

FREE SKEW FIELDS HAVE MANY *-ORDERINGS

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ABSTRACT. The notion of a *-ordering on a skew-field with involution was introduced by S. Holland in [H] as an analogue to the notion of a total ordering on a skew-field and developed further in a series of papers of T. Craven, I. Idris, M. Marshall and T. Smith. While it is well known that every free skew-field has a total ordering (see [Le]), it has not been known so far whether every free skew-field with some natural involution has a *-ordering. The aim of this paper is to give an explicit construction of a class of *-orderings on free associative algebras and to prove that *-orderings from this class extend uniquely to the corresponding free skew fields. The problem was posed by T. Craven and T. Smith in [CS2].

1. *-ORDERINGS AND *-VALUATIONS

A *domain* is an associative unital ring without zero divisors. Let R be a domain, Γ an ordered group and $\infty \notin \Gamma$. A mapping $v : R \rightarrow \Gamma \cup \{\infty\}$ is a *valuation* if

- (1) for every $x \in R$, $v(x) = \infty$ if and only if $x = 0$,
- (2) $v(x + y) \geq \min\{v(x), v(y)\}$ for every $x, y \in R^\times$,
- (3) $v(xy) = v(x) + v(y)$ for every $x, y \in R^\times$,

where $R^\times := R \setminus \{0\}$. A valuation $v : R \rightarrow \Gamma \cup \{\infty\}$ is *quasi-commutative* if $v(xy - yx) > v(x) + v(y)$ for every $x, y \in R$.

A **-domain* is a domain with involution. A valuation v on a *-domain R is a **-valuation* if it has an additional property

- (4) $v(x^*) = v(x)$ for every $x \in R$.

Note that $v(x) + v(y) = v(xy) = v((xy)^*) = v(y^*x^*) = v(y^*) + v(x^*) = v(y) + v(x)$ for every *-valuation v on R and every $x, y \in R$.

For any valuation v on a domain R , we define a relation \sim_v on R^\times by

$$x \sim_v y \iff v(x) < v(x - y).$$

Clearly, $x \sim_v y$ implies that $v(x) = v(y)$. Let us recall Lemma 5.11 from [H] :

Key words and phrases. free skew fields, ordered rings, rings with involution.

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Lemma 1. *Let R be a domain and v a valuation on R . The relation \sim_v is an equivalence relation on R^\times and it has the following properties:*

- (1) *If $x \sim_v y$, and x, y are invertible, then $x^{-1} \sim y^{-1}$. If $x \sim_v y$ and v is a $*$ -valuation, then $x^* \sim_v y^*$.*
- (2) *If $x_1 \sim_v y_1$ and $x_2 \sim_v y_2$, then $x_1 x_2 \sim_v y_1 y_2$. If also $x_1 \not\sim_v -x_2$, then $x_1 + x_2 \sim_v y_1 + y_2$ as well.*
- (3) *If $x \sim_v y$ and $v(z) > v(x)$, then $x + z \sim_v y$.*

Let R be a $*$ -domain and $S(R)$ the set of all symmetric elements of R . A subset $P \subseteq S(R)$ is a $*$ -ordering if

- (1) $1 \in P$,
- (2) $P + P \subseteq P$,
- (3) $rPr^* \subseteq P$ for every $r \in R$,
- (4) $S(R) = P \cup -P$,
- (5) $P \cap -P = \{0\}$,
- (6) If $a, b \in P$, then $ab + ba \in P$.

A $*$ -ordering P is *compatible* with a $*$ -valuation v if for every $x \in P$ and every $y \in S(R)$ such that $y \sim_v x$ we have that $y \in P$.

We will need the following proposition in the proof of the main theorem.

Proposition 2. *Let K be a skew field, $v : K \rightarrow \mathbb{Z} \cup \{\infty\}$ a quasi-commutative valuation and R a subring of K such that K is generated by R . Then for every $z \in K^\times$ there exist $a, b \in R^\times$ such that $x \sim_v ab^{-1}$.*

Proof. We define recursively a sequence $R_0 \subseteq R_1 \subseteq R_2 \subseteq \dots$ of subrings of K by $R_0 = R$ and $R_{n+1} = \Sigma(G_n)$, where G_n is the set of all finite products of elements from $R_n \cup R_n^{-1}$ and $\Sigma(G_n)$ is the set of all finite sums of elements from G_n . Note that $\cup R_n$ is a subfield of K containing R . Since R generates K , it follows that $K = \cup R_n$.

