ORDERINGS AND $*$-ORDERINGS ON COCOMMUTATIVE HOPF ALGEBRAS

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Abstract. Our aim is to construct new examples of totally ordered and $*$-ordered noncommutative integral domains. We will discuss the following classes of rings: enveloping algebras $U(L)$, group rings $\Bbbk G$ and smash products $U(L)\#_\varphi \Bbbk G$. All of them are examples of Hopf algebras. Characterizations of orderability for enveloping algebras and group rings and of $*$-orderability for enveloping algebras have been found before and will be recalled in the article. Our main results are: for $\Bbbk = \mathbb{R}$ and L finite-dimensional, we characterize the orderability of $U(L)\#_\varphi \Bbbk G$; for $\Bbbk = \mathbb{C}$, we give a necessary and a sufficient condition for $*$-orderability of $\Bbbk G$ (G orderable, resp., G residually 'torsion-free nilpotent'). Moreover, for $\Bbbk = \mathbb{C}$ and L finite-dimensional, we reduce the problem of characterizing the $*$-orderability of $U(L)\#_\varphi \Bbbk G$ to the problem of characterizing the $*$-orderability of $\Bbbk G$. The latter remains open.

1. Introduction

Let R be a ring. A subset $P \subset R$ is called an *ordering* if $P + P \subset P$, $P \cdot P \subset P$, $P \cup -P = R$, and $\operatorname{supp} P := P \cap -P$ is a prime ideal of R. The set of all orderings of R is called the *real spectrum* of R. The study of real spectra of noncommutative rings is known as *noncommutative real algebraic geometry*. Rings with nonempty real spectrum are called *semireal*. Orderings with zero support are of special importance. Rings that admit such an ordering are called *real*.

We observed that many real rings carry the additional structure of a Hopf algebra, e.g., group rings, universal enveloping algebras (see [8]), quantum affine rings, quantized enveloping algebras, and quantized function algebras (see [3]). This motivates the question of finding criteria for reality and semireality of an arbitrary Hopf algebra (viewed as a ring). In the present paper we set ourselves a more modest task of determining when a cocommutative Hopf algebra is real. The results will be given in Section 3. The basics about Hopf algebras are recalled below and the basics about ordered structures in Section 2.

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In the context of rings with involution it seems more natural to work with the so called $*$-orderings. We will recall the basic facts from [7],[4] in Section 4. Although quantum groups have several interesting involutions, they almost never carry a $*$-ordering. We will explain this phenomenon in Section 6. However, we are able to construct a large class of cocommutative Hopf algebras with involution which admit $*$-orderings — see Sections 5 and 6.

A few words about notation. $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ have their usual meaning. $\mathbb{N}$ and $\mathbb{Z}_+$ denote the sets of positive and nonnegative integers, resp. Throughout the paper $\Bbbk$ will be a fixed ground field. All vector spaces, algebras, tensor products, etc. will be assumed over $\Bbbk$ unless indicated otherwise. Since we are interested in orderings, almost everywhere $\Bbbk$ will be of characteristic zero, and for some of our results we will have to take $\Bbbk = \mathbb{R}$ or $\mathbb{C}$.

For general theory of Hopf algebras we refer the reader to [11].

Definition. $(H, m, u, \Delta, \varepsilon)$ is called a *bialgebra* if

1) (H, m, u) is a unital associative algebra, i.e., $m : H \otimes H \to H$ (multiplication) and $u : \Bbbk \to H$ (unit) are linear maps such that $m \circ (m \otimes \mathrm{id}_H) = m \circ (\mathrm{id}_H \otimes m)$ and $m \circ (u \otimes \mathrm{id}_H) = m \circ (\mathrm{id}_H \otimes u) = id_H$,
2) (H, Δ, ε) is a counital coassociative coalgebra, i.e., $\Delta : H \to H \otimes H$ (comultiplication) and $\varepsilon : H \to \Bbbk$ (counit) are linear maps such that $(\Delta \otimes \mathrm{id}_H) \circ \Delta = (\mathrm{id}_H \otimes \Delta) \circ \Delta$ and $(\varepsilon \otimes \mathrm{id}_H) \circ \Delta = (\mathrm{id}_H \otimes \varepsilon) \circ \Delta = \mathrm{id}_H$, and
3) Δ, ε are homomorphisms of unital algebras, or, equivalently, m, u are homomorphisms of counital coalgebras.

H is called a *Hopf algebra* if there exists a linear map $\mathcal{S} : H \to H$ (antipode) such that $m \circ (\mathcal{S} \otimes \mathrm{id}_H) \circ \Delta = m \circ (\mathrm{id}_H \otimes \mathcal{S}) \circ \Delta = u \circ \varepsilon$. This map is uniquely determined and it is an anti-homomorphism of bialgebras. H is *commutative* if $m \circ \tau = m$ and *cocommutative* if $\tau \circ \Delta = \Delta$ where τ is the flip $a \otimes b \mapsto b \otimes a$.

A common notation for $\Delta : H \to H \otimes H$ is $\Delta h = \sum h_{(1)} \otimes h_{(2)}$ (the so called "Σ-notation"). The simplest examples of Hopf algebras are group algebras and universal enveloping algebras.

Example 1.1. Let G be a unital semigroup and $H = \Bbbk G$. Then Δ and ε defined by $\Delta : g \mapsto g \otimes g$ and $\varepsilon : g \mapsto 1$ for $g \in G$ make H a bialgebra. H is a Hopf algebra iff G is a group. Then $\mathcal{S}(g) = g^{-1}$ for $g \in G$.

Example 1.2. Let L be a Lie algebra and $H = U(L)$. The maps $\Delta : x \mapsto x \otimes 1 + 1 \otimes x$ and $\varepsilon : x \mapsto 0$ for $x \in L$ extend uniquely to the entire H. They make H a Hopf algebra with $\mathcal{S}(x) = -x$ for $x \in L$.

The above two examples are cocommutative and in fact every pointed cocommutative Hopf algebra can be built from them as follows.

Definition. Let H be a Hopf algebra.

1) An nonzero element $g \in H$ is *group-like* if $\Delta g = g \otimes g$. The group-like elements of H form a multiplicative subgroup denoted by $G(H)$.
2) An element $x \in H$ is called *primitive* if $\Delta x = x \otimes 1 + 1 \otimes x$. The primitive elements of H form a Lie subalgebra denoted by $P(H)$.
3) H is *pointed* if every simple subcoalgebra of H is one-dimensional. (This condition is automatic if H is cocommutative and $\Bbbk$ is algebraically closed.) Every one-dimensional subcoalgebra is spanned by a group-like element.
4) H is *cosemisimple* if H is the sum of its simple subcoalgebras.

Definition. Let H be a Hopf algebra and A a left H-module algebra, i.e., A is a left H-module via $\varphi : H \to \mathrm{End}_{\Bbbk}(A) : h \mapsto \varphi_h$ such that $\varphi_h(ab) = \sum \varphi_{h_{(1)}}(a)\varphi_{h_{(2)}}(b)$ and $\varphi_h(1) = \varepsilon(h)1$ for $h \in H$ and $a, b \in A$. Then the *smash product* $A\#_{\varphi}H$ is the vector space $A \otimes H$ endowed with multiplication

$$(a\#h)(b\#k) = \sum a\varphi_{h_{(1)}}(b)\#h_{(2)}k \text{ for } a, b \in A \text{ and } h, k \in H,$$

where we write $a\#h$ for $a \otimes h$, etc.

It is convenient to identify the algebras A and H with their isomorphic copies $A\#1$ and $1\#H$, resp., inside $A\#H$. Then the multiplication on $A\#H$ is defined by the commutation rule $hb = \sum \varphi_{h_{(1)}}(b)\#h_{(2)}$ for $h \in H$, $b \in A$.

In the case $H = \Bbbk G$, the definition of an H-module algebra just says that elements of G act as algebra automorphisms on A, i.e., $\varphi : G \to \mathrm{Aut}(A)$, and the commutation rule for $A\#_{\varphi}H$ simplifies to $gb = \varphi_g(b)g$, i.e., $\varphi_g(b) = gbg^{-1}$ for $g \in G$, $b \in A$.

Smash products arise very frequently in the theory of Hopf algebras. A classical example is the following structure theorem for pointed cocommutative Hopf algebras (see e.g. [11, Section 5.6]).

Theorem. *Let H be a pointed cocommutative Hopf algebra over a field $\Bbbk$ of characteristic zero. Then H is isomorphic to $U(L)\#_{\varphi}\Bbbk G$ as an algebra, where $G = G(H)$, $L = P(H)$, and $\varphi(G) \subset \mathrm{Aut}(U(L))$ preserves $L \subset U(L)$. The isomorphism $U(L)\#_{\varphi}\Bbbk G \to H$ is defined by $x\#g \mapsto xg$ for $x \in L$, $g \in G$.* □

2. Orderings and Valuations

The aim of this section is to recall the definitions and basic facts about ordered algebraic structures. We also introduce some examples that we will need later. Most of the results in this section are well-known to specialists.

A *(total) order on a semigroup S* is a total order on the set S which is preserved by left and right translations, i.e., $a \le b$ implies $ac \le bc$ and $ca \le cb$, for all $a, b, c \in S$.

An *ordering of a group G* is a subset P of G such that $P \cap P^{-1} = \{1\}$, $P \cup P^{-1} = G$, $P \cdot P \subset P$, and $gPg^{-1} \subset P$ for every $g \in G$. There is a one-to-one correspondence between the orderings of the group G and the total orders on the group G, given by $a \leq b$ iff $ba^{-1} \in P$.

If A is an abelian group (written additively), then the axioms for an ordering P simplify to $P \cap -P = \{0\}$, $P \cup -P = A$, $P + P \subset P$.

A subset P of a prime ring R is a *ring ordering* (with zero support) if P is an ordering of the additive group $(R, +)$ and $P \cdot P \subset P$.

Let $\Bbbk$ be a field with fixed ordering $\Bbbk_+$. An ordering of a $\Bbbk$-vector space V is an ordering of the abelian group $(V, +)$ which is closed under multiplication by $\Bbbk_+$.

An ordering of a $\Bbbk$-algebra R is an ordering of both the ring R and the $\Bbbk$-vector space R.

Orderable semigroups. Clearly, a subsemigroup of an orderable semigroup is orderable. The direct product of a family of orderable semigroups is orderable. Indeed, the index set can be well-ordered by the axiom of choice, hence the direct product can be ordered lexicographically.

We will be interested only in semigroups S with *cancellation property*: $ac = bc \Rightarrow a = b$ and $ca = cb \Rightarrow a = b$, for all $a, b, c \in S$. We will also assume that our semigroups have the identity element.

If a cancellation semigroup S satisfies the *right Ore condition*:

$$\forall a, b \in S \quad \exists c, d \in S \, : \, ac = bd,$$

then S embeds into its group of right quotients $Q_r(S) = \{ab^{-1} \mid a, b \in S\}$. By a result of Weinert [14, Corollary to Theorem 2], any order on S can be uniquely extended to an order on $Q_r(S)$.

Orderable groups. In general, orderability of groups is not preserved under extensions. Namely, the group $\langle x, y | xyx^{-1} = y^{-1}\rangle$ is an extension of $\mathbb{Z}$ by $\mathbb{Z}$, but it is not orderable. However, Lemma 2.1 implies that orderability is preserved by central extensions.

Lemma 2.1. *Let G be a group, $N \lhd G$. If G/N is orderable and N has an ordering which is invariant under conjugation by elements of G, then G is also orderable.*

Proof. For $a, b \in G$, define $a < b$ iff either $\pi(a) < \pi(b)$ or $\pi(a) = \pi(b)$ and $ab^{-1} < 1$ in N where $\pi : G \to G/N$ is the natural homomorphism. The verification that this ordering of G is invariant under left and right multiplications is straightforward. □

Every orderable group is torsion-free. The converse fails for $\langle x, y | xyx^{-1} = y^{-1}\rangle$. However, we have the following well-known partial converse (we include a proof for completeness).

Proposition 2.2. *Every torsion-free nilpotent group is orderable.*

Proof. Torsion-free abelian groups are orderable (see the following subsection).

Suppose now that G is torsion-free nilpotent of class c. Let $\zeta_i G$, $i = 0, \dots, c$ be the upper central series of G, i.e. $\zeta_0 G = \{1\}$ and $\zeta_{i+1}G/\zeta_i G = Z(G/\zeta_i G)$ and $\zeta_c G = G$. Clearly, $\zeta_1 G/\zeta_0 G$ is torsion-free. Assume that $\zeta_i G/\zeta_{i-1}G$ is torsion-free and pick $x \in \zeta_{i+1}G$ such that $x^n \in \zeta_i G$. For every $y \in G$ we have that $[x, y] \in \zeta_i G$ and $[x, y]^n \equiv [x^n, y] \equiv 1 \mod \zeta_{i-1}G$. Hence, $[x, y] \in \zeta_{i-1}G$ for every y by the induction hypothesis. It follows that $x \in \zeta_i G$. Therefore, $\zeta_{i+1}G/\zeta_i G$ is torsion-free.

