A VARIANT OF HIGHER PRODUCT LEVELS OF INTEGRAL DOMAINS AND $*$-DOMAINS

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Abstract. For every domain R and every even integer n we define $\text{ms}_n(R)$ (resp. $\text{ps}_n(R)$) as the smallest number k such that 0 is a sum of $k+1$ products (resp. permuted products) of n-th powers of nonzero elements from R. There are many results about ps_n in the literature but nothing is known about ms_n.

We prove two results about ms_n of twisted Laurent series rings $R((x,\omega))$. The first result is that if $\text{ms}_2(R)=\infty$ and ω has order $n/2$ in $\text{Aut}(R)$, then $\text{ms}_n(R((x,\omega)))=\infty$. The second result is that there exist R and ω such that $\text{ms}_n(R((x,\omega)))=\infty$ and $\text{ps}_n(R((x,\omega)))<\infty$. (Take $k=\frac{n}{2}-1$, $R=\mathbb{R}(t_1,\ldots,t_k)$ and $\omega(f(t_1,\ldots,t_k))=f(-t_k,t_1-t_k,\ldots,t_{k-1}-t_k)$.)

Finally, we define ms_n and ps_n of a domain R with involution. For a certain involution on $R((x,\omega))$, we prove analogues of the first and the second result.

1. Introduction

The n-th level of a field F, denoted by $s_n(F)$, is the length of the shortest representation of -1 as a sum of n-th powers of elements from F. The study of $s_n(F)$ was initiated by J.-P. Joly in [9] and continued by E. Becker in a series of papers (e.g. [2], [3]). The case $n=2$ is classical, see [14].

The results about $s_n(F)$ do not carry over to skew-fields because of the fact that the product of two n-th powers need not be an n-th power. Therefore, for a skew-field D and number n, we have besides $s_n(D)$ at least two other related invariants, $\text{ms}_n(D)$ and $\text{ps}_n(D)$, which are defined as the number of terms in the shortest representation of -1 as a sum of products (in the case of ms_n) or permuted products (in the case of ps_n) of n-th powers of elements from D. The invariant $\text{ps}_n(D)$ for $n>2$ was introduced in [4] and studied later in [10] and [6]. The motivation for studying ps_n comes from the theory of higher level orderings. Namely, skew-fields with $\text{ps}_n(D)=1$ are furthest away from having an ordering of level $n/2$ while skew-fields with $\text{ps}_n(D)=\infty$ always have such an ordering, see [7] and [15]. Our motivation for

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studying ms_n will be explained in a separate paper where we will define a variant of higher level orderings. It measures how close is D to having this type of ordering. For $n > 2$, the invariants $s_n(D)$ and $\mathrm{ms}_n(D)$ have not yet been studied. If $n = 2$ then $\mathrm{ps}_2(D) = \mathrm{ms}_2(D)$ because every multiplicative commutator is a product of three squares, the standard notation in this case is $s_\pi(D)$. Important references are [17], [18] and [12] for $s_\pi(D) < \infty$ and [16], [1] for $s_\pi(D) = \infty$. Our $s_2(D)$ is usually denoted by $s(D)$. It has been studied by many authors mostly for quaternion algebras, the results are summarized in [13].

The next step is to pass from skew fields to integral domains (=associative rings without zero divisors). Let R_n be the set of n-th powers of elements from R, $P_n(R)$ the set of products of elements from R_n and $\Pi_n(R)$ for the set of all permutations of products from $P_n(R)$. For example, if $x, y, z \in R$, then $(xyz)(zxy)^3 \in \Pi_4(R)$, since it is a permutation of $x^4y^4z^4 \in P_4(R)$. We define

$$\begin{aligned} \underline{s}_n(R) &= \min\{k \mid \exists z_0, \ldots, z_k \in R_n \setminus \{0\} : 0 = \textstyle\sum_{i=0}^k z_i\}, \\ \mathrm{ms}_n(R) &= \min\{k \mid \exists z_0, \ldots, z_k \in P_n(R) \setminus \{0\} : 0 = \textstyle\sum_{i=0}^k z_i\}, \\ \mathrm{ps}_n(R) &= \min\{k \mid \exists z_0, \ldots, z_k \in \Pi_n(R) \setminus \{0\} : 0 = \textstyle\sum_{i=0}^k z_i\}, \end{aligned}$$