We claim that for every element $z \in R_{n+1}^\times$ there exist elements $a, b \in R_n^\times$ such that $z \sim_v ab^{-1}$. We will prove the claim by constructing a sequence of representations $z = u_1^{(k)} + u_2^{(k)} + \dots + u_{m^{(k)}}^{(k)}$ (where $k = 0, \dots, t$ and $u_i^{(k)} \in G_n$ for $i = 1, \dots, m^{(k)}$) such that the last representation satisfies $u_1^{(t)} = a_t b_t^{-1}$, $a_t, b_t \in R_n^\times$ and $v(u_1^{(t)}) < \min\{v(u_2^{(t)}), \dots, v(u_{m^{(t)}}^{(t)})\}$. It follows that $z \sim_v a_t b_t^{-1}$.

By the definition of R_{n+1} , there exist elements $u_1, \dots, u_m \in G_n$ such that $z = u_1 + \dots + u_m$. Set $m^{(0)} = m$ and $u_i^{(0)} = u_i$ for $i = 1, \dots, m$. Assume now that $u_1^{(k-1)}, u_2^{(k-1)}, \dots, u_{m^{(k-1)}}^{(k-1)}$ have been constructed. Write $m = m^{(k-1)}$. We may assume that $v(u_1^{(k-1)}) = \dots = v(u_p^{(k-1)}) < v(u_{p+1}^{(k-1)}) \leq \dots \leq v(u_m^{(k-1)})$. For every $i = 1, \dots, p$ there exist elements

$x_{ij}, y_{ij} \in R_n$ ($j = 1, \dots, q_i$) such that $u_i^{(k-1)} = \prod_{j=1}^{q_i} x_{ij} y_{ij}^{-1}$. Write $x_i = \prod_{j=1}^{q_i} x_{ij}$ and $y_i = \prod_{j=1}^{q_i} y_{ij}$ for every $i = 1, \dots, p$. Since v is quasi-commutative, we have for every $i = 1, \dots, p$ that $u_i^{(k-1)} = x_i y_i^{-1} + r_i = (x_i \prod_{l=1, l \neq i}^p y_l) (\prod_{l=1}^p y_l)^{-1} + o_i$, where o_i and r_i are sums of elements from G_n with value $> v(u_i^{(k-1)})$. Write $a_k = \sum_{i=1}^p (x_i \prod_{l=1, l \neq i}^p y_l) \in R_n$ and $b_k = \prod_{l=1}^p y_l \in R_n^\times$. It follows that $z = a_k b_k^{-1} + \sum_{i=p+1}^m u_i + \sum_{i=1}^p o_i$. Write $u_1^{(k)} = a_k b_k^{-1}$, $u_2^{(k)} = \dots = u_p^{(k)} = 0$, $u_{p+1}^{(k)} = u_{p+1}^{(k-1)}, \dots, u_m^{(k)} = u_m^{(k-1)}$ and pick $u_{m+1}^{(k)}, \dots, u_{m^{(k)}}^{(k)} \in G_n$ such that $\sum_{i=1}^p o_i = \sum_{i=m+1}^{m^{(k)}} u_i^{(k)}$. If $v(u_1^{(k)}) < \min\{v(u_2^{(k)}), \dots, v(u_{m^{(k)}}^{(k)})\}$ then set $t = k$ and stop. Else, repeat the cycle with k replaced by $k + 1$.

Write $L_k = \min\{v(u_i^{(k)}), i = 1, \dots, m^{(k)}\}$. Note that $L_k \leq v(z)$ for every $k = 0, \dots, t$. If $1 < k < t$, then $L_k > L_{k-1}$. It follows that the algorithm terminates in at most $v(z) - L_0$ steps.

By induction on n using the claim and Lemma 1 we prove that for every $z \in R_n$ there exist elements $a, b \in R$ such that $z \sim_v ab^{-1}$. Now, the proposition follows from the fact that $K = \bigcup_n R_n$. Q.E.D.

Note that Proposition 2 implies that Corollary 1 in [Li2] is true for any Lie algebra L .