Clearly $G/\zeta_c G = \{1\}$ is orderable. Suppose that $G/\zeta_i G$ is orderable for some i. Since $\zeta_i G/\zeta_{i-1}G$ is a torsion-free abelian group, it is orderable by the first paragraph. Note that $G/\zeta_{i-1}G$ is a central extension of $\zeta_i G/\zeta_{i-1}G$ by $G/\zeta_i G$, hence it is orderable by the remark above. By induction, it follows that $G = G/\zeta_0 G$ is orderable. □

Example 2.3. Let $\Bbbk$ be an ordered field and n a positive integer. Let $UT_n(\Bbbk)$ be the group of upper unitriangular $n \times n$ matrices over $\Bbbk$ (i.e., upper triangular matrices with diagonal entries equal to 1). It is well-known that $UT_n(\Bbbk)$ is torsion-free nilpotent, hence it is orderable by Proposition 2.2. To construct an explicit ordering on $G = UT_n(\Bbbk)$, we can use the fact that $\zeta_i G$ consists of all upper unitriangular matrices whose k-th superdiagonals are zero for $k < i$. In particular, $\zeta_i G/\zeta_{i-1}G$ is isomorphic to the additive group $\Bbbk^i$, which can be ordered lexicographically. Then the ordering given by Proposition 2.2 can be described explicitly: $[a_{rs}] > [b_{rs}]$ if and only if the first nonzero element in the sequence

$$a_{12}-b_{12}, a_{23}-b_{23}, \dots, a_{n-1,n}-b_{n-1,n}; a_{13}-b_{13}, \dots, a_{n-2,n}-b_{n-2,n}; \dots; a_{1n}-b_{1n}$$

is positive.

Example 2.4. Let $\Bbbk$ be an ordered field and n a positive integer. Let $PT_n(\Bbbk)$ be the group of upper triangular matrices whose diagonal entries are positive. Note that $UT_n(\Bbbk)$ is a normal subgroup of $PT_n(\Bbbk)$ and that $PT_n(\Bbbk)/UT_n(\Bbbk)$ is isomorphic to the multiplicative group $\Bbbk^n_{>0}$, which can be ordered lexicographically. The ordering of $UT_n(\Bbbk)$ constructed in Example 2.3 is invariant under the conjugation by the elements from $PT_n(\Bbbk)$. Hence $PT_n(\Bbbk)$ is orderable by Lemma 2.1. As above, we have $[a_{rs}] > [b_{rs}]$ if and only if the first nonzero element in the sequence

$$a_{11} - b_{11}, a_{22} - b_{22}, \dots, a_{nn} - b_{nn}; a_{12} - b_{12}, \dots, a_{n-1,n} - b_{n-1,n}; \dots; a_{1n} - b_{1n}$$

is positive.

Example 2.5. For every positive integer n, $G_n := \langle x, y \mid xyx^{-1} = y^n \rangle$ is an orderable group.

Proof. We will construct a realization of G_n which is easier to work with. Let Q_n be the subgroup of $(\mathbb{Q}, +)$ that consists of the elements of the form mn^l where $m, l \in \mathbb{Z}$. Consider the semidirect product $\mathbb{Z} \ltimes Q_n$ where $k \in \mathbb{Z}$ acts on Q_n by $q \mapsto qn^k$. Obviously, $x \mapsto (0, 1)$ and $y \mapsto (1, 0)$ define a homomorphism $\varphi : G_n \to \mathbb{Z} \ltimes Q_n$. Replacing xy^l by $y^{ln}x$ and $y^l x^{-1}$ by

$x^{-1}y^{ln}$, we can rewrite every word in $x^{\pm 1}$ and $y^{\pm 1}$ in the form $x^{-k}y^l x^m$ where $k, m \geq 0$. Since $\varphi(x^{-k}y^l x^m) = (m-k, \frac{l}{n^k})$, it follows that φ is one-to-one and onto. We can order $\mathbb{Z} \ltimes Q_n$ as in Lemma 2.1: $(a,b) > (c,d)$ if and only if either $a > c$ or $a = c$ and $b > d$. Hence G_n is orderable. □

Recall that if $\mathfrak{A}$ is a property of groups, then we say that a group G is *residually* $\mathfrak{A}$ if there is a family $\{N_i\}$ of normal subgroups of G such that $\cap_i N_i = \{1\}$ and G/N_i have the property $\mathfrak{A}$ for all i.

Remark 2.6. Every residually orderable group is orderable. In particular, every residually 'torsion-free nilpotent' group is orderable.

Proof. Let N_i be a family of normal subgroups such that $\cap_i N_i = \{1\}$ and G/N_i is orderable for each i. Then the product $\Pi_i G/N_i$ is also orderable. Since the natural mapping $G \to \Pi_i G/N_i$ is an embedding, it follows that G is orderable. □

Let $\gamma_i(G)$, $i = 1, 2, \ldots$, be the lower central series of G, i.e., $\gamma_1(G) = G$, $\gamma_{i+1}(G) = (\gamma_i(G), G)$. Then the sets

$$\sqrt{\gamma_i(G)} := \{g \in G \mid \exists m \in \mathbb{N} : \; g^m \in \gamma_i(G)\}$$

are normal subgroups of G (see e.g. [12, Lemma IV.1.3]). Clearly, the quotient groups $G/\sqrt{\gamma_i(G)}$ are nilpotent and torsion-free. Therefore, G is residually 'torsion-free nilpotent' iff $\cap_{i=1}^{\infty}\sqrt{\gamma_i(G)} = \{1\}$.

It is well-known that free groups are residually 'torsion-free nilpotent'. However, the groups $PT_n(\Bbbk)$ and G_n are not. Indeed, for $G = PT_n(\Bbbk)$, $n > 1$, we have $\gamma_i(G) = UT_n(\Bbbk)$ for all $i > 1$, so $PT_n(\Bbbk)$ is not even residually nilpotent. As to $G = G_n = \langle x, y \,|\, xyx^{-1} = y^n \rangle$, $n > 1$, the relations $y^{l(n-1)} = (x, y^l)$ imply that $y^{(n-1)^i} \in \gamma_{i+1}(G)$ for every i. Hence $y \in \cap_{i=1}^{\infty}\sqrt{\gamma_i(G)}$ and G is not residually 'torsion-free nilpotent'. (In fact, for $n = 2$, G is not even residually nilpotent.)

Ordered abelian groups. An abelian group is orderable if and only if it is torsion-free. Every torsion-free abelian group A can be embedded into its divisible hull $A \otimes_{\mathbb{Z}} \mathbb{Q}$, which is a vector space over $\mathbb{Q}$. Moreover, the mapping $P \mapsto \mathbb{Q}_+ P$ defines a one-to-one correspondence between orderings of A and vector space orderings of $A \otimes_{\mathbb{Z}} \mathbb{Q}$.

Let A be an abelian group and Γ a totally ordered set. Fix the element $\infty \notin \Gamma$ and declare $\gamma < \infty$ for all $\gamma \in \Gamma$. A mapping $v : A \to \Gamma \cup \{\infty\}$ is a *valuation* if for any $a, b \in A$:

1) $v(a) = \infty$ if and only if $a = 0$,
2) $v(a+b) \geq \min\{v(a), v(b)\}$.

We say that a valuation v is *compatible* with an ordering P if for any $a, b \in A$ such that $a \in P$ and $v(b) > v(a)$ we have $a + b \in P$.

If v, w are two valuations on A (not necessarily with the same Γ), we will say that w is *finer* than v (or, equivalently, v is *coarser* than w) if

$w(a) \geq w(b) \Rightarrow v(a) \geq v(b)$. For every ordering P, there exists the finest valuation v_P compatible with P. It is constructed in the following way.

Let P be an ordering of an abelian group A. For any $a \in A$ write $|a| = a$ if $a \in P$ and $|a| = -a$ if $a \in -P$. For $a, b \in A$, write $a \preceq b$ if and only if $|b| \leq n|a|$ for some $n \in \mathbb{N}$. Write $a \sim b$ iff $a \preceq b$ and $b \preceq a$. Then $\sim$ is an equivalence relation on A. It is called the *Archimedean equivalence* and its classes are called the *Archimedean classes* of the ordering. The Archimedean class of zero is denoted by ∞, it has only one element. The set of Archimedean classes of nonzero elements is denoted by Γ_P. The relation $\preceq$ defines a total order on the set $\Gamma_P \cup \{\infty\}$, denoted by $\leq$. The *natural valuation* of P is the map $v_P : A \to \Gamma_P \cup \{\infty\}$ that sends each element to its Archimedean class. By construction, a valuation v on A is compatible with P iff v is coarser than v_P. The group A with ordering P is called *Archimedean* if all nonzero elements of A are Archimedean equivalent, i.e., Γ_P consists of a single element.

Remark 2.7. Any commutative cancellation semigroup S is canonically embedded into its group of quotients $A = Q(S)$. As noted earlier, any order on S uniquely extends to A. Consequently, the above definitions of Archimedean classes, natural valuation, etc. can be extended to ordered commutative cancellation semigroups.

Remark 2.8. For any elements $a, b \in P$ with $v_P(a) = v_P(b)$, there exists a unique real number $r \neq 0$ such that $r \in [\frac{m}{n}, \frac{m+1}{n}]$ implies $mb \leq na \leq (m+1)b$, for any $m \in \mathbb{Z}$, $n \in \mathbb{N}$.

Ordered vector spaces. Let $\mathbb{k}$ be a field with a fixed ordering $\mathbb{k}_+$ and let V be a $\mathbb{k}$-vector space. Every ordered basis $\{e_i\}_{i\in I}$ of V defines an ordering P by $0 \neq \sum_{i\in I} c_i e_i \in P$ (finite sum) if and only if the first nonzero c_i belongs to $\mathbb{k}_+$. Note that $\Gamma_P = I \times \Gamma_{\mathbb{k}_+}$ with lexicographic order and that $v_P(\sum_{i\in I} c_i e_i) = (i_0, v_{\mathbb{k}_+}(c_{i_0}))$ where $i_0 = \min\{i \,|\, c_i \neq 0\}$. If $\mathbb{k}$ is Archimedean (i.e., a subfield of $\mathbb{R}$), then $\Gamma_{\mathbb{k}_+}$ is a singleton, hence $\Gamma_P = I$.

If V is a finite-dimensional vector space over $\mathbb{R}$, then the construction above gives all orderings of V. Namely, let Q be an ordering on V. For any $a, b \in V$ with $v_Q(a) = v_Q(b)$, there is $r \in \mathbb{R}$ such that $v_Q(a - rb) > v_Q(a)$ (see Remark 2.8). Therefore, starting with any basis of V, we can transform it into a basis $e_1, \ldots, e_n$ such that $v_Q(e_1) < \ldots < v_Q(e_n)$. Since $v_Q(-x) = v_Q(x)$ for every $x \in V$, we may assume that $e_1, \ldots, e_n \in Q$. Since v_Q is compatible with Q, an element $\sum_{i=1}^n c_i e_i \in V$ belongs to Q if and only if the first nonzero c_i is positive.

Ordered rings. Every orderable prime ring is a domain by [6, Proposition 2.1]. So in this paper we will be interested in domains only. By [5], a domain R is orderable if and only if for any $a_1, \ldots, a_k \in R$ which are permuted products of squares, $a_1 + \ldots + a_k = 0$ implies that $a_1 = \ldots = a_k = 0$. (An example of a permuted product of squares: $xyzyxz$, which is a permutation of $x^2y^2z^2$.) Clearly, R must be of characteristic zero. The mapping $P \mapsto$

$\mathbb{Q}_+P$ defines a one-to-one correspondence between orderings of a ring R and orderings of the $\mathbb{Q}$-algebra $R \otimes_{\mathbb{Z}} \mathbb{Q}$. If the multiplicative semigroup $R \setminus \{0\}$ satisfies the right Ore condition, then the domain R embeds in its skew-field of right quotients $Q_r(R) = \{ab^{-1} \mid a, b \in R,\, b \neq 0\}$. By a result of Albert [1], any ordering of R can be uniquely extended to an ordering of $Q_r(R)$.

Let R be a domain and $(\Gamma, +, \leq)$ an ordered semigroup with cancellation property (not necessarily abelian, but written additively). As in the previous subsection, pick $\infty \notin \Gamma$ and extend the ordering of Γ to $\Gamma \cup \{\infty\}$ so that ∞ is the largest element. A mapping $v : R \to \Gamma \cup \{\infty\}$ is a *valuation* of the domain R if v is a valuation of the abelian group $(R, +)$ and $v(ab) = v(a) + v(b)$ for all nonzero $a, b \in R$. Replacing Γ with $v(R \setminus \{0\})$, we can assume that $v : R \setminus \{0\} \to \Gamma$ is onto. Then Γ is called the *value semigroup* of v.

For every ordering P of a domain R, there exists the finest valuation on R compatible with P. It is constructed in the same way as for abelian groups. Note that the set Γ_P of nonzero Archimedean classes is a semigroup for $v(a) + v(b) := v(ab)$. Clearly, Γ_P has the cancellation property and the ordering of Γ_P is preserved by left and right translations. If R is unital (as we will always assume), then $v(1)$ is the zero element of Γ_P.

Now let R be a domain, Γ a totally ordered semigroup and $v : R \to \Gamma \cup \{\infty\}$ a valuation. The *associated graded ring* $\mathrm{gr}(R, v)$ is defined by

$$\mathrm{gr}(R, v) = \oplus_{\gamma \in \Gamma} \overline{R}_\gamma,$$

where $\overline{R}_\gamma = R_\gamma / R_\gamma^+$, $R_\gamma = \{a \in R \mid v(a) \geq \gamma\}$, $R_\gamma^+ = \{a \in R \mid v(a) > \gamma\}$, with componentwise addition and multiplication induced by $(\overline{a}, \overline{b}) \mapsto \overline{ab}$ for $a \in R_\alpha$, $b \in R_\beta$ (here $\overline{a}$ denotes the coset $a + R_\alpha^+$, etc.). Clearly, $\mathrm{gr}(R, v)$ is a domain with valuation $\overline{v} = \mathrm{gr}(v) : \mathrm{gr}(R, v) \to \Gamma \cup \{\infty\}$ defined by $\overline{v}(\sum_\alpha \overline{a}_\alpha) = \gamma$ where γ is the least α such that $\overline{a}_\alpha \neq 0$. The following observation is very useful [9, Theorem 2.1].