where $\min \emptyset = \infty$. Obviously, we have $\mathrm{ps}_n(R) \le \mathrm{ms}_n(R) \le \underline{s}_n(R)$ for every n and R. If n is odd then all are equal to 1. This definition of ms_n and ps_n extends the definition from the skew-field case, however $s_n(D)$ need not be equal to $\underline{s}_n(D)$. The number $\underline{s}_2(D)$ is usually denoted by $\underline{s}(D)$ and called *sublevel*, see [13].

We will prove two results about ms_n for twisted Laurent series rings:

Theorem. *If* $\mathrm{ms}_2(R) = \infty$ *and* ω *has order* m *in* $\mathrm{Aut}(R)$*, then*

$$\mathrm{ms}_{2m}(R((x, \omega))) = \infty.$$

Theorem. *For every* $m \ge 2$*, there exist* R *and* ω *such that*

$$\mathrm{ms}_{2m}(R((x, \omega))) = \infty \text{ and } \mathrm{ps}_{2m}(R((x, \omega))) < \infty.$$

Let $(D, *)$ be a skew-field with involution. An element of the form xx^*, $x \in D$, is called a *hermitian square* in D. For every natural number m we define *the* $2m$*-th hermitian level* $s_{2m}(D, *)$ as the number of terms in the shortest representation of -1 as a sum of m-th powers of hermitian squares. The results about $s_2(D, *)$ are surveyed in [13]. Again, the product of two hermitian squares need not be a hermitian square. Therefore, we have at least two related invariants, $\mathrm{ms}_{2m}(D, *)$

and $\mathrm{ps}'_{2m}(D,*)$, which are defined as the number of terms in the shortest representation of -1 as a sum of products (resp. permuted products) of m-th powers of hermitan squares. We do not break up hermitian squares when we permute them. Note that $\mathrm{ms}_4(D,*) = \mathrm{ps}'_4(D,*)$. The invariants $\mathrm{ms}_{2m}(D,*)$ and $\mathrm{ps}'_{2m}(D,*)$ have not yet been studied.

The theory of $*$-orderings of higher level (see [5] which is motivated by [8]) suggests that another invariant, $\mathrm{ps}_{2m}(D,*)$, may be more natural than $\mathrm{ps}'_{2m}(D,*)$. Here, the set of nonzero permuted products of m-th powers of hermitian squares is defined as the subgroup of $D \setminus \{0\}$ generated by all elements of the form $(dd^*)^m$ and $dsd^{-1}s^{-1}$, where $d, s \in D \setminus \{0\}$ and $s = s^*$. If we require also that s is a hermitian square, we get $\mathrm{ps}'_{2m}(D,*)$. Hence, $\mathrm{ps}_{2m}(D,*) \le \mathrm{ps}'_{2m}(D,*)$. The invariant $\mathrm{ps}_{2m}(D,*)$ will not be considered in the sequel.

For every integer m and every domain with involution $(R,*)$ write $H_{2m}(R,*)$ for the set of all m-th powers of hermitian squares in R, $P_{2m}(R,*)$ for the set of all products of elements from $H_{2m}(R,*)$ and $\Pi'_{2m}(R,*)$ for the set of permuted products of elements from $H_{2m}(R,*)$ (hermitian squares do not break up when permuted). We define

$$\begin{aligned} \underline{s}_{2m}(R,*) &= \min\{k \mid \exists z_0,\dots,z_k \in H_{2m}(R,*) \setminus \{0\} \colon 0 = \textstyle\sum_{i=0}^k z_i\}, \\ \mathrm{ms}_{2m}(R,*) &= \min\{k \mid \exists z_0,\dots,z_k \in P_{2m}(R,*) \setminus \{0\} \colon 0 = \textstyle\sum_{i=0}^k z_i\}, \\ \mathrm{ps}'_{2m}(R,*) &= \min\{k \mid \exists z_0,\dots,z_k \in \Pi'_{2m}(R,*) \setminus \{0\} \colon 0 = \textstyle\sum_{i=0}^k z_i\}, \end{aligned}$$