Lemma 3. *Let K be a $*$ -field and $v : K \rightarrow \Gamma \cup \{\infty\}$ a quasi-commutative $*$ -valuation on K . For any $s_1, s_2 \in S(K)$ write $\langle s_1, s_2 \rangle = \frac{1}{2}(s_1 s_2 + s_2 s_1)$.*

- (1) *If $s_1, s_2 \in S(K)^\times$, then $\langle s_1, s_2 \rangle \neq 0$ and $v(\langle s_1, s_2 \rangle) = v(s_1) + v(s_2)$.*
- (2) *If $s_1, s_2, t_1, t_2 \in S(K)^\times$, $s_1 \sim_v t_1$ and $s_2 \sim_v t_2$ then $\langle s_1, s_2 \rangle \sim_v \langle t_1, t_2 \rangle$.*
- (3) *If $s_1, s_2, s_3 \in S(K)^\times$ then $\langle \langle s_1, s_2 \rangle, s_3 \rangle \sim_v \langle s_1, \langle s_2, s_3 \rangle \rangle$.*

Proof. For every $s_1, s_2 \in S(K)^\times$, we have that $\langle s_1, s_2 \rangle = s_1 s_2 + \frac{1}{2}(s_2 s_1 - s_1 s_2)$. Since v is quasi-commutative, it follows that $v(\frac{1}{2}(s_2 s_1 - s_1 s_2)) > v(s_1 s_2)$. Hence, $v(\langle s_1, s_2 \rangle) = v(s_1 s_2)$. It follows that $\langle s_1, s_2 \rangle \neq 0$. This proves the first assertion.

To prove the second assertion, note that $\langle s_1, s_2 \rangle - \langle t_1, t_2 \rangle = \langle s_1, s_2 - t_2 \rangle + \langle s_1 - t_1, t_2 \rangle$ for any $s_1, s_2, t_1, t_2 \in S(K)$. If $s_1 \sim_v t_1$ and $s_2 \sim_v t_2$ then $v(\langle s_1, s_2 \rangle - \langle t_1, t_2 \rangle) \geq \min\{v(s_1) + v(s_2 - t_2), v(s_1 - t_1) + v(t_2)\} > \min\{v(s_1) + v(s_2), v(t_1) + v(t_2)\} = v(\langle s_1, s_2 \rangle) = v(\langle t_1, t_2 \rangle)$. Hence, $\langle s_1, s_2 \rangle \sim_v \langle t_1, t_2 \rangle$.

Since $\langle \langle s_1, s_2 \rangle, s_3 \rangle - \langle s_1, \langle s_2, s_3 \rangle \rangle = \frac{1}{4}(s_2(s_1 s_3 - s_3 s_1) - (s_1 s_3 - s_3 s_1)s_2)$, it follows that $v(\langle \langle s_1, s_2 \rangle, s_3 \rangle - \langle s_1, \langle s_2, s_3 \rangle \rangle) > v(s_1) + v(s_2) + v(s_3) = v(\langle \langle s_1, s_2 \rangle, s_3 \rangle) = v(\langle s_1, \langle s_2, s_3 \rangle \rangle)$. The third assertion follows. Q.E.D.

Theorem 4. *Let K be a $*$ -field, $v : K \rightarrow \mathbb{Z} \cup \{\infty\}$ a quasi-commutative $*$ -valuation on K and R a $*$ -subring of K such that K is generated by R . Then every $*$ -ordering on R which is compatible with $v|_R$ extends uniquely to a $*$ -ordering on K compatible with v .*

Proof. We claim that for every $z \in S(K)^\times$, there exist elements $s, t \in S(R)^\times$ such that $z \sim_v \langle s, t^{-1} \rangle$. By Proposition 2, there exist elements $a, b \in R^\times$ such that $z \sim_v ab^{-1}$. Write $c = ab^*$ and $t = bb^* \in S(R)^\times$ and note that $ab^{-1} = ct^{-1}$. We have $ct^{-1} \sim_v t^{-1}c^*$ since $z = z^*$ and $ct^{-1} \sim_v t^{-1}c$ since v is quasi-commutative. It follows that $c \sim_v c^*$. Write $s = \frac{1}{2}(c + c^*) \in S(R)^\times$ and note that $z \sim_v ct^{-1} \sim_v st^{-1} \sim_v \langle s, t^{-1} \rangle$. This proves the claim.