Proposition 2.9. *There is a natural one-to-one correspondence $P \mapsto \overline{P}$ between orderings of R compatible with v and orderings of $\mathrm{gr}(R, v)$ compatible with $\overline{v}$. Namely, $\overline{P} \setminus \{0\}$ consists of all nonzero $a = \sum_\alpha \overline{a}_\alpha$ such that $a_\gamma \in P$ where $\gamma = \overline{v}(a)$ and, conversely, $P \setminus \{0\}$ consists of all nonzero b such that $\overline{b} := b + R_\beta^+$, where $\beta = v(b)$, belongs to $\overline{P}$.* □

Remark 2.10. Let α be an automorphism of R that preserves v, i.e., $v \circ \alpha = v$. Then α induces an automorphism $\overline{\alpha}$ of $\mathrm{gr}(R, v)$ such that $\overline{v} \circ \overline{\alpha} = \overline{v}$. Since $\overline{\alpha}(\overline{a}) = \overline{\alpha(a)}$ for every $a \in R$, it follows that $\overline{\alpha}(\overline{P}) \subset \overline{P}$ if and only if $\alpha(P) \subset P$.

Ordered algebras. We consider two examples that are of particular interest in the context of Hopf algebras.

Example 2.11. Let $\Bbbk$ be a domain and G a unital semigroup with cancellation property. The semigroup algebra $\Bbbk G$ is orderable if and only if $\Bbbk$ and G are orderable.

Proof. If P is an ordering of $\Bbbk G$, then $P \cap \Bbbk$ is an ordering of $\Bbbk$, and $g_1 \leq g_2 \Leftrightarrow |g_2| - |g_1| \in P$ defines an ordering of G. Conversely, suppose G is an ordered unital semigroup with cancellation and $\Bbbk$ is an ordered domain, then we can construct an ordering P on $\Bbbk G$ in the following way. Every nonzero $a \in \Bbbk G$ can be expressed uniquely as $a = a_1 g_1 + \cdots + a_r g_r$ where $g_1 < \cdots < g_r$ are in G and $a_k \neq 0$, $k = 1, \ldots r$, are in $\Bbbk$. We declare $a \in P$ iff $a_1 > 0$. Note that $\Gamma_P = G \times \Gamma_{\Bbbk_+}$ with lexicographic order, and $v_P(\sum_{k=1}^r a_k g_k) = (g_1, v_{\Bbbk_+}(a_1))$. □

Example 2.12. Let L be a Lie algebra over a field $\Bbbk$. Then the universal enveloping algebra $U(L)$ is orderable iff $\Bbbk$ is orderable.

Proof. If $\Bbbk$ is ordered, then we can always construct an ordering on $U(L)$ as follows. Pick a totally ordered basis $\{x_i\}_{i \in I}$ of L. By Poincaré-Birkhoff-Witt Theorem, the monomials

$$x_{i_1} \ldots x_{i_n} \text{ where } x_{i_1} \leq \cdots \leq x_{i_n}, \quad n \geq 0,$$

form a basis of $U(L)$. We define a total ordering on the monomials as follows. We declare $x_{i_1} \ldots x_{i_n} < x_{j_1} \ldots x_{j_m}$ to hold if either $n > m$ (note the reversed inequality!) or if $n = m$ and $x_{i_1} \ldots x_{i_n} <_{\text{lex}} x_{j_1} \ldots x_{j_m}$ where $<_{\text{lex}}$ stands for the usual lexicographic order on words.

Now using this ordering of the PBW basis, we can order $U(L)$ by the sign of the lowest coefficient. Namely, every nonzero element $z \in U(L)$ can be written uniquely as $z = c_1 M_1 + \cdots + c_r M_r$ where $c_k \in \Bbbk$ are nonzero and $M_1 < \cdots < M_r$ are PBW monomials. We declare $z \in P$ iff $c_1 > 0$. One can verify directly that P is indeed an ordering of $U(L)$. Alternatively, one can use Proposition 2.9 as follows. Observe that $-\deg : U(L) \to \mathbb{Z}_- \cup \{\infty\}$ is a valuation, where $\deg z$ is the highest degree of PBW monomials appearing in the expression for nonzero $z \in U(L)$ and $\deg(0) := -\infty$. (Of course, deg does not depend on the choice of a basis for L.) Clearly, $\operatorname{gr}(U(L), -\deg)$ is isomorphic to the algebra of polynomials $\Bbbk[x_i | i \in I]$. The ordering of monomials that we constructed gives rise to an ordering $\overline{P}$ of $\Bbbk[x_i]$. Since $\overline{P}$ is compatible with $-\deg$, we conclude that P is indeed an ordering of $U(L)$ by Proposition 2.9.

Note that $\Gamma_P = \Gamma \times \Gamma_{\Bbbk_+}$ with lexicographic order, where $(\Gamma, +, <)$ is the free commutative semigroup generated by the symbols $w(x_i)$ with the ordering induced by the monomial ordering, i.e., $\sum_{s=1}^n k_s w(x_{i_s}) < \sum_{t=1}^m l_t w(x_{j_t})$ iff $x_{i_1}^{k_1} \ldots x_{i_n}^{k_n} < x_{j_1}^{l_1} \ldots x_{j_m}^{l_m}$. (In other words, Γ is just the semigroup of monomials, but written additively.) The natural valuation is given by $v_P(\sum_{k=1}^r c_k M_k) = (w(M_1), v_{\Bbbk_+}(c_1))$ where the map w sends each monomial $M = x_{i_1}^{k_1} \cdots x_{i_n}^{k_n}$ to $\sum_{s=1}^n k_s w(x_{i_s})$. □

In Section 4, we will develop an analog of the above construction of an ordering for $\mathbb{N}$-graded Lie algebras in such a way that the ordering will be compatible with the valuation $v : U(L) \to \mathbb{Z}_+ \cup \{\infty\}$ determined by the grading, i.e., $v(z)$ is the lowest degree of the homogeneous components of z with respect to the grading of $U(L)$ induced by the given grading of L.

3. Orderability of Smash Products

The aim of this section is to find necessary and sufficient conditions for the orderability of smash products. Proposition 3.1 is a general result and Theorem 3.7 gives a more precise result in a special case.

Proposition 3.1. *Let G be a group, A a $\Bbbk$-algebra and φ an action of G on A. Then $A\#_\varphi \Bbbk G$ is an orderable domain if and only if*

1) *G is an orderable group, and*
2) *A is a domain that admits an ordering P_0 such that $\varphi_g(P_0) \subset P_0$ for every $g \in G$.*

Proof. The necessity of condition 1) is clear. Also if $A\#_\varphi \Bbbk G$ is a domain, then so is its subalgebra A. If P is an ordering of $A\#_\varphi \Bbbk G$, then $P_0 = P \cap A$ is an ordering of A. For every $x \in A$ and every $g \in G$, xg and gx have the same sign with respect to P. It follows that $x \in P_0$ if and only if $\varphi_g(x) = gxg^{-1} \in P_0$.

Assume now that 1) and 2) hold. Every nonzero element $z \in A\#_\varphi \Bbbk G$ can be written uniquely as $z = a_1\#g_1 + \cdots + a_k\#g_k$ with $g_1 < \cdots < g_k$ and $a_1, \ldots, a_k \in A$ nonzero. If $0 \neq x = \sum_{i=1}^k a_i\#g_i$ and $0 \neq y = \sum_{j=1}^l a'_j\#g'_j$, then $xy = a_1(g_1 a'_1 g_1^{-1})\#g_1 g'_1 + o$, where $a_1(g_1 a'_1 g_1^{-1}) \neq 0$ and o is a sum of terms $b\#g$ with $g > g_1 g'_1$. Thus $A\#_\varphi \Bbbk G$ is a domain.

Set $P = \{0\} \cup \{z \in A\#_\varphi \Bbbk G \setminus \{0\} \mid a_1 \in P_0\}$. It is clear that $P + P \subset P$, $P \cap -P = \{0\}$ and $P \cup -P = A\#_\varphi \Bbbk G$. Now suppose $x, y \in P$, then $a_1, a'_1 \in P_0$. Since $P_0 \cdot P_0 \subset P_0$ and $g_1 P_0 g_1^{-1} \subset P_0$, it follows that $a_1(g_1 a'_1 g_1^{-1}) \in P_0$, so that $xy \in P$. Therefore, P is an ordering of $A\#_\varphi \Bbbk G$. □

Now we want to examine condition 2) for the case $A = U(L)$, the universal enveloping algebra of a Lie algebra L over $\mathbb{R}$.

Proposition 3.2. *Let L be a real Lie algebra and $U(L)$ its universal enveloping algebra. Every ordering Q of the vector space L can be extended to an ordering $\tilde{Q}$ of the algebra $U(L)$. Moreover, $\tilde{Q}$ can be chosen so that $\tilde{\alpha}(\tilde{Q}) \subset \tilde{Q}$ for every Lie algebra automorphism α of L such that $\alpha(Q) \subset Q$, where $\tilde{\alpha}$ is the extension of α to $U(L)$.*

Proof. Applying Proposition 2.9 and Remark 2.10, we can pass from $U(L)$ to $\mathrm{gr}(U(L), -\deg)$, which is isomorphic the symmetric algebra $S(L)$. So without loss of generality we may assume that L is abelian.

Suppose that we have elements $e_1, \ldots, e_n$ of L such that $v_Q(e_1) < \cdots < v_Q(e_n)$ and $e_1, \cdots, e_n \in Q$. Then $e_1, \ldots, e_n$ form an ordered basis for their span L_0. This basis gives rise to a valuation $w : U(L_0) \to \Gamma \cup \{\infty\}$ as in Example 2.12. Write Q_0 for the corresponding ordering of $U(L_0)$, i.e.,

$Q_0 \setminus \{0\}$ is the set of elements whose w-lowest term has positive coefficient. Note that $Q_0 \cap L_0 = Q \cap L_0$.

Now pick any finite-dimensional subspaces L_1 and L_2 of L and let $\eta : L_1 \to L_2$ be an injective linear map such that $\eta(Q \cap L_1) \subset Q \cap L_2$. Since L_1 and L_2 are finite-dimensional real vector spaces, we can find a basis $e_1, \ldots, e_m$ of L_1 such that $v_Q(e_1) < \cdots < v_Q(e_m)$ and $e_1, \ldots, e_m \in Q$, and a basis $f_1, \ldots, f_n$ of L_2 such that $v_Q(f_1) < \cdots < v_Q(f_n)$ and $f_1, \ldots, f_n \in Q$. Let Q_1 and Q_2 be the corresponding orderings of $U(L_1)$ and $U(L_2)$, respectively. We claim that $\tilde{\eta}(Q_1) \subset Q_2$. For each $i = 1, \ldots, m$, pick k_i such that $v_Q(f_{k_i}) = v_Q(\eta(e_i))$. Since $\eta(Q \cap L_1) \subset Q \cap L_2$, we conclude that $k_1 < \cdots < k_m$ and, for each $i = 1, \ldots, m$, $\eta(e_i) = \sum_{j=k_i}^n c_{ij} f_j$ where $c_{i,k_i} > 0$. It follows that for any $l_1, \ldots, l_n$, we have $\tilde{\eta}(e_1^{l_1} \cdots e_m^{l_m}) = c f_{k_1}^{l_1} \cdots f_{k_n}^{l_n} + o$ where $c > 0$ and o is a sum of terms with larger w. This proves the claim.

Finally, $U(L)$ a the direct limit of $U(L_i)$ where L_i runs through all finite-dimensional subspaces of L. By the second paragraph, each $U(L_i)$ has an ordering extending $Q \cap L_i$. By the third paragraph, these orderings are compatible with each other. Hence the direct limit $U(L)$ has an ordering $\tilde{Q}$ extending Q. Moreover, if α is an automorphism of L such that $\alpha(Q) \subset Q$, then $\tilde{\alpha}(\tilde{Q} \cap U(L_1)) \subset \tilde{Q} \cap U(L_2)$, for any finite-dimensional subspaces L_1, L_2 of L such that $\alpha(L_1) \subset L_2$. It follows that $\tilde{\alpha}(\tilde{Q}) \subset \tilde{Q}$. □

Proposition 3.1 and Proposition 3.2 imply:

Corollary 3.3. *Let φ be an action of a group G on a real Lie algebra L. Then $U(L)\#_\varphi \mathbb{R}G$ is an orderable domain if and only if G is orderable and L has a vector space ordering Q such that $\varphi_g(Q) \subset Q$ for every $g \in G$.* □

If $\dim L < \infty$, we can give a more explicit characterisation:

Proposition 3.4. *Let φ be an action of a group G on a finite-dimensional Lie algebra L over $\mathbb{R}$. The following assertions are equivalent:*

1) *L has a vector space ordering Q such that $\varphi_g(Q) \subset Q$ for every $g \in G$.*
2) *There exists a basis of L in which all φ_g are lower triangular with positive diagonal entries.*
3) *$\varphi(G)$ is a solvable group and every $\varphi_g \in \varphi(G)$ has positive spectrum.*

The implications 2) $\Rightarrow$ 1) and 2) $\Leftrightarrow$ 3) are true over any ordered field $\Bbbk$.

