Finding Shortest Non-Contractible and Non-Separating Cycles for Topologically Embedded Graphs

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Overview

• surfaces and graphs
• old and new results
• other similar work
• key points for the non-separating case
• key point for the non-contractible case
Surfaces and graphs

Surface: compact set, locally like the plane

Genus $g$ of $\Sigma$: nb of holes $= \text{nb of merged torus}$
Surfaces and graphs

Contractible, non-contracible, and non-separating loops.
Surfaces and graphs

Contractible, non-contracible, and non-separating loops.

Contractible $\Rightarrow$ separating  
Non-separating $\Rightarrow$ non-contractible  
Contractible $\Leftrightarrow$ Zero in the homotopy group  
Separating $\Leftrightarrow$ Zero in the $\mathbb{Z}_2$-homology group
Surfaces and graphs

(Weighted) graph $G$ on $\Sigma$:

Cycles/loops in $G$ are curves in $\Sigma$. 

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Cycles/loops in $G$ are curves in $\Sigma$.

**Problem:** Find shortest non-contractible cycle.

**Problem:** Find shortest non-separating cycle.
Old and new results

$G$ a graph with $V$ vertices in a surface of genus $g$

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Better if $g = O(1)$

Better if $g = o(V^{1/3})$
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[Thomassen]             | $O(g^{O(g)}V^{3/2})$  
[Thomassen]             |                                                     |
| Shortest non-separating cycle   | $O^*(V(V + g))$  
[Erickson, Har-Peled] | $O(g^{O(g)}V^{4/3})$  
[Erickson, Har-Peled] |                                                     |

SODA’06
Why non-contractible/non-separating?

3-path property

\[ P_1 + P_2 \text{ non-separating} \]
\[ \Downarrow \]
\[ P_1 + P_3 \text{ or } P_2 + P_3 \text{ non-separating} \]

Are there polynomial time algorithms for shortest separating? More difficult for shortest contractible.
Other similar work

- Erickson and Har-Peled (2004, 2005) find minimum-length cut subgraph $C$ s.t. $\Sigma \setminus C$ planar.


- Erickson and Whittlesey (2005) find shortest system of loops with given basepoint.

- Colin de Verdière and Erickson (SODA’06) find shortest loop/cycle/path in a homotopy class.
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Key points for the non-separating cycle

Crossings

crossing

intersection without crossing
Key points for the non-separating cycle

\[ C = \{C_1, \ldots, C_{\Theta(g)}\} \text{ system of fundamental loops.} \]

\[ C_1, \ldots, C_{\Theta(g)} \text{ through a common vertex.} \]

Surface cut along \( C_1, \ldots, C_{\Theta(g)} \) is a disk.
Key points for the non-separating cycle

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- \[ C_1, \ldots, C_{\Theta(g)} \] through a common vertex.
- Surface cut along \( C_1, \ldots, C_{\Theta(g)} \) is a disk.

Fix \( x \in V(G) \) and construct from-\( x \)-shortest-path tree \( T_x \).

For edge \( e \notin T_x \), \( \text{loop}(T_x, e) \) is . . .

**Theorem:** There is always a system of fundamental loops of the form \( \text{loop}(T_x, e_1), \ldots, \text{loop}(T_x, e_{\Theta(g)}) \).

Easy to compute it (tree-cotree decomposition)
Key points for the non-separating cycle

\[ \text{loop}(T_x, e_1), \ldots, \text{loop}(T_x, e_{\Theta(g)}) \] system of fund. loops.

Lem: \( \exists \) shortest non-separ cycle crossing \( \leq 2 \) each \( \text{loop}(T_x, e_i) \).

Lem: each non-sep cycle crosses some \( \text{loop}(T_x, e_i) \) odd times.

\[ \Rightarrow \exists \text{ shortest non-sep cycle } C^* \text{ and } \text{loop}(T_x, e_i) \]
holding \( cr(C^*, \text{loop}(T_x, e_i)) = 1 \)
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Algorithm:

for each cycle \( C_i = \text{loop}(T_x, e_i) \) in the system
find a shortest cycle crossing \( C_i \) exactly once;
report the shortest one
Key points for the non-separating cycle

\[ C = \{\text{loop}(T_x, e_1) \ldots \text{loop}(T_x, e_{\Theta(g)})\} \] system of fund. loops.

Algorithm: for each cycle \( C_i \in C \)
- find a shortest cycle crossing \( C_i \) exactly once;
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\[ |C_i| \text{ pairs of shortest paths in } (G\text{-cut-by-}C_i) \]
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Algorithm: for each cycle \( C_i \in C \)

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Solvable in \( O^*((V + g)\sqrt{gV}) \) time via separators
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The algorithm takes \( O^*(g^{3/2}V^{3/2} + g^{5/2}V^{1/2}) \) time.
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**Thm:** Let \( \tilde{V} = O(g^{O(g)}V) \). Finding a shortest non-contractible cycle can be reduced in \( O(\tilde{V}) \) time to: computing \( O(\tilde{V}) \) distances in a planar graph with \( O(\tilde{V}) \) vertices.
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Solvable in \( O(\tilde{V}^{3/2}) \) via separators.

Solvable in \( O^*(\tilde{V}^{4/3}) \) [SODA’06].
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- shortest non-separating cycle: $O^*(g^{3/2}V^{3/2} + g^{5/2}V^{1/2})$;
- shortest non-contractible cycle: $O^*(g^{O(g)}V^{3/2})$.

Main techniques:

- system of fundamental loops made of 2 geodesics;
- reduce the problem to computing $V$ distances in graphs.

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- better results for face-width in $\mathbb{P}^2$ and $\mathbb{T}$
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