Let P' be a v -compatible $*$ -ordering on K . Then $P := P' \cap S(R)$ is a $v|_R$ -compatible $*$ -ordering on R . We claim that

$$P' = \{z \in S(K) \mid \exists s, t \in P : z \sim_v \langle s, t^{-1} \rangle\}.$$

If $z \in (P')^\times$, then there exist $s, t \in S(R)^\times$ such that $z \sim_v \langle s, t^{-1} \rangle$. Changing signs if necessary, we may assume that $t \in P$. Since P' is v -compatible and $s \sim_v \langle \langle s, t^{-1} \rangle, t \rangle \sim_v \langle z, t \rangle \in P'$ by Lemma 3, it follows that $s \in P' \cap S(R) = P$. So, z belongs to the set on the right side. The other inclusion is trivial. This proves uniqueness.

To prove existence, take any $v|_R$ -compatible $*$ -ordering P on R and define P' as above. We claim that P' is a v -compatible $*$ -ordering which extends P .

Compatibility follows directly from the definition. Clearly, $P \subseteq P' \cap S(R)$. Conversely, if $x \in P' \cap S(R)$, then there exist elements $s, t \in P$ such that $x \sim_v \langle s, t^{-1} \rangle$. Since P is v -compatible and $\langle x, t \rangle \sim_v s$, it follows that $\langle x, t \rangle \in P$. If $x \in -P$, then $\langle x, t \rangle \in P \cap -P = \{0\}$, so that $x = 0$. It follows that $x \in P$. Hence, $P = P' \cap S(R)$. We have to verify that P' satisfies properties 1-6 from the definition of a $*$ -ordering.

Clearly, $1 \in P'$. If $x, y \in (P')^\times$ then there exist $s_1, s_2, t_1, t_2 \in P^\times$ such that $x \sim_v \langle s_1, t_1^{-1} \rangle$ and $y \sim_v \langle s_2, t_2^{-1} \rangle$. It follows that $\langle x, \langle t_1, t_2 \rangle \rangle \sim_v \langle s_1, t_2 \rangle \in P$ and $\langle y, \langle t_1, t_2 \rangle \rangle \sim_v \langle s_2, t_1 \rangle \in P$. Since $\langle s_1, t_2 \rangle \not\sim_v -\langle s_2, t_1 \rangle$, it follows that $\langle x + y, \langle t_1, t_2 \rangle \rangle \sim_v \langle s_1, t_2 \rangle + \langle s_2, t_1 \rangle \in P$ by assertion 2 of Lemma 1. Hence, $x + y \in P'$.

If $x \in (P')^\times$ and $d \in K^\times$, then there exist $s, t \in S(R)^\times$ and $a, b \in R^\times$ such that $x \sim_v \langle s, t^{-1} \rangle$ and $d \sim_v ab^{-1}$. By assertion 1 of Lemma 1 we have that $d^* \sim_v (b^*)^{-1}a^*$. It follows that $dxd^* \sim_v \langle asa^*, (btb^*)^{-1} \rangle$. Since $asa^* \in P$ and $btb^* \in P$, it follows that $dxd^* \in P'$.

If $x \in S(K)^\times$, then by the claim in the first paragraph of this proof there exist $s, t \in S(R)^\times$ such that $x \sim_v \langle s, t^{-1} \rangle$. If s and t have the same sign with respect to P , then $x \in P'$, otherwise $x \in -P'$.

If $x \in P' \cap -P'$ is nonzero, then there exist $s_1, s_2, t_1, t_2 \in P^\times$ such that $x \sim_v \langle s_1, t_1^{-1} \rangle$ and $-x \sim_v \langle s_2, t_2^{-1} \rangle$. It follows that $\langle x, \langle t_1, t_2 \rangle \rangle \sim_v \langle s_1, t_2 \rangle \in P$ and $\langle -x, \langle t_1, t_2 \rangle \rangle \sim_v \langle s_2, t_1 \rangle \in P$. It follows that $\langle s_1, t_2 \rangle \sim_v -\langle s_2, t_1 \rangle$, a contradiction with $P \cap -P = \{0\}$.