Proof. Suppose 2) holds. Let $e_1, \ldots, e_n$ be a basis of L in which all φ_g are lower triangular. Write Q for the set of all elements $\sum_{i=1}^n c_i e_i$ such that either all c_i are zero or the first nonzero c_i is positive. Then Q satisfies 1). Conversely, if 1) holds, then we can pick a basis $e_1, \ldots, e_n$ of L such that $v_Q(e_1) < \cdots < v_Q(e_n)$ (here we use that $\Bbbk = \mathbb{R}$). For every automorphism α of L such that $\alpha(Q) \subset Q$, we have $v_Q(\alpha(e_1)) < \cdots < v_Q(\alpha(e_n))$. Since $\{v_Q(e_1), \ldots, v_Q(e_n)\} = \Gamma_Q = \{v_Q(\alpha(e_1)), \ldots, v_Q(\alpha(e_n))\}$, we conclude that $v_Q(\alpha(e_i)) = v_Q(e_i)$ for $i = 1, \ldots, n$. It follows that $\alpha(e_i) = \sum_{j=i}^n c_{ij} e_j$. Since v_Q is compatible with Q, $c_{ii} > 0$. So $e_1, \ldots, e_n$ satisfies 2).

Clearly, 2) implies 3). Conversely, if 3) holds, then $\tilde{G} := \varphi(G)$ is a solvable matrix group and its elements have positive spectrum. We claim that $\tilde{G}$ has a common eigenvector in L. The usual argument by induction on $\dim L$ then implies that $\tilde{G}$ is triangularizable over $\Bbbk$, proving 2). By Malcev's theorem (see [13, Theorem 3.6]), $\tilde{G}$ contains a (normal) subgroup G_0 of finite index that has a common eigenvector u in $L \otimes \bar{\Bbbk}$ where $\bar{\Bbbk}$ is the algebraic closure of $\Bbbk$. Fix a basis $\{\xi_i\}_{i\in I}$, with $\xi_0 = 1$, of $\bar{\Bbbk}$ as a $\Bbbk$-vector space. Then we can write: $u = \sum_{i\in I} u_i \otimes \xi_i$ with $u_i \in L$. Without loss of generality, $u_0 \neq 0$. Since all elements of G_0 have matrices with entries and eigenvalues in $\Bbbk$, we conclude that u_0 is also a common eigenvector of G_0. To prove that u_0 is a common eigenvector for the entire group $\tilde{G}$, take any element $g \in \tilde{G} \setminus G_0$. Pick $k \in \mathbb{N}$ such that $g^k \in G_0$ and thus $g^k u_0 = \lambda u_0$ for some positive $\lambda \in \Bbbk$. Let U be the span of $\{u_0, gu_0, \dots, g^{k-1}u_0\}$. Let $\mu(t)$ and $\mu_U(t)$ be the minimal polynomials of g and $g|_U$, respectively. Note that both $\mu(t)$ and $t^k - \lambda$ annihilate $g|_U$, hence $\mu_U(t)$ divides both. Since all roots of $\mu(t)$ are positive and $t^k - \lambda$ has at most one positive root $\lambda^{\frac{1}{k}}$, it follows that $\mu_U(t) = t - \lambda^{\frac{1}{k}}$. Hence u_0 is an eigenvector of g. □

Remark 3.5. Example 2.4 and Proposition 3.4 imply that a necessary condition for the orderability of $U(L)\#_\varphi\mathbb{R}G$ is the orderability of $\varphi(G)$.

The following example shows that the orderability of G and $\Bbbk$ alone is not enough to ensure the orderability of $U(L)\#_\varphi\Bbbk G$.

Example 3.6. Recall the noncommutative orderable groups

$$G_{n+1} := \langle x, y |\ xyx^{-1} = y^{n+1}\rangle$$

considered in Example 2.5. Let L_n be the abelian real Lie algebra with basis $e_1, \dots, e_n$. We define a representation of G_{n+1} on L_n by $\varphi_x(e_i) = e_i$ and $\varphi_y(e_i) = e_{i+1 \pmod n}$. Since $\varphi(G_{n+1})$ is isomorphic to the cyclic group of order n, which is not orderable, it follows from the remark above that the ring $U(L_n)\#_\varphi\mathbb{R}G_{n+1}$ is not orderable.

Combining Propositions 3.1 and 3.4, we obtain:

Theorem 3.7. *Let φ be an action of a group G on a finite-dimensional real Lie algebra L. Then $U(L)\#_\varphi\mathbb{R}G$ is an orderable domain if and only if*

1) *G is an orderable group, and*
2) *$\varphi(G)$ is a solvable group and every $\varphi_g \in \varphi(G)$ has positive spectrum.*

□

To conclude this section, we observe that because of the following proposition, the orderings of $U(L)\#_\varphi\Bbbk G$ often give rise to orderings on the skew-field of quotients.

Proposition 3.8. *Let L be a locally finite Lie algebra over a field $\Bbbk$ and G a torsion-free locally nilpotent group. Let φ be a locally finite action of G on*

L, i.e., for every $x \in L$ and $g \in G$, the span of the orbit $\{\varphi_g^n(v) \mid n \in \mathbb{Z}\}$ is finite-dimensional. Then $U(L)\#_\varphi \Bbbk G$ is an Ore domain.

Proof. Let G_0 be any finitely generated subgroup of G. Then G_0 is torsion-free nilpotent, hence orderable. It follows that $U(L)\#_\varphi \Bbbk G_0$ is a domain. Since G_0 is arbitrary, we conclude that $U(L)\#_\varphi \Bbbk G$ is a domain.

To verify the (right) Ore condition, let $x = \sum_i a_i\#g_i$ and $y = \sum_j b_j\#h_j$ be nonzero elements of $U(L)\#_\varphi \Bbbk G$. Let G_0 be the subgroup generated by g_i and h_j. Since G_0 is finitely-generated torsion-free nilpotent, it is polycyclic. It follows that every finite subset of L is contained in a finite-dimensional $\varphi(G_0)$-invariant subspace, which, in its turn, generates a finite-dimensional $\varphi(G_0)$-invariant Lie subalgebra. Let L_0 be a finite-dimensional G_0-invariant Lie subalgebra such that $U(L_0)$ contains a_i and b_j. Then $x, y \in U(L_0)\#_\varphi \Bbbk G_0$. Since $\dim L_0 < \infty$ and G_0 is polycyclic, $U(L_0)$ and $\Bbbk G_0$ are noetherian, hence $U(L_0)\#_\varphi \Bbbk G_0$ is also noetherian by [10, Theorem 5.12]. Therefore, $U(L_0)\#_\varphi \Bbbk G_0$ is an Ore domain and we can find nonzero $z, w \in U(L_0)\#_\varphi \Bbbk G_0$ such that $xz = yw$. □

4. $*$-Orderings and $*$-Valuations

In this section we recall the generalities on $*$-orderings and $*$-valuations and construct certain $*$-orderings on universal enveloping algebras that we will need in Section 5.

Let R be a domain with involution $*$, i.e $* : R \to R$ is such that $(a+b)^* = a^* + b^*$, $(ab)^* = b^*a^*$, and $a^{**} = a$, for all $a, b \in R$. An element $a \in R$ is called *symmetric* if $a^* = a$, *skew* if $a^* = -a$. We will denote by $S = S(R)$ the set of symmetric elements: $S = \{a \in R | a^* = a\}$. Clearly, $a, b \in S$ implies $ab + ba \in S$, so S is a Jordan ring. The following is the definition of a $*$-ordering (with zero support) given in [7].

Definition. A *$*$-ordering* (also called a *Jordan ordering*) on R is a subset $P \subset S$ such that

1) $P + P \subset P$,
2) $a, b \in P \Rightarrow ab + ba \in P$,
3) $P \cap -P = \{0\}$,
4) $P \cup -P = S$,
5) $rPr^* \subset P$ for any $r \in R$.

Note that it follows from this definition that P is an ordering of the abelian group S, $1 \in P$, and R has zero characteristic.

It is often convenient to extend a $*$-ordering so that it becomes closed under multiplication.

Definition. A subset $Q \subset R$ is called an *extended $*$-ordering* if

1) $Q + Q \subset Q$,
2) $Q \cdot Q \subset Q$,

3) $Q^* = Q$,
4) $Q \cap -Q = \{0\}$,
5) $Q \cup -Q \supset S$,
6) $rQr^* \subset Q$ for any $r \in R$.

We will say that Q is an *extension* of a $*$-ordering P if $Q \cap S = P$.

By [7, Theorem 2.2], every $*$-ordering P has such an extension. Moreover, there exists a unique minimal extension of P, which is referred to as the *extended $*$-ordering generated by P* [7, Proposition 2.4]. Generally speaking, the notion of $*$-ordering seems more natural, but extended $*$-orderings are easier to work with.

Consider for a moment the case when R is commutative. Then an involution on R is the same as an automorphism of order ≤ 2 and $S = S(R)$ is a subring. Also every $*$-ordering is automatically an extended $*$-ordering, i.e. closed under multiplication. Therefore, a $*$-ordering on R is the same as an ordering on S that contains the elements of the form rr^*, $r \in R$.

Let $\Bbbk$ be a field with involution and let $\Bbbk_0$ be the subfield of symmetric elements of $\Bbbk$. Then either $\Bbbk_0 = \Bbbk$ or $\Bbbk$ is a quadratic extension of $\Bbbk_0$, generated by some ξ, which we can choose so that $\xi^* = -\xi$. In the first case, a $*$-ordering of $\Bbbk$ is of course the same as an ordering of $\Bbbk$ (and thus $\sqrt{-1} \notin \Bbbk$). In the second case, a $*$-ordering of $\Bbbk$ is the same as an ordering of $\Bbbk_0$ such that ξ^2 is negative. In particular, if $\Bbbk_0$ contains square roots of positive elements, we can make $\xi = \sqrt{-1}$.

Suppose now that R is a $\Bbbk$-algebra with involution (a "$*$-algebra" for short). We require in this case that $(\lambda a)^* = \lambda^* a^*$, for all $a \in R$, $\lambda \in \Bbbk$, i.e. $*$ must agree with the given involution on $\Bbbk$. Also if P is a $*$-ordering on R, we require that $\lambda P \subset P$ for all positive $\lambda \in \Bbbk$, i.e., $P \cap \Bbbk$ is the given $*$-ordering of $\Bbbk$.

Remark 4.1. In the case $\Bbbk_0 = \Bbbk$, if R is a $\Bbbk$-algebra with involution, consider the $\Bbbk(\sqrt{-1})$-algebra $\tilde{R} = R \otimes_{\Bbbk} \Bbbk(\sqrt{-1})$ where the involution is extended to $\Bbbk(\sqrt{-1})$ and $\tilde{R}$ by $\sqrt{-1} \mapsto -\sqrt{-1}$. Clearly, if P is a $*$-ordering on $\tilde{R}$, then $P \cap R$ is a $*$-ordering on R. This observation reduces the problem of constructing a $*$-ordering in the case $\Bbbk_0 = \Bbbk$ to the case $\Bbbk_0 \neq \Bbbk$.

If R is a commutative $*$-algebra and $\Bbbk_0 \neq \Bbbk$, then $S = S(R)$ is a $\Bbbk_0$-subalgebra and $R = S \otimes_{\Bbbk_0} \Bbbk$. As we know, a $*$-ordering on R is the same as an ordering of S that contains rr^* for all $r \in R$. Writing $r = a + b\xi$, $a, b \in S$ (where, as before, $\Bbbk = \Bbbk_0(\xi)$, $\xi^* = -\xi$), we obtain: $rr^* = a^2 - b^2\xi^2$. Since ξ^2 is negative in $\Bbbk$, we see that every ordering of S contains the elements rr^*, $r \in R$. Therefore, $*$-orderings of R are precisely orderings of S.

We will need the notion of a $*$-valuation on a ring R with involution.

Definition. A valuation $v : R \to \Gamma \cup \{\infty\}$ is called a $*$-valuation if $v(a^*) = v(a)$ for all $a \in R$. This forces the value semigroup Γ to be commutative, so Γ can be canonically embedded into an ordered abelian group, which is called the *value group* of v.

A $*$-valuation v on R is said to be *compatible* with a $*$-ordering $P \subset S = S(R)$ if for all $a \in P$ and $b \in S$, $v(b) > v(a)$ implies that $a + b \in P$.

It is shown in [7] that for every $*$-ordering P, there exists the finest $*$-valuation compatible with P, which is called the *natural $*$-valuation* associated to P. It is denoted v_P and constructed in the following way.

The $*$-ordering P gives an order relation $\leq$ on $S = S(R)$, which induces the Archimedean equivalence $\sim$ on S. We extend the latter to the whole R by declaring, for all $0 \neq a, b \in R$, that $a \preceq b$ if $aa^* \leq nbb^*$ for some integer n, and $a \sim b$ if $a \preceq b$ and $b \preceq a$ (by [7, Proposition 3.1], this is equivalent to our earlier definition of $a \sim b$ for $a, b \in S$). Denote $v_P(a)$ the equivalence class of $0 \neq a \in R$ (and $v_P(0) := \infty$). Then the relation $\preceq$ induces a total order on the set $\Gamma_P = v_P(R \setminus \{0\})$. By [7, Theorem 3.3], the binary operation $v_P(a) + v_P(b) := v_P(ab)$ is well-defined on Γ_P, so Γ_P becomes an ordered commutative cancellation semigroup. It is also shown that v_P is a $*$-valuation and

$$v_P(ab - ba) > v_P(a) + v_P(b) \text{ for all } 0 \neq a, b \in S(R). \tag{1}$$

Remark 4.2. If R is a $\mathbb{C}$-algebra, then applying Remark 2.8 to the ordered $\mathbb{R}$-vector space $S(R)$ we see that, for every $a, b \in S(R)$ such that $v_P(a) = v_P(b)$, there exists $r \in \mathbb{R}$ such that $v_P(a - rb) > v_P(a)$. This holds true, with $r \in \mathbb{C}$, even if $a \notin S(R)$, because we can write $a = a_1 + a_2\sqrt{-1}$ where $a_1, a_2 \in S(R)$.

