^4}{3} - \tfrac{t^5}{3} + \tfrac{\mathbf{t^6}}{\mathbf{6}} + \tfrac{\mathbf{5\,t^7}}{\mathbf{6}} + \tfrac{\mathbf{5\,t^8}}{\mathbf{12}} - \tfrac{\mathbf{5\,t^9}}{\mathbf{12}}\}\}.$$

In step II) we determine

$$\gamma = \gamma_0 + \gamma_1 t + \gamma_2 t^2 + \gamma_3 t^3 + \gamma_4 t^4 + \gamma_5 t^5.$$

Since

$$\sum_{i=0}^{11} \sigma_{12-i}\, t^i$$

equals

$$\alpha - \gamma\, \frac{1-t^{12}}{(1-t)(1+t)(1+t^2)(1+t+t^2)}$$

we get

$$\begin{aligned} &\sigma_0 = -\tfrac{7}{12} - \gamma_0 \\ &\sigma_{11} = -\tfrac{1}{12} + \gamma_0 - \gamma_1 \\ &\sigma_{10} = \tfrac{1}{4} + \gamma_1 - \gamma_2 \\ &\sigma_9 = -\tfrac{5}{12} - \gamma_0 + \gamma_2 - \gamma_3 \\ &\sigma_8 = -\tfrac{5}{12} - \gamma_1 + \gamma_3 - \gamma_4 \\ &\sigma_7 = \tfrac{5}{12} + \gamma_0 - \gamma_2 + \gamma_4 - \gamma_5 \\ &\sigma_6 = -\gamma_0 + \gamma_1 - \gamma_3 + \gamma_5 \\ &\sigma_5 = -\gamma_1 + \gamma_2 - \gamma_4 \\ &\sigma_4 = -\gamma_2 + \gamma_3 - \gamma_5 \\ &\sigma_3 = -\gamma_3 + \gamma_4 \\ &\sigma_2 = -\gamma_4 + \gamma_5 \\ &\sigma_1 = -\gamma_5. \end{aligned}$$

Recall that by (i)

$\sigma_i = \sigma_{1+2+3+4-i} = \sigma_{10-i}$ holds for $i = 0, 1, \ldots, 11.$

Next apply (ii) on the following multiples of $\frac{B}{\mathrm{hcf}(A,B)}$:

$$\begin{aligned} \tfrac{B}{\mathrm{hcf}(A,B)} &= \tfrac{1-t^{12}}{(1-t)(1+t)(1+t^2)(1+t+t^2)} \\ &= 1 - t + t^3 - t^5 + t^6, \end{aligned}$$

$$\tfrac{B}{\mathrm{hcf}(A,B)}(1+t+t^2) = 1 + t^4 + t^8,$$

$$\begin{aligned} &\tfrac{B}{\mathrm{hcf}(A,B)}(1+t)(1+t^2) \\ &= \tfrac{B}{\mathrm{hcf}(A,B)}(1+t+t^2+t^3) \\ &= 1 + t^3 + t^6 + t^9 \end{aligned}$$

and $B$.

Therefore

$$\begin{aligned} &\sigma_0 - \sigma_1 + \sigma_3 - \sigma_5 + \sigma_6 = 0, \\ &\sigma_0 + \sigma_4 + \sigma_8 = 0, \\ &\sigma_0 + \sigma_3 + \sigma_6 + \sigma_9 = 0, \\ &\textstyle\sum_{i=0}^{11} \sigma_i = 0. \end{aligned}$$

The system of the above equations has a unique solution

$$\gamma = -\frac{37}{72} - \frac{11\,t}{24} - \frac{5\,t^2}{36} + \frac{t^3}{18} + \frac{t^4}{8} + \frac{7\,t^5}{72}.$$

Finally, in step III) we obtain $\sigma_i$ as the coefficient at $t^{12-i}$ in

$$\texttt{In}[2] := \quad \alpha - \gamma\mathbf{PolynomialQuotient}[\mathbf{1-t^{12}}, \mathbf{(1-t)(1+t)(1+t^2)(1+t+t^2)}, \mathbf{t}]$$

$$\texttt{Out}[2] := \quad -\tfrac{5}{72} - \tfrac{5\,t}{36} - \tfrac{5\,t^2}{72} - \tfrac{7\,t^3}{72} - \tfrac{t^4}{36} + \tfrac{5\,t^5}{72} + \tfrac{7\,t^6}{72} + \tfrac{7\,t^7}{36} + \tfrac{7\,t^8}{72} + \tfrac{5\,t^9}{72} - \tfrac{t^{10}}{36} - \tfrac{7\,t^{11}}{72}.$$

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