If $x, y \in (P')^\times$ and $x \sim_v \langle s_1, t_1^{-1} \rangle, y \sim_v \langle s_2, t_2^{-1} \rangle$, then $\langle \langle x, y \rangle, \langle t_1, t_2 \rangle \rangle \sim_v \langle \langle x, t_1 \rangle, \langle y, t_2 \rangle \rangle \sim_v \langle s_1, s_2 \rangle \in P$. Hence $\langle x, y \rangle \in P'$. Q.E.D.

2. ENVELOPING ALGEBRAS AND THEIR SKEW-FIELDS OF FRACTIONS

Let L be a complex Lie algebra and $U(L)$ its enveloping algebra. In [Li1] Lichtmann gives a construction of a skew field $D(L)$ which contains $U(L)$ and is generated by it. We recall the basic steps of the construction for later reference.

The *standard filtration* $\dots \subset F_1 \subset F_0 \subset F_{-1} \subseteq \dots$ of $U(L)$ is defined by $F_i = \{0\}$ if $i > 0$, $F_0 = \mathbb{C}$ and for $i = -k < 0$, F_i is a vector subspace of $U(L)$ generated by all products containing $\leq k$ elements from L . The mapping $v : U(L) \rightarrow \mathbb{Z} \cup \{+\infty\}$ defined by $v(x) = \sup\{n \mid x \in F_n\}$ is a valuation called the *standard valuation*. Let $U(L)[t, t^{-1}]$ be the ring of Laurent polynomials in a central variable t . Extend v to a valuation on $U(L)[t, t^{-1}]$ by $v(\sum c_i t^i) = \min\{v(c_i) + i\}$. Write $R := \{x \in U(L)[t, t^{-1}] \mid v(x) \geq 0\} \subset U(L)[t]$. Note that $U(L)[t, t^{-1}]$ is canonically isomorphic to the localization $R[1/t]$ and that the valuation v on $U(L)[t, t^{-1}]$ corresponds to the t -adic valuation on $R[1/t]$. For every $n = 1, 2, \dots$ write $R_n = R/t^n R$ and U_n for the projection of $U = R \setminus tR$ in R_n . It turns out that U_n is a regular Ore set in R_n for every n and we denote by $S_n = (R_n)_{U_n}$ its Ore localization. Write S for the inverse limit of S_n . The localization $D = S[1/t]$ is a skew field. There exists a natural embedding of R into S which extends to a natural embedding of $U(L)[t, t^{-1}] = R[1/t]$ into $D = S[1/t]$. Write $D(L)$ for the minimal subfield of D containing $U(L)$. The restriction of the t -adic valuation from $D = S[1/t]$ to $D(L)$ will be called the *standard valuation* on $D(L)$. It extends the standard valuation on $U(L)$ and it is known to be quasi-commutative.

Let L be a complex Lie algebra. A mapping $* : L \rightarrow L$ is an *involution* if for all $x, y \in L$ and $c \in \mathbb{C}$, $(x + y)^* = x^* + y^*$, $(cx)^* = \bar{c}x^*$, $[x, y]^* = [y^*, x^*]$, $x^{**} = x$.

Proposition 5. *Every involution on L has a canonical extension to $D(L)$ such that the standard valuation on $D(L)$ is a *-valuation.*

Proof. Every involution extends uniquely from L to its enveloping algebra $U(L)$ and the standard valuation on $U(L)$ is a *-valuation.

Setting $t^* = t$ we get an involution on $U(L)[t, t^{-1}]$ which induces an involution on R . Since $(t^n R)^* \subseteq t^n R$ for every n , we have an induced involution on $R/t^n R$ which will also be denoted by $*$. Note that the natural epimorphisms $\phi_n : R_{n+1} \rightarrow R_n$ are $*$ -homomorphisms. The same proof as in [CS2], Lemma 2.2 shows that the involution on R_n extends uniquely to an involution of $S_n = (R_n)_{U_n}$. It is easy to verify that the natural epimorphisms $\phi'_n : S_{n+1} \rightarrow S_n$ are $*$ -homomorphisms. It follows that the termwise involution on the inverse system of S_n and ϕ'_n induces an involution on its inverse limit S . Since $t^* = t$, this involution extends uniquely to an involution of $D = S[1/t]$. It is clear that the t -adic valuation on $S[1/t]$ is a $*$ -valuation which extends the t -adic valuation on $R[1/t]$. Since $D(L) \cap D(L)^*$ is a subfield of D containing $U(L) = U(L)^*$ and since $D(L)$ is the smallest subfield of D containing $U(L)$, it follows that $D(L) = D(L)^*$. Hence, $D(L)$ is a $*$ -subfield of D . Q.E.D.