Suppose now that R is a domain with involution and $v : R \to \Gamma \cup \{\infty\}$ is a $*$-valuation. Then the graded ring $\mathrm{gr}(R, v)$ is also a domain with involution $\overline{a} \mapsto \overline{a^*}$, and $\overline{v} = \mathrm{gr}(v)$ is a $*$-valuation on $\mathrm{gr}(R, v)$. Decomposing $a \in R$ as $a = s + t$ with s symmetric, t skew, we see that $v(a^* - a) > v(a)$ iff $v(a - s) > v(a)$. Thus the symmetric elements of $\mathrm{gr}(R, v)$ have the form $a = \sum_\alpha \overline{a}_\alpha$, with a_α symmetric. We will make use of the following analog of Proposition 2.9 from [8].

Proposition 4.3. *There is a natural one-to-one correspondence $P \mapsto \overline{P}$ between $*$-orderings on R compatible with v and $*$-orderings on $\mathrm{gr}(R, v)$ compatible with $\overline{v}$. Namely, $\overline{P} \setminus \{0\}$ consists of all nonzero symmetric $a = \sum_\alpha \overline{a}_\alpha$ such that $a_\gamma \in P$ where $\gamma = \overline{v}(a)$ and, conversely, $P \setminus \{0\}$ consists of all nonzero symmetric b such that $\overline{b} := b + R_\beta^+$, where $\beta = v(b)$, belongs to $\overline{P}$.* □

Now we turn our attention to Hopf algebras.

Definition. Let H be a Hopf algebra with multiplication m and comultiplication Δ over a field $\Bbbk$ with involution. A *Hopf involution* of H is a mapping $* : H \to H$ such that

1) $(\lambda x + \mu y)^* = \lambda^* x^* + \mu^* y^*$ and $x^{**} = x$, for all $x, y \in H$ and $\lambda, \mu \in \Bbbk$.
2) $* \circ m = m \circ \tau \circ (* \otimes *)$, where $\tau(u \otimes v) = v \otimes u$,
3) $\Delta \circ * = (* \otimes *) \circ \tau \circ \Delta$,

Note that conditions 1) and 2) simply say that (H, m) is a $*$-algebra. Condition 3) is the formal dual of 2). It can be written in Σ-notation as follows:

$$\Delta(x) = \sum x_{(1)} \otimes x_{(2)} \quad \Rightarrow \quad \Delta(x^*) = \sum x^*_{(2)} \otimes x^*_{(1)}.$$

Since the counit ε and antipode $\mathcal{S}$ are uniquely determined by m and Δ, one checks that $\varepsilon(x^*) = \varepsilon(x)^*$ and $\mathcal{S}(x^*) = \mathcal{S}(x)^*$ for all $x \in H$. Also 3) implies that $G(H)^* = G(H)$ and $P(H)^* = P(H)$.

Now we look at our simplest examples of Hopf algebras: group algebras and universal enveloping algebras.

Example 4.4. A *group involution* on a group G is a group anti-automorphism of order ≤ 2. The *standard involution* on G is $g \mapsto g^{-1}$. Every group involution $*$ of G can be extended uniquely to a Hopf involution of $H = \Bbbk G$ by $(\sum \lambda_g g)^* = \sum \lambda_g^* g^*$. Conversely, every Hopf involution of H preserves $G(H) = G$ and its restriction to G is a group involution. This gives a one-to-one correspondence between Hopf involutions of $H = \Bbbk G$ and group involutions of G.

Example 4.5. Suppose L is a Lie algebra over a field $\Bbbk$ with involution and $\operatorname{char} \Bbbk \neq 2$. A *Lie involution* on L is a map $* : L \to L$ such that $(\lambda x + \mu y)^* = \lambda^* x^* + \mu^* y^*$, $x^{**} = x$, and $[x, y]^* = [y^*, x^*]$ for all $x, y \in H$ and $\lambda, \mu \in \Bbbk$. Every Hopf involution of $H = U(L)$ preserves $L = P(H)$ and its restriction to L is a Lie involution. The universal property of $U(L)$ implies that every involution of L can be lifted to $U(L)$. This gives a one-to-one correspondence between Hopf involutions of $H = U(L)$ and Lie involutions of L.

It should be noted that there is no standard Lie involution on a Lie algebra L. Depending on whether or not the involution on the ground field is trivial, we have the following two possibilities.

If $\Bbbk_0 = \Bbbk$, set

$$L_0 = \{x \in L | x^* = -x\} \text{ and } L_1 = \{x \in L | x^* = x\}.$$

Then L_0 and L_1 are $\Bbbk$-subspaces and $L = L_0 \oplus L_1$ is a $\mathbb{Z}_2$-grading of the Lie algebra L. Conversely, every $\mathbb{Z}_2$-grading of the Lie algebra L gives a Lie involution.

If $\Bbbk_0 \neq \Bbbk$, then L_0, defined as above, is only a $\Bbbk_0$-subspace. In fact, it is a $\Bbbk_0$-subalgebra of L and $L_1 = \xi L_0$ where $\xi \in \Bbbk$ is such that $\xi^* = -\xi$. Therefore, $L = L_0 \otimes_{\Bbbk_0} \Bbbk$. Conversely, if we can write L as $L_0 \otimes_{\Bbbk_0} \Bbbk$ for some Lie $\Bbbk_0$-algebra L, then we can define a Lie involution on L by $(\sum \lambda_i x_i)^* = -\sum \lambda_i^* x_i$ for any $x_i \in L_0$ and $\lambda_i \in \Bbbk$. We conclude that a Lie algebra over $\Bbbk$ has a Lie involution iff it admits a basis such that all "structure constants" lie in $\Bbbk_0$.

Recall that if H is a commutative $*$-algebra over a $*$-ordered field $\Bbbk$ and $\Bbbk_0 \neq \Bbbk$, then $*$-orderings of H are just orderings of $S(H)$. If H is a commutative and cocommutative Hopf algebra and $*$ is a Hopf involution, then one checks that $S(H)$ is again a Hopf algebra (over $\Bbbk_0$).

Example 4.6. Consider the Hopf algebra of complex polynomial functions on the unit circle $H = \mathbb{C}[s,c]/(s^2+c^2-1)$, $\Delta s = s\otimes c + c\otimes s$, $\Delta c = c\otimes c - s\otimes s$. Then the usual complex conjugation is a Hopf involution and $S(H)$ is the Hopf algebra of real polynomial functions on the unit circle:

$$H_{\mathbb{R}} = \mathbb{R}[s,c]/(s^2+c^2-1) \text{ with } \Delta s = s\otimes c + c\otimes s, \quad \Delta c = c\otimes c - s\otimes s.$$

We can construct an ordering on $S(H)$ as follows. Define an embedding of $S(H)$ into the algebra of power series $\mathbb{R}[[t]]$ by $s \mapsto \sin(t)$ and $c \mapsto \cos(t)$. Then the ordering of $\mathbb{R}[[t]]$ by the sign of the lowest coefficient induces an ordering P on $S(H)$. Now observe that H is isomorphic (as a Hopf algebra) to the group algebra $\mathbb{C}\langle g\rangle$ of the infinite cyclic group, where $g = c + is$, $g^* = c - is = g^{-1}$, $i = \sqrt{-1}$. Thus P is a $*$-ordering on $\mathbb{C}\langle g\rangle$ with standard involution. In terms of g, this ordering can be described as follows: $\sum_n \lambda_n g^n \in P$ iff $\overline{\lambda}_n = \lambda_{-n}$ and the (real) power series $\sum_n \lambda_n \exp(int)$ has a positive lowest coefficient. We will revisit this example in Section 5.

More generally, if $H_{\mathbb{R}}$ is a real Hopf algebra that is commutative, cocommutative and cosemisimple, then so is its complexification $H = H_{\mathbb{R}} \otimes \mathbb{C}$. Moreover, since $\mathbb{C}$ is algebraically closed, H is also pointed, which implies by the structure theorem that $H = \mathbb{C}G$ where $G = G(H)$ is an abelian group. Complex conjugation on $\mathbb{C}$ induces a Hopf involution on $H_{\mathbb{R}}\otimes\mathbb{C} = H$ with respect to which $H_{\mathbb{R}} = S(H)$. The restriction of this involution to G is an automorphism σ of order ≤ 2. Thus $H_{\mathbb{R}}$ is determined by the pair (G,σ), and orderings of $H_{\mathbb{R}}$ are precisely $*$-orderings of $\mathbb{C}G$ with involution $\sum_g \lambda_g g \mapsto \sum_g \overline{\lambda}_g \sigma(g)$.

Now consider $H = U(L)$. If L has a Lie involution and $\Bbbk_0 \neq \Bbbk$, then we can pick a basis $\{x_i\}_{i\in I}$ of L consisting of symmetric elements (see the discussion after Example 4.5). Invoking Proposition 4.3 and the PBW Theorem, we see that the $*$-orderings of $U(L)$ compatible with the valuation $-\deg : U(L) \to \mathbb{Z}_- \cup \{\infty\}$ are in one-to-one correspondence with the $*$-orderings of the polynomial algebra $\Bbbk[x_i | i \in I]$ compatible with $-\deg$, but the latter are the same as the orderings of $\Bbbk_0[x_i | i\in I]$ compatible with $-\deg$. Thus we obtain

Corollary 4.7. 1) *If the field $\Bbbk$ is ordered and L is a Lie algebra over $\Bbbk$, then there exists an ordering on $U(L)$ extending the ordering on $\Bbbk$ and compatible with the valuation $-\deg$ on $U(L)$.*

2) *If the $*$-field $\Bbbk$ is $*$-ordered and $(L,*)$ is a Lie algebra with involution over $\Bbbk$, then there exists a $*$-ordering on $U(L)$ extending the $*$-ordering on $\Bbbk$ and compatible with the $*$-valuation $-\deg$ on $U(L)$.*

Proof. 1) This is done in Example 2.12.

2) If $\Bbbk_0 \neq \Bbbk$, any ordering of $\Bbbk_0[x_i | i \in I]$ compatible with $-\deg$ gives rise to a $*$-ordering of $U(L)$ by the discussion above. In the case $\Bbbk_0 = \Bbbk$, we can obtain a $*$-ordering of $U(L)$ by restricting a $*$-ordering on $U(L) \otimes \Bbbk(\sqrt{-1}) = U(L \otimes \Bbbk(\sqrt{-1}))$ — see Remark 4.1. □

Remark 4.8. Assertion 1) is well-known. Assertion 2) was proved in [8] in the case $\Bbbk = \mathbb{C}$ and $\dim L < \infty$.

The problem of $*$-orderability of group algebras of groups with involution seems more difficult. We give some partial results in Section 5. We will need the following analog of the above construction of orderings on $U(L)$ for the case when L is an $\mathbb{N}$-graded Lie algebra and the valuation $-\deg$ is replaced by $v : U(L) \to \mathbb{Z}_+ \cup \{\infty\}$, where $v(f)$ is the lowest degree of the homogeneous components of $f \in U(L)$ with respect to the grading on $U(L)$ induced by the given grading on L.

Let $L = \oplus_{i \in \mathbb{N}} L_i$ be a graded Lie algebra over a field $\Bbbk$. Let

$$\{x_{ij} | i \in \mathbb{N},\, j \in J_i\}$$

be a basis for L chosen so that, for each i, $\{x_{ij} | j \in J_i\}$ is a basis for L_i. Order this basis by fixing a total ordering $<$ on each J_i and declaring $x_{ij} < x_{i'j'}$ to mean that either $i < i'$ or ($i = i'$ and $j < j'$).

By the PBW Theorem, the monomials $x_{i_1 j_1} \dots x_{i_n j_n}$, $x_{i_1 j_1} \leq \dots \leq x_{i_n j_n}$, $n \geq 0$, form a basis for $U(L)$, i.e., as a $\Bbbk$-vector space, $U(L)$ is identical to the polynomial algebra $\Bbbk[x_{ij}]$. The multiplication on $U(L)$ is determined by the relations $x_{ij} x_{i'j'} - x_{i'j'} x_{ij} = [x_{ij}, x_{i'j'}]$. Since L is graded, $[x_{ij}, x_{i'j'}]$ is some finite linear combination of elements $x_{i+i',s}$, $s \in J_{i+i'}$.

We define a total ordering $<$ on the monomials

$$x_{i_1 j_1} \dots x_{i_n j_n} \text{ where } x_{i_1 j_1} \leq \dots \leq x_{i_n j_n}, \quad n \geq 0$$

by declaring $x_{i_1 j_1} \dots x_{i_n j_n} < x_{p_1 q_1} \dots x_{p_m q_m}$ to hold if either $i_1 + \dots + i_n < p_1 + \dots + p_m$ or if $i_1 + \dots + i_n = p_1 + \dots + p_m$ and $x_{i_1 j_1} \dots x_{i_n j_n} <_{\text{lex}} x_{p_1 q_1} \dots x_{p_m q_m}$, where $<_{\text{lex}}$ is the lexicographic order on words.

This is an ordering on the multiplicative semigroup of monomials in the commuting variables $\{x_{ij}\}$. Consequently, it determines a valuation, call it w, on the polynomial algebra $\Bbbk[x_{ij}]$ as follows. The value semigroup of w is the free commutative semigroup $(\Gamma, +, <)$ generated by the symbols $w(x_{ij})$ with the ordering induced by the monomial ordering, i.e., $\sum_{s=1}^{n} k_s w(x_{i_s j_s}) < \sum_{t=1}^{m} l_t w(x_{p_t q_t})$ iff $x_{i_1 j_1}^{k_1} \dots x_{i_n j_n}^{k_n} < x_{p_1 q_1}^{l_1} \dots x_{p_m q_m}^{l_m}$. Define $w(x_{i_1 j_1}^{k_1} \dots x_{i_n j_n}^{k_n}) := \sum_{s=1}^{n} k_s w(x_{i_s j_s})$. For an arbitrary non-zero $f \in k[x_{ij}]$, set $w(f) := w(x_{i_1 j_1}^{k_1} \dots x_{i_n j_n}^{k_n})$, where $x_{i_1 j_1}^{k_1} \dots x_{i_n j_n}^{k_n}$ is the least monomial appearing in f. As usual, $w(0) := \infty$.