where $\min\emptyset = \infty$. We have $\mathrm{ps}'_{2m}(R,*) \le \mathrm{ms}_{2m}(R,*) \le \underline{s}_{2m}(R,*)$. If R is a skew-field, then $s_2(R,*) = \underline{s}_2(R,*)$ and the definitions of $\mathrm{ms}_{2m}(R,*)$ and $\mathrm{ps}'_{2m}(R,*)$ are compatible with above.

If ω is an automorphism of a domain R with involution $*$ such that $\omega(r)^* = \omega^{-1}(r^*)$ for every $r \in R$, then $(\sum a_i x^i)^* = \sum \omega^i(a_i^*) x^i$ defines an involution on the twisted Laurent series ring $R((x,\omega))$. We will prove two results about $\mathrm{ms}_n(R((x,\omega)),*)$.

Theorem. *Let R be a $*$-domain with $\mathrm{ms}_2(R,*) = \infty$ and ω an automorphism of R compatible with $*$. If ω has order m in* $\mathrm{Aut}(R)$ *then*

$$\mathrm{ms}_{2m}(R((x,\omega)),*) = \infty.$$

Theorem. *For every $m \ge 3$, there exist a $*$-domain R and an automorphism ω of R compatible with $*$ such that*

$$\mathrm{ms}_{2m}(R((x,\omega)),*) = \infty \text{ and } \mathrm{ps}'_{2m}(R((x,\omega)),*) < \infty.$$

2. The Mal'cev Neumann construction

If R is a ring, Γ an ordered group and $\omega : \Gamma \to \mathrm{Aut}(R)$ an antihomomorphism of groups, then the Mal'cev Neumann ring $R((\Gamma,\omega))$ consists of all formal expressions $a = \sum a_\gamma \gamma$, where $\mathrm{supp}(a)$ is a well-ordered subset of Γ. Addition is defined termwise and multiplication by

$$(a_\gamma \gamma)(b_\delta \delta) = a_\gamma \omega_\gamma(b_\delta)\gamma\delta.$$

If R is a skew-field, then $R((\Gamma,\omega))$ is also a skew-field by [11], Lemma 14.17. If R is an integral domain, then $R((\Gamma,\omega))$ is clearly also an integral domain. If $\Gamma = \mathbb{Z}$, then ω is uniquely determined by ω_1 and $R((\Gamma,\omega))$ is just the usual twisted Laurent series ring $R((x,\omega_1))$.

Theorem 1. *Let $A = R((\Gamma,\omega))$, where R, Γ, ω are as above. If R is an integral domain with $\mathrm{ms}_2(R) = \infty$ and n is an even number such that ω_γ has order $n/2$ in $\mathrm{Aut}(R)$ for every $\gamma \in \Gamma$, then $\mathrm{ms}_n(A) = \infty$.*

Proof. If $a = c\gamma$, then $a^n = (c\gamma)^n = c\omega_\gamma(c)\omega_{\gamma^2}(c)\cdots\omega_{\gamma^{n-1}}(c)\gamma^n$. Since $\omega_{\gamma^{\frac{n}{2}+i}}(c) = \omega_{\gamma^i}(c)$ for every $i = 0,\ldots,\frac{n}{2}-1$, it follows that $a^n = r^2\gamma^n$, where $r = c\omega_\gamma(c)\cdots\omega_{\gamma^{\frac{n}{2}-1}}(c)$. It follows that $(c_1\gamma_1)^n\cdots(c_k\gamma_k)^n = (r_1^2\gamma_1^n)\cdots(r_k^2\gamma_k^n) = r_1^2\cdots r_k^2\gamma_1^n\cdots\gamma_k^n$. The last equality follows from the fact that $(r\gamma^n)(s\delta^n) = r\omega_{\gamma^n}(s)\gamma^n\delta^n = rs\gamma^n\delta^n$ by induction on n.

If $\mathrm{