Any involution on a complex Lie algebra L preserves the standard filtration of $U(L)$ and induces an involution on the corresponding graded ring $\bigoplus_i F_i/F_{i+1}$ which is isomorphic to the complex polynomial ring in $\dim L$ variables by the Poincare-Birkhoff-Witt Theorem.

Theorem 6. *Let L be a complex Lie algebra with involution. There exists a canonical 1-1-1 correspondence between*

- (1) *$*$ -orderings on $D(L)$ compatible with the standard valuation,*
- (2) *$*$ -orderings on $U(L)$ compatible with the standard valuation,*
- (3) *$*$ -orderings on the complex polynomial ring in $\dim L$ variables compatible with the total degree.*

Proof. The standard valuation v on $D(L)$ is a quasi-commutative $*$ -valuation. Hence the 1-1 correspondence between $*$ -orderings in 1 and 2 follows from Theorem 4. The 1-1 correspondence between $*$ -orderings in 2 and 3 follows from Proposition 2.5 in [M2]. Q.E.D.

If L is a finite dimensional Lie algebra, then $U(L)$ is an Ore domain and $D(L)$ is its skew field of fractions. Hence, Theorem 6 is a special case of Corollary 2.5 in [CS2] and Corollary 4.3 in [M2].

Let $L_{\mathcal{S}}$ be the free Lie algebra on a set \mathcal{S} , then its enveloping algebra $U(L_{\mathcal{S}})$ is isomorphic to the free associative algebra $\mathbb{C}\langle \mathcal{S} \rangle$ on \mathcal{S} (Theorem 0.5 in [R] or a remark after Theorem 2.6.6 in [Co]) and the field $D(L_{\mathcal{S}})$ is isomorphic to the free field $\Delta_{\mathcal{S}}$ on \mathcal{S} . (Theorem 1 in [Li2]). Note that the group ring $\mathbb{C}[F_{\mathcal{S}}]$ of the free group $F_{\mathcal{S}}$ on \mathcal{S} contains $\mathbb{C}\langle \mathcal{S} \rangle$ and is contained in $\Delta_{\mathcal{S}}$.

We can define an involution on $L_{\mathcal{S}}$ by $x^* = -x$ for every $x \in \mathcal{S}$ and $c^* = \bar{c}$ for every $c \in \mathbb{C}$. This involution extends to $\mathbb{C}\langle \mathcal{S} \rangle$, $\mathbb{C}[F_{\mathcal{S}}]$ and $\Delta_{\mathcal{S}}$ by Proposition 5. We have the following corollary of Theorem 6.

Corollary 7. *Let \mathcal{S} be an arbitrary set, $L_{\mathcal{S}}$ the free Lie algebra on \mathcal{S} with the involution defined above. The standard valuation v on $\Delta_{\mathcal{S}} = D(L_{\mathcal{S}})$ is a $*$ -valuation. We have a canonical 1-1-1-1 correspondence between*

- (1) $*$ -orderings on $\Delta_{\mathcal{S}}$ compatible with v ,
- (2) $*$ -orderings on $\mathbb{C}[F_{\mathcal{S}}]$ compatible with v ,
- (3) $*$ -orderings on $\mathbb{C}\langle \mathcal{S} \rangle$ compatible with v ,
- (4) $*$ -orderings on the complex polynomial ring in $\dim L_{\mathcal{S}}$ variables compatible with the total degree.

The following example solves a problem posed by T. Craven and T. Smith in [CS2].