Since w is a valuation on $\Bbbk[x_{ij}]$, $w(f+g) \geq \min\{w(f), w(g)\}$ and $w(fg) = w(f) + w(g)$ for the multiplication on $\Bbbk[x_{ij}]$. The point is that $w(fg) = w(f) + w(g)$ also holds for the multiplication on $U(L)$, i.e., w is also a valuation on $U(L)$. The proof of this reduces to establishing the following

Lemma 4.9. *Suppose $x_{i_1j_1} \le \cdots \le x_{i_nj_n}$ and π is any permutation of $1,\dots,n$. Then $x_{i_{\pi(1)}j_{\pi(1)}}\dots x_{i_{\pi(n)}j_{\pi(n)}} \equiv x_{i_1j_1}\dots x_{i_nj_n}$ modulo a linear combination of monomials strictly greater than $x_{i_1j_1}\dots x_{i_nj_n}$.*

Proof. By induction on n. The cases $n = 0$, $n = 1$ are trivial. For $n = 2$ the result follows from the fact that $[x_{i_1j_1}, x_{i_2j_2}]$ is a linear combination of $x_{i_1+i_2,s}$, $s \in S_{i_1+i_2}$ plus the fact that $x_{i_1j_1}x_{i_2j_2} < x_{i_1+i_2,s}$ for any $s \in S_{i_1+i_2}$. Suppose now that $n \ge 3$. Since π is a product of adjacent interchanges, it suffices to check what happens when we make one additional interchange, replacing $x_{i_{\pi(t)}j_{\pi(t)}}x_{i_{\pi(t+1)}j_{\pi(t+1)}}$ by $x_{i_{\pi(t+1)}j_{\pi(t+1)}}x_{i_{\pi(t)}j_{\pi(t)}}$ say. We have

$$(\dots x_{i_{\pi(t)}j_{\pi(t)}}x_{i_{\pi(t+1)}j_{\pi(t+1)}}\dots) - (\dots x_{i_{\pi(t+1)}j_{\pi(t+1)}}x_{i_{\pi(t)}j_{\pi(t)}}\dots) = \sum_s d_s y_s,$$

where $y_s = \dots x_{i_{\pi(t)}+i_{\pi(t+1)},s}\dots$, $d_s \in k$. Denote by $y_s{}'$ the monomial obtained from y_s by writing the factors in non-decreasing order. Let $u = \pi(t)$, $u' = \pi(t+1)$. The factors appearing in y'_s are the $x_{i_rj_r}$, $r \notin \{u, u'\}$ and $x_{i_u+i_{u'},s}$. Thus y'_s is obtained from $x := x_{i_1j_1}\dots x_{i_nj_n}$ by removing two factors $x_{i_uj_u}$ and $x_{i_{u'}j_{u'}}$ and inserting one new factor $x_{i_u+i_{u'},s}$. Since $x_{i_uj_u}$ and $x_{i_{u'}j_{u'}}$ are both strictly less than $x_{i_u+i_{u'},s}$ one checks that in all possible cases ($x_{i_uj_u} < x_{i_{u'}j_{u'}}$, $x_{i_uj_u} = x_{i_{u'}j_{u'}}$, $x_{i_uj_u} > x_{i_{u'}j_{u'}}$), the definition of the monomial ordering implies that $x < y'_s$. At the same time, by induction on n, $y_s - y'_s$ is a linear combination of monomials strictly greater than y'_s (so also strictly greater than x). Finally, this implies that $\sum_s d_s y_s = \sum_s d_s(y_s - y'_s) + \sum_s d_s y'_s$ is a linear combination of monomials each strictly greater than x. □

Using Lemma 4.9, we see that fg and gf both have the same lowest term, i.e., not only do we have $w(fg) = w(f) + w(g) = w(gf)$, but we also have $w(fg-gf) > w(fg) = w(gf)$, i.e., the associated graded algebra $\mathrm{gr}(U(L), w)$ is commutative. One checks that, in fact, $\mathrm{gr}(U(L), w) = \Bbbk[x_{ij}]$, where the grading on $\Bbbk[x_{ij}]$ is the one induced by the valuation w on $\Bbbk[x_{ij}]$.

For clarity of exposition it is useful to distinguish between w, viewed as a valuation on the $\Bbbk$-algebra $U(L)$ and w, viewed as a valuation on the $\Bbbk$-algebra $\Bbbk[x_{ij}]$. Following the notation of Propositions 2.9 and 4.3, we denote the former by w and the latter by $\overline{w}$, i.e., $\overline{w} = \mathrm{gr}(w)$.

Now recall the valuation v associated to the grading on $U(L)$. We have $v(x_{i_1j_1}^{k_1}\dots x_{i_nj_n}^{k_n}) = \sum_{s=1}^n k_s i_s$ and, for arbitrary nonzero $f \in U(L)$, $v(f)$ is the minimum of the $v(x_{i_1j_1}^{k_1}\dots x_{i_nj_n}^{k_n})$, $x_{i_1j_1}^{l_1}\dots x_{i_nj_n}^{l_n}$ a monomial appearing in f. It is clear from the definition of w that w is a refinement of v. Since the value semigroup of v is commutative, v also satisfies $v(fg) = v(f) + v(g) = v(gf)$, but $\mathrm{gr}(U(L), v) = U(L)$, which is not commutative in general.

It remains to put the involution into picture. So suppose $(L, *)$ is a $\mathbb{N}$-graded Lie algebra with involution respecting the grading, i.e., $* : L_i \to L_i$ for $i \in \mathbb{N}$. Then the extension of $*$ to an involution on $U(L)$ also respects the grading and therefore, for each $f \in U(L)$, $v(f^*) = v(f)$, i.e., v is a $*$-valuation.

We choose the basis $\{x_{ij} : i \geq 1, j \in J_i\}$ so that each x_{ij} is either symmetric or skew, say $x_{ij}^* = \epsilon_{ij} x_{ij}$, $\epsilon_{ij} \in \{1, -1\}$. By Lemma 4.9, $(x_{i_1 j_1} \dots x_{i_n j_n})^* = x_{i_n j_n}^* \dots x_{i_1 j_1}^* = (\pm 1) x_{i_n j_n} \dots x_{i_1 j_1} \equiv (\pm 1) x_{i_1 j_1} \dots x_{i_n j_n}$ modulo a linear combination of monomials strictly greater than $x_{i_1 j_1} \dots x_{i_n j_n}$. This implies $w(f^*) = w(f)$ for any $f \in U(L)$, i.e., w is a $*$-valuation. The induced involution on $\mathrm{gr}(U(L), w) = \Bbbk[x_{ij}]$ is the one defined by $x_{ij}^* = \epsilon_{ij} x_{ij}$.

Thus we obtain the following

Proposition 4.10. 1) *If the field* $\Bbbk$ *is ordered and* $L = \oplus_{i \in \mathbb{N}} L_i$ *is a graded Lie algebra over* $\Bbbk$*, then there exists an ordering on* $U(L)$ *extending the ordering on* $\Bbbk$ *and compatible with the valuation* v *on* $U(L)$ *determined by the grading.*

2) *If the* $*$*-field* $\Bbbk$ *is* $*$*-ordered and* $(L, *)$ *is a graded Lie algebra with involution over* $\Bbbk$ *such that* $*$ *respects the grading, then there exists a* $*$*-ordering on* $U(L)$ *extending the* $*$*-ordering on* $\Bbbk$ *and compatible with the* $*$*-valuation* v *on* $U(L)$ *determined by the grading.*

Proof. 1) Pick any ordering on $\Bbbk[x_{ij}]$ extending the given ordering on $\Bbbk$ and compatible with the valuation $\overline{w} = \mathrm{gr}(w)$ on $\Bbbk[x_{ij}]$. According to Proposition 2.9, this yields an ordering of $U(L)$ extending the given ordering on $\Bbbk$ and compatible with the valuation w on $U(L)$. Since v is a coarsening of w, this ordering is also compatible with v.

The proof of 2) is similar. We pick a $*$-ordering on $\Bbbk[x_{ij}]$ extending the $*$-ordering on $\Bbbk$ and compatible with the $*$-valuation $\overline{w}$ on $\Bbbk[x_{ij}]$ and then use Proposition 4.3. That the required $*$-ordering on $\Bbbk[x_{ij}]$ exists is clear in the case $\Bbbk_0 \neq \Bbbk$, because then we can choose all x_{ij} symmetric. In the case $\Bbbk_0 = \Bbbk$, use Remark 4.1. □

Remark 4.11. Note that the valuation $\overline{w}$ is so fine that there are not so many orderings (resp., $*$-orderings) compatible with it. Every such ordering (resp., $*$-ordering) P is determined by prescribing a sign to each variable x_{ij} (resp., to each y_{ij} where $y_{ij} = x_{ij}$ if x_{ij} is symmetric and $y_{ij} = \sqrt{-1} x_{ij}$ if x_{ij} is skew). Note also that the natural valuation v_P is given by $v_P(cM + o) = (w(M), v_{\Bbbk_+}(c)) \in \Gamma \times \Gamma_{\Bbbk_+}$ (with lexicographic order) where M is a PBW monomial, $0 \neq c \in \Bbbk$, and o is a linear combination of monomials M' with $w(M') > w(M)$.

5. $*$-Orderability of Group Algebras

Let G be a group with involution and $\Bbbk$ a field with involution. In this section we investigate the problem when the group algebra $\Bbbk G$ is $*$-orderable. Theorem 5.1 gives a sufficient condition for $*$-orderability of $\Bbbk G$ where the $*$-field $\Bbbk$ and the group involution are arbitrary. Theorem 5.5 is a necessary condition in the case of $\Bbbk = \mathbb{C}$ and the standard group involution, i.e., $g \mapsto g^{-1}$.

We will use the following general facts in the proof of Theorem 5.1. Let G be a group, $\Bbbk$ a field, and $\mathfrak{d}$ the augmentation ideal of $\Bbbk G$. Define the "dimension subgroups" $\mathcal{D}_n \subset G$, $n \in \mathbb{N}$, by

$$\mathcal{D}_n := (1 + \mathfrak{d}^n) \cap G.$$

Then by [12, Theorem IV.1.5] the subgroups $\mathcal{D}_n$ depend only on the characteristic of $\Bbbk$ and in the case $\operatorname{char} \Bbbk = 0$ (which we assume from now on) we have

$$\mathcal{D}_n = \sqrt{\gamma_n(G)} \tag{2}$$

where $\gamma_n(G)$ is the lower central series of G, i.e., $\gamma_1(G) = G$, $\gamma_{n+1}(G) = (\gamma_n(G), G)$, and $\sqrt{\gamma_n(G)} := \{g \in G \mid \exists m \in \mathbb{N} : \ g^m \in \gamma_n(G)\}$. It follows that the quotients $\mathcal{D}_n/\mathcal{D}_{n+1}$ are abelian and torsion-free.

The graded Lie ring associated to G is constructed as follows:

$$L_{\mathbb{Z}}(G) := \oplus_{n=1}^{\infty} \mathcal{D}_n/\mathcal{D}_{n+1}$$

as an abelian group (written additively), with the bracket defined by

$$[x, y] := (g, h)\mathcal{D}_{n+m+1} \tag{3}$$

where $x = g\mathcal{D}_{n+1}$, $y = h\mathcal{D}_{m+1}$, $g \in \mathcal{D}_n$, $h \in \mathcal{D}_m$, and $(g, h) = ghg^{-1}h^{-1}$.

Set $L(G) := L_{\mathbb{Z}}(G) \otimes_{\mathbb{Z}} \Bbbk$. Then $L(G)$ is a graded Lie algebra over $\Bbbk$. Let $\operatorname{gr}(\Bbbk G)$ be the associated graded algebra of $\Bbbk G$ filtered by the powers of $\mathfrak{d}$, i.e.,

$$\operatorname{gr}(\Bbbk G) := \oplus_{n=0}^{\infty} \mathfrak{d}^n/\mathfrak{d}^{n+1}.$$

Consider the mapping

$$\theta : U(L(G)) \to \operatorname{gr}(\Bbbk G) : \quad x_1 \cdots x_m \mapsto (g_1 - 1) \cdots (g_m - 1) + \mathfrak{d}^{n+1} \tag{4}$$

where $x_i \in \mathcal{D}_{n_i}/\mathcal{D}_{n_i+1}$, $x_i = g_i\mathcal{D}_{n_i+1}$ and $n = n_1 + \ldots + n_m$.

According to Quillen's result (see [12, Theorem VIII.5.2]), θ is an isomorphism of graded algebras. Recall from Section 4 that the grading of $U(L(G))$ determines a valuation, which we called v. Transporting v by the isomorphism θ, we obtain a valuation on $\operatorname{gr}(\Bbbk G)$, which by abuse of notation we also denote by v.

Now assume that $\cap_{n=0}^{\infty} \mathfrak{d}^n = \{0\}$. By [12, Theorem VI.2.26], this is equivalent to the assumption that G is residually 'torsion-free nilpotent'. Then we have a valuation u on $\Bbbk G$ defined by $u(a) =$ the greatest n such that $a \in \mathfrak{d}^n$, $u(0) = \infty$. (The fact that u is a valuation, i.e., $u(ab) = u(a) + u(b)$ follows from the fact that v is a valuation.) Clearly, $\operatorname{gr}(\Bbbk G) = \operatorname{gr}(\Bbbk G, u)$ and $v = \operatorname{gr}(u)$.

Now we are ready to prove our sufficient condition for ∗-orderability of $\Bbbk G$.

Theorem 5.1. *Suppose G is a group which is residually 'torsion-free nilpotent'.*

1) *If the field* $\Bbbk$ *is ordered, then there exists an ordering on the group algebra* $\Bbbk G$ *extending the ordering on* $\Bbbk$ *and compatible with the valuation* u *on* $\Bbbk G$ *determined by the augmentation ideal.*
2) *If the* $*$*-field* $\Bbbk$ *is* $*$*-ordered, then, for any involution* $*$ *of* G*, the group algebra* $\Bbbk G$ *with the induced involution admits a* $*$*-ordering extending the given* $*$*-ordering on* $\Bbbk$ *and compatible with the* $*$*-valuation* u *on* $\Bbbk G$ *determined by the augmentation ideal.*

Proof. 1) Using Proposition 4.10 1) and the isomorphism θ, we can construct an ordering $\overline{P}$ on $\operatorname{gr}(\Bbbk G)$ extending the ordering on $\Bbbk$ and compatible with the valuation v on $\operatorname{gr}(\Bbbk G)$. By Proposition 2.9, $\overline{P}$ pulls back to an ordering P on $\Bbbk G$ compatible with u.

2) First we notice that by (2), any group involution preserves the subgroups $\mathcal{D}_n$ and, therefore, induces a mapping on $L_{\mathbb{Z}}(G)$. It follows from (3) that this mapping will be an involution of the Lie ring and thus induces an involution of the Lie algebra $L(G) = L_{\mathbb{Z}}(G) \otimes_{\mathbb{Z}} \Bbbk$ compatible with the given involution on $\Bbbk$. Then it follows from (4) that θ is a $*$-isomorphism.

Now by Proposition 4.10 2) we can produce a $*$-ordering $\overline{P}$ on $\operatorname{gr}(\Bbbk G)$ extending the given $*$-ordering on $\Bbbk$ and compatible with the $*$-valuation v. Finally, we use Proposition 4.3 to pull $\overline{P}$ back to a $*$-ordering P on $\Bbbk G$ compatible with the $*$-valuation u. □

Remark 5.2. We know in general that for $\Bbbk$ orderable, $\Bbbk G$ is orderable iff G is orderable. This, taken together with Theorem 5.1 1), gives another proof that every residually 'torsion-free nilpotent' group is orderable.

Now we consider the case $\Bbbk = \mathbb{C}$ in more detail. We will use the notation $i = \sqrt{-1}$ and bars for complex conjugates. The following example illustrates Theorem 5.1 in the simplest possible case.

Example 5.3. Let $G = \langle g \rangle$ be the infinite cyclic group. Consider $\mathbb{C}G$ with the standard involution. Then $L(G)$ is the 1-dimensional Lie algebra $\mathbb{C}t$ with involution $\lambda t \mapsto -\overline{\lambda} t$. Thus $U(L(G)) = \mathbb{C}[t]$ with involution $f(t)^* = \overline{f(-t)}$ and the usual grading, so the valuation $v(f)$ is equal to the lowest degree of t occuring in f. The isomorphism θ sends t^n to $(g-1)^n + (g-1)^{n+1}\mathbb{C}G$. Clearly, the symmetric elements of $U(L(G))$ are of the form $f(it)$, $f \in \mathbb{R}[t]$. Order them by the sign of the lowest coefficient. This is a $*$-ordering of $U(L(G))$ compatible with v. The corresponding $*$-ordering of $\mathbb{C}G$ declares a symmetric element $\sum_n \lambda_n g^n$ positive or negative according to the sign of the lowest coefficient of the power series $\sum_n \lambda_n (1+it)^n \in \mathbb{R}[[t]]$. One checks that it is the same $*$-ordering as in Example 4.6 and that the valuation u is its natural valuation.

Suppose now that G is any group such that $\mathbb{C}G$ is $*$-orderable and fix a $*$-ordering P on $\mathbb{C}G$. Let $v_P : \mathbb{C}G \to \Gamma_P \cup \{\infty\}$ be the natural $*$-valuation associated to P. First we observe that for $z \in \mathbb{C}$, $z \neq 0$, we have $v_P(z) = 0$. Indeed, by the definition of the natural valuation, we must show that $z \sim 1$, i.e., that there exists a positive integer n such that $nz\bar{z} \geq 1$ and $n \geq z\bar{z}$. This is clear.

Now assume that $\Gamma_P \geq 0$, i.e., $v_P(a) \geq 0$ holds for all $a \in \mathbb{C}G$. Then the set $\mathfrak{m}$ defined by

$$\mathfrak{m} = \{a \in \mathbb{C}G \,|\, v_P(a) > 0\}$$

is an ideal in $\mathbb{C}G$. Also by Remark 4.2, for each $a \in \mathbb{C}G$, there exists a unique $z \in \mathbb{C}$ such that $v_P(a-z) > 0$. It follows that the natural embedding $\mathbb{C} \hookrightarrow \mathbb{C}G/\mathfrak{m}$ is an isomorphism and, in particular, $\mathfrak{m}$ is a maximal ideal of $\mathbb{C}G$. Then the natural homomorphism $\mathbb{C}G \to \mathbb{C}G/\mathfrak{m} = \mathbb{C}$ restricts to a group homomorphism $\chi : G \to \mathbb{C}^*$. In other words, χ is defined by $\chi(g) = z$ where z is the unique element of $\mathbb{C}$ satisfying $v_P(g - z) > 0$.

Define $\tilde{G} \subset \mathbb{C}G$ by $\tilde{G} = \{g/\chi(g) \,|\, g \in G\}$. Clearly, $\tilde{G}$ is a multiplicative group and $G \cong \tilde{G}$ via $g \mapsto g/\chi(g)$. Furthermore, $\mathbb{C}G = \mathbb{C}\tilde{G}$. Consequently, replacing G by $\tilde{G}$, we can assume without loss of generality that $\chi(g) = 1$, i.e., $v_P(g - 1) > 0$ for all $g \in G$ (so now $\mathfrak{m}$ is the augmentation ideal $\mathfrak{d}$ of $\mathbb{C}G$).

Corollary 5.4. *Let G be a group with involution. Then the following are equivalent:*

1) *G is residually 'torsion-free nilpotent'.*
2) *There exists a $*$-ordering P of $\mathbb{C}G$ such that the value semigroup Γ_P of v_P has the properties: a) $\Gamma_P \geq 0$, b) there exists a least positive element $\gamma_0 \in \Gamma_P$, c) the multiples of γ_0 are cofinal in Γ_P.*

Proof. To prove 1) $\Rightarrow$ 2), we apply the proof of Theorem 5.1 2) to construct a $*$-ordering P and observe that by Remark 4.11 the value semigroup Γ_P of v_P is isomorphic to the semigroup of monomials in a certain set of variables, with a degree-lexicographic order. So a) and c) are clear. Condition b) will also hold if the set of variables has the least element (which is then also the least monomial $\neq 1$). The choice of the order on the variables allows enough freedom to achieve this.

Conversely, suppose 2) holds. Let $\mathfrak{m} = \{a \in \mathbb{C}G \,|\, v_P(a) > 0\}$. As we showed, a) implies that replacing G with the isomorphic group $\tilde{G} = \{g/\chi(g) \,|\, g \in G\}$, we can assume without loss of generality that $\mathfrak{m}$ equals the augmentation ideal $\mathfrak{d}$. Now if $a \in \mathfrak{d}^n$, then from b) it follows that $v_P(a) \geq n\gamma_0$. So if $a \in \cap_{n=0}^{\infty}\mathfrak{d}^n$, then $v_P(a) \geq n\gamma_0$ for all $n \in \mathbb{N}$, which implies by c) that $v_P(a) = \infty$, i.e., $a = 0$. Hence G is residually 'torsion-free nilpotent'. □

Now we prove our necessary condition for $*$-orderability of $\mathbb{C}G$.

Theorem 5.5. *If $\mathbb{C}G$ with the standard involution is $*$-orderable, then G is orderable.*

Proof. Suppose P is a $*$-ordering of $\mathbb{C}G$ and $v = v_P$ is its natural valuation. By assumption, the involution $*$ on $\mathbb{C}G$ is defined by $\sum_g c_g g \mapsto$

$\sum_g \bar{c}_g g^{-1}$. Then $v(g) = v(g^*) = v(g^{-1}) = -v(g)$, so $v(g) = 0$ for $g \in G$. It follows that $v(a) \geq 0$ for all $a \in \mathbb{C}G$. Consequently, we have a group homomorphism $\chi : G \to \mathbb{C}^*$ defined by $v(g - \chi(g)) > 0$ for $g \in G$. Note that if $\chi(g) = z$, then $\chi(g^{-1}) = \chi(g^*) = z^* = \overline{z}$, so $z\overline{z} = \chi(g)\chi(g^{-1}) = \chi(1) = 1$. Thus, in the case of the standard involution, the image of G under χ is a subgroup of the unit circle. As before, we replace G by $\tilde{G} = \{g/\chi(g) \mid g \in G\}$.

Each $g \in G$ decomposes in $\mathbb{C}G$ as

$$g = \frac{g + g^{-1}}{2} + i\frac{i(g^{-1} - g)}{2}$$

with $g + g^{-1}$ and $i(g^{-1} - g)$ symmetric. Since $g + g^{-1} \equiv 2 \pmod{\mathfrak{d}}$, $g + g^{-1}$ is strictly positive, i.e., belongs to $P \setminus \{0\}$. However, $i(g^{-1} - g)$ may be either positive or negative. It cannot be zero, because $\mathbb{C}G$ is a domain and, consequently, the group G is torsion-free.

We claim that if $i(g^{-1} - g)$ and $i(h^{-1} - h)$ are both strictly positive, then so is $i(g^{-1}h^{-1} - hg)$. Indeed,

$$i(g^{-1} - g)(h^{-1} + h) + (h^{-1} + h)i(g^{-1} - g)$$

and

$$i(h^{-1} - h)(g^{-1} + g) + (g^{-1} + g)i(h^{-1} - h)$$

are both strictly positive. Adding yields

$$i(g^{-1}h^{-1} - hg) + i(h^{-1}g^{-1} - gh).$$

At the same time,

$$gi(g^{-1}h^{-1} - hg)g^{-1} = i(h^{-1}g^{-1} - gh),$$

so $i(g^{-1}h^{-1} - hg)$ and $i(h^{-1}g^{-1} - gh)$ have the same sign. Consequently, $i(g^{-1}h^{-1} - hg)$ and $i(h^{-1}g^{-1} - gh)$ are both strictly positive.

Now define

$$T = \{g \in G \mid i(g^{-1} - g) \in P\}. \tag{5}$$

It is immediate that $G = T \cup T^{-1}$, $T \cap T^{-1} = \{1\}$, and $gTg^{-1} \subset T$ for each $g \in G$. By the claim that we have just proved, $T \cdot T \subset T$. Therefore, T is an ordering on G. □

To illustrate Theorem 5.5, consider the infinite cyclic group $G = \langle g \rangle$. Recall that in Example 4.6 we constructed a $*$-ordering on $\mathbb{C}G$ by declaring that a symmetric $a = \sum_n \lambda_n g^n$ positive or negative according to the sign of the lowest coefficient of the power series $\sum_n \lambda_n \exp(int) \in \mathbb{R}[[t]]$. Clearly, the natural valuation $v(a)$, for arbitrary a, is equal to the lowest degree of t appearing in the corresponding series. So we see that in this case $v(g-1) > 0$ and thus $\chi(g) = 1$. Further, $i(g^{-n} - g^n)$ has the same sign as n, because the corresponding power series is $2\sin(nt)$, whose lowest term is $2nt$. Thus we see that the ordering on $G \cong \mathbb{Z}$ defined by (5) is just the standard ordering of $\mathbb{Z}$.

Example 5.6. There exists an orderable group G such that the group algebra $\mathbb{C}G$ is not $*$-orderable. Take

$$G = \langle x, y | \; xy = y^2 x \rangle$$

with the standard involution (this is the group of Example 2.3 with $n = 2$).

Proof. Assume that there exists a $*$-ordering P on $\mathbb{C}G$. Let $v = v_P$ be its natural valuation. As in the proof of Theorem 5.5, it follows that $v(g) = 0$ for all $g \in G$. Replacing x and y by $x/\chi(x)$ and $y/\chi(y)$, we may assume that $v(x - 1) > 0$ and $v(y - 1) > 0$. Writing $x = 1 + s$ and $y = 1 + t$, the defining relation $xy = y^2 x$ gives

$$st = t(2s + 1) + t^2(s + 1).$$

Since $v(t^2(s + 1)) = 2v(t) > v(t) = v(t(2s + 1))$, it follows that $v(st) = v(t(2s + 1))$. Hence $v(s) = 0$, a contradiction. □

6. $*$-Orderability of Smash Products

The aim of this section is to find necessary and sufficient conditions for the $*$-orderability of the smash products of the form $U(L)\#_\varphi \mathbb{C}G$ under some natural assumptions. Theorem 6.2 is the $*$-analog of Theorem 3.7.

Let H be a cocommutative Hopf algebra over an algebraically closed field $\Bbbk$ of characteristic 0. Then $H = U(L)\#_\varphi \Bbbk G$ where $G = G(H)$, $L = P(H)$, and $\varphi : G \to \mathrm{Aut}(L)$ is a group homomorphism. Suppose now $\Bbbk$ is a $*$-field. We want to find all Hopf involutions of H.