Example. Let the involution on $A = \mathbb{Z}\langle a, b \rangle$ be defined by $a^* = b$ and $b^* = a$. By Example 3.2. in [CS2], we know that $*$ -orderings on A exist. We are able to give an explicit description of a class of $*$ -ordering on A . Let the involution on $A' = \mathbb{C}\langle a_1, a_2 \rangle$ be defined by $a_1^* = -a_1$ and $a_2^* = -a_2$ and $c^* = \bar{c}$ for every $c \in \mathbb{C}$. The homomorphism $\phi : A \rightarrow A'$ defined by $\phi(a) = \frac{1}{2}(a_1 + ia_2)$, $\phi(b) = \frac{1}{2}(-a_1 + ia_2)$ satisfies $\phi(a^*) = \phi(a)^*$ and $\phi(b^*) = \phi(b)^*$, hence it is an embedding of $*$ -rings. For any $*$ -ordering P on A' the set $\phi^{-1}(P)$ is a $*$ -ordering on A . Corollary 7 reduces the construction of $*$ -orderings on A' to the construction of $*$ -orderings on the complex polynomial ring $\mathbb{C}[x_k]_{k \in \mathbb{N}}$. Replacing variables x_k with $y_k = ix_k$, we get $S(\mathbb{C}[x_k]_{k \in \mathbb{N}}) = \mathbb{R}[y_k]_{k \in \mathbb{N}}$. Hence, there is a natural 1-1 correspondence between $*$ -orderings on $\mathbb{C}[x_k]_{k \in \mathbb{N}}$ and total orderings on $\mathbb{R}[y_k]_{k \in \mathbb{N}}$.

The following example shows that there exist $*$ -orderings on $\mathbb{C}\langle a, b \rangle$ which are not compatible with the standard valuation

Example. Let $R = \mathbb{C}\langle a, b \rangle$ with involution $*$ defined by $a^* = -a, b^* = -b$ and $c^* = \bar{c}$ for $c \in \mathbb{C}$. Let v be the standard valuation on R and P a $*$ -ordering on R compatible with v . Let $\deg : R \rightarrow \mathbb{Z} \cup \{-\infty\}$ be the total degree function in a and b . For every element $x \in R$ write $x = \text{lt}(x) + o$, where $\text{lt}(x)$ is a sum of all terms in x of total degree $\deg(x)$ and $\deg(o) < \deg(x)$. Write $Q = \{x \in S(R) \mid \text{lt}(x) \in P\}$. We claim that Q is a $*$ -ordering on R which is not compatible with v . We must verify properties 1-6 of $*$ -orderings.

Clearly, $1 \in Q$. Take any $x, y \in Q$. Note that $\text{lt}(x)$ and $\text{lt}(y)$ cannot cancel each other, since they both belong to P . If $\deg(x) = \deg(y)$, then

$\text{lt}(x+y) = \text{lt}(x) + \text{lt}(y) \in P$. If $\deg(x) \neq \deg(y)$, then $\text{lt}(x+y)$ is either $\text{lt}(x)$ or $\text{lt}(y)$. In all cases $\text{lt}(x+y) \in P$, so that $x+y \in Q$. If $x \in Q$ and $r \in R$, then $\text{lt}(rxr^*) = \text{lt}(r)\text{lt}(x)\text{lt}(r)^* \in \text{lt}(r)P\text{lt}(r)^* \subseteq P$, so that $rxr^* \in Q$. For every $x \in S(R)$, $\text{lt}(x) \in P \cup -P$, hence $x \in Q \cup -Q$. If $x \in Q \cap -Q$, then $\text{lt}(x) \in P \cap -P = \{0\}$. It follows that $x = 0$. If $x, y \in Q$, then $\text{lt}(xy) = \text{lt}(x)\text{lt}(y)$ and $\text{lt}(yx) = \text{lt}(y)\text{lt}(x)$ cannot cancel each other. Hence $\text{lt}(xy + yx) = \text{lt}(x)\text{lt}(y) + \text{lt}(y)\text{lt}(x) \in P$, so that $xy + yx \in Q$.

Finally Q is not v -compatible. Take $x = [a, b]^2$ and $y = i[a, [a, [a, [a, b]]]]$. We have $x, y \in S(R)$, $v(x) = -2 < -1 = v(y)$ and $\deg(x) = 4 < 5 = \deg(y)$. Changing signs if necessary, we may assume that $x, y \in P$. It follows that $\text{lt}(y-x) = y \in P$, so that $y-x \in Q$. If Q were compatible with v , then $x-y \in Q$, a contradiction.

3. OPEN PROBLEMS

- (1) Do $*$ -orderings on a free algebra which are not compatible with the standard valuation extend to the corresponding free field? Is the extension unique?
- (2) Do $*$ -orderings always extend from $*$ -domains to their universal fields of fractions.
- (3) Is every $*$ -domain which has a $*$ -ordering embeddable in a $*$ -field? If not find nontrivial sufficient conditions for embeddability.

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