Lemma 6.1. *There exists a natural one-to-one correspondence between Hopf algebra involutions of $H = U(L)\#_\varphi \Bbbk G$ and pairs of group and Lie involutions on G and L, respectively, that satisfy*

$$\varphi_{g^*}(x^*) = \varphi_{g^{-1}}(x)^* \text{ for all } g \in G \text{ and } x \in L. \tag{6}$$

Proof. Every Hopf involution on H preserves $1\#G$ and $L\#1$, hence we can define involutions on G and L respectively by

$$1\#g^* := (1\#g)^*, \quad x^*\#1 := (x\#1)^*.$$

We can express $(x\#g)^*$ in two ways:

$$(x\#g)^* = ((x\#1)(1\#g))^* = (1\#g)^*(x\#1)^* = (1\#g^*)(x^*\#1) = \varphi_{g^*}(x^*)\#g^*$$

and

$$\begin{aligned}(x\#g)^* &= ((1\#g)(\varphi_{g^{-1}}(x)\#1))^* = (\varphi_{g^{-1}}(x)\#1)^*(1\#g)^* \\ &= (\varphi_{g^{-1}}(x)^*\#1)(1\#g^*) = \varphi_{g^{-1}}(x)^*\#g^*.\end{aligned}$$

Condition (6) follows.

Conversely, suppose we have a pair of involutions on L and G such that (6) holds. Then we can lift the first involution (as well as φ) from L to $U(L)$. Note that (6) now holds for all $x \in U(L)$. Set

$$(x\#g)^* := \varphi_{g^*}(x^*)\#g^* = \varphi_{g^{-1}}(x)^*\#g^* \text{ for all } g \in G \text{ and } x \in U(L)$$

and extend to the entire H by additivity. Clearly, $*$ agrees with the involution on $\Bbbk$. For all $x, y \in U(L)$ and $g, h \in G$, we have

$$(x\#g)^{**} = (\varphi_{g^*}(x^*)\#g^*)^* = \varphi_{(g^*)^{-1}}(\varphi_{g^*}(x^*))^*\#g = x\#g$$

and

$$\begin{aligned} ((x\#g)(y\#h))^* &= (x\varphi_g(y)\#gh)^* = \varphi_{h^{-1}g^{-1}}(x\varphi_g(y))^*\#h^*g^* \\ &= \varphi_{h^{-1}}(y)^*\varphi_{h^{-1}g^{-1}}(x)^*\#h^*g^* \\ &= (\varphi_{h^{-1}}(y)^*\#h^*)(\varphi_{g^{-1}}(x)^*\#g^*) = (y\#h)^*(x\#g)^*. \end{aligned}$$

It remains to verify that $\Delta \circ * = (* \otimes *) \circ \tau \circ \Delta$. Since both maps are anti-homomorphisms of algebras, it suffices to check the equality on any set of generators of H. Clearly, $L\#1$ and $1\#G$ generate H, and the desired equality holds for these elements (see Examples 4.4 and 4.5). □

Condition (6) can be restated in the following way. Suppose L is an algebra (not necessarily Lie or associative) with involution $*$. If α is an automorphism of L, then so is the composition $(* \circ \alpha^{-1} \circ *)$. In fact, the map $\alpha \mapsto (* \circ \alpha^{-1} \circ *)$ is a group involution on $\mathrm{Aut}(L)$. Then condition (6) simply says that $\varphi : G \to \mathrm{Aut}(L)$ is a homomorphism of groups with involution.

Theorem 6.2. *Let L be a finite-dimensional complex Lie algebra with involution, G a group with involution and $\varphi : G \to \mathrm{Aut}(L)$ a homomorphism of groups with involution. Then $U(L)\#_\varphi\mathbb{C}G$ with the induced involution is $*$-orderable if and only if*

1) *$\mathbb{C}G$ is $*$-orderable, and*
2) *$\varphi(G)$ is a unipotent matrix group.*

Proof. Observe first of all that condition 2) is equivalent to 2′) that says that L has a basis consisting of symmetric elements in which all φ_g, $g \in G$, have a lower unitriangular matrix. Clearly, 2′) implies 2). Suppose that 2) holds. Then all $\varphi_g \in \varphi(G)$ have a common eigenvector $x \in L$ (with eigenvalue 1). If we can show that we can always find a symmetric common eigenvector, then condition 2′) will follow by induction on $\dim L$. Write $x = x_1 + ix_2$ where x_1, x_2 are symmetric. Without loss of generality, $x_1 \neq 0$. Using (6), we compute: $\varphi_g(w^*) = \varphi_{(g^*)^{-1}}(w)^* = w^*$. Hence $w^* = x_1 - ix_2$ is also a common eigenvector for $\varphi_g \in \varphi(G)$ (with eigenvalue 1). It follows that x_1 is a symmetric common eigenvector.

Now suppose 1) and 2′) hold. Fix an extended $*$-ordering Q of $\mathbb{C}G$ and a basis $e_1, \ldots, e_n$ of L consisting of symmetric elements in which the matrices of φ_g are lower unitriangular. Define an ordering on PBW monomials as in Example 2.12. Every nonzero element $z \in H := U(L)\#_\varphi\mathbb{C}G$ can be expressed uniquely as $z = \sum_{k=1}^r M_k\#a_k$ where M_k are PBW monomials such that $M_1 < \ldots < M_r$ and $a_k \in \mathbb{C}G$. Define $\mathrm{lc}(z) := a_1$ and $\mathrm{lc}(0) := 0$. We claim that the set

$$P := \{z \in H |\ \mathrm{lc}(z) \in Q\}$$

is an extended $*$-ordering on H. Therefore, 1) is true.

It is clear that $P+P\subset P$ and $P\cap -P=\{0\}$. In the verification of other properties we will use the following observations. Let $w:U(L)\to\Gamma\cup\{\infty\}$ be the valuation determined by our monomial ordering (see Example 2.12). Then $\mathrm{gr}(U(L),w)$ is the algebra of polynomials in $e_1,\ldots,e_n$. Since $e_1,\ldots,e_n$ are symmetric, we also have $M^*=M+o$ where $w(o)>w(M)$. For every PBW monomial M and every $g\in G$ we have that $\varphi_g(M)=M+o$ where $w(o)>w(M)$, because φ_g is unitriangular.

Now we extend w to the entire H by setting $w(z):=w(M_1)$ for nonzero $z=\sum_{k=1}^r M_k\#a_k\in H$ with $M_1<\ldots<M_r$. Then $w:H\to\Gamma\cup\{\infty\}$ is a vector space valuation (but we do not know at this point that w is a ring valuation).

To prove that $P^*\subset P$, it suffices to show that $\mathrm{lc}(z^*)=\mathrm{lc}(z)^*$ for every $z\in H$. If $z=M\#(\sum c_g g)+o=\sum c_g(M\#g)+o$ where $w(o)>w(M)$, then $z^*=\sum\bar{c}_g\varphi_{g^*}(M^*)\#g^*+o^*=\sum\bar{c}_g M\#g^*+o'=M\#(\sum c_g g)^*+o'$ where $w(o')>w(M)$. Hence $\mathrm{lc}(z^*)=\sum c_g g=\mathrm{lc}(z)^*$. This computation also shows that $w(z^*)=w(z)$.

To prove that $P\cdot P\subset P$, it suffices to show that $\mathrm{lc}(z_1z_2)=\mathrm{lc}(z_1)\,\mathrm{lc}(z_2)$. If $z_1=M\#a+o=M\#(\sum c_g g)+o_1=\sum c_g(M\#g)+o_1$ with $w(o_1)>w(M)$ and $z_2=N\#b+o_2$ with $w(o_2)>w(N)$, then

$$\begin{aligned} z_1z_2 &= (M\#a)(N\#b)+o'=\sum c_g(M\#g)(N\#b)+o' \\ &= \sum c_g M\varphi_g(N)\#gb+o'=\sum c_g MN\#gb+o'' \\ &= MN\#ab+o'' \text{ where } w(o'),w(o'')>w(MN). \end{aligned}$$

Hence $\mathrm{lc}(z_1z_2)=ab=\mathrm{lc}(z_1)\,\mathrm{lc}(z_2)$. This also shows that $w(z_1z_2)=w(M)+w(N)=w(z_1)+w(z_2)$, so w is a $*$-valuation on H.

Since $\mathrm{lc}(uzu^*)=\mathrm{lc}(u)\,\mathrm{lc}(z)\,\mathrm{lc}(u)^*$, it follows that $uPu^*\subset P$ for any $u\in H$. Every symmetric element can be written as a sum of elements of the form $M\#a+(M\#a)^*$. As we already computed, $(M\#a)^*=M\#a^*+o$, so $M\#a+(M\#a)^*=M\#(a+a^*)+o$ where $w(o)>w(M)$. Since $a+a^*$ is symmetric, it belongs to $Q\cup -Q$. It follows that $M\#a+(M\#a)^*\in P\cup -P$. This completes the proof that P is an extended $*$-ordering (compatible with the $*$-valuation w).

Conversely, suppose H is $*$-orderable. Let P be a $*$-ordering on H. Then $P\cap\mathbb{C}G$ is a $*$-ordering on $\mathbb{C}G$, so 1) holds. Pick a basis $e_1,\ldots,e_n$ of L consisting of positive symmetric elements such that $v_P(e_1)<\cdots<v_P(e_n)$. For $g\in G$ and $x\in L$ we have $\varphi_g(x)=gxg^{-1}$ and thus $v_P(\varphi_g(x))=v_P(x)$ (recall that the value semigroup of v_P is commutative!). In particular, $v_P(\varphi_g(e_i))=v_P(e_i)$ for $i=1,\ldots,n$. Therefore, we can write

$$\varphi_g(e_k)=\sum_{l=k}^{n} c_{kl}(g)e_l=c_{kk}(g)e_k+o_k,\quad v_P(o_k)>v_P(e_k). \tag{7}$$

In other words, the matrices of φ_g, $g \in G$, are lower triangular. We claim that $c_{kk}(g) = 1$ for every $g \in G$ and every $k = 1, \ldots, n$. Indeed, from (1) it follows that symmetric elements commute in $\overline{H} := \mathrm{gr}(H, v_P)$. Since every $\overline{z} \in \overline{H}$ can be written $\overline{z} = \overline{z}_1 + i\overline{z}_2$ with $\overline{z}_1, \overline{z}_2$ symmetric, we conclude that $\overline{H}$ is commutative. Therefore, $v_P(\varphi_g(x) - x) = v_P(gxg^{-1} - x) > v_P(x)$ for $x \in L$ and $g \in G$. Comparing this with (7), we obtain $v_P(c_{kk}(g)e_k - e_k) > v_P(e_k)$. Hence $v_P(c_{kk}(g) - 1) > 0$, so $c_{kk}(g) = 1$, proving 2′). □

Example 6.3. Consider $H = \mathbb{C}\langle x^{\pm 1}, y\rangle/(xy - qyx)$ where $q \in \mathbb{C}^*$. Clearly, $H = U(L)\#_\varphi \mathbb{C}G$ where $L = \langle y \rangle$, $G = \langle x \rangle$, and $\varphi_x(y) = qy$. For any Lie involution on L, we can scale y so that $y^* = y$. There are two involutions on G: $x^* = x^{-1}$ and $x^* = x$. In the first case, condition (6) is satisfied iff $q \in \mathbb{R}$. In the second case, it is satisfied iff $|q| = 1$. In both cases, by Theorem 6.2, H is not $*$-orderable unless $q = 1$.

7. Open Problems

1) Find exact conditions on G for $\mathbb{C}G$ to be $*$-orderable, at least in the case of the standard involution. (It is possible for $\mathbb{C}G$ to be $*$-orderable without G being residually 'torsion-free nilpotent' — see I. Klep & P. Moravec, $*$-Orderable groups, a work in progress.)
2) If G is orderable, then $U(L)\#\Bbbk G$ is a domain. When can $U(L)\#\Bbbk G$ be embedded in a skew field? Can every ordering (resp., $*$-ordering) be extended from $U(L)\#\Bbbk G$ to a skew-field containing it? (The answer is yes in the Ore case since every ordering (resp., $*$-ordering) on an Ore domain can be extended to its skew-field of quotients — see [1], [4], and our Proposition 3.8.)
3) Is Corollary 3.3 true if $\mathbb{R}$ is replaced by $\mathbb{Q}$ (or any other Archimedean field)? What about Theorem 3.7?
4) If H is a pointed Hopf algebra over $\Bbbk$ (not necessarily cocommutative), then it is filtered by the so called coradical filtration. The associated graded Hopf algebra $\mathrm{gr}(H)$ is isomorphic to the biproduct $R\#_\varphi^\rho \Bbbk G$ (which is just the smash product $R\#_\varphi \Bbbk G$ as far as the algebra structure is concerned), where $G = G(H)$. If $\mathrm{char}\,\Bbbk = 0$ and H is generated by its group-like and skew-primitive elements, then R is the so called Nichols algebra $\mathcal{B}(V)$ of a braided vector space V — see e.g. [2]. What are the necessary and sufficient conditions for the smash product $\mathcal{B}(V)\#_\varphi \Bbbk G$ to be orderable (resp., $*$-orderable)? Can one construct orderings (resp., $*$-orderings) on $\mathrm{gr}(H) = \mathcal{B}(V)\#_\varphi \Bbbk G$ in such a way that they can be pulled back to H?
5) In this paper we constructed orderings and $*$-orderings (with zero support) of Hopf algebras viewed just as algebras, i.e., forgetting the comultiplication Δ. Should one impose any compatibility conditions between the ($*$-)ordering and Δ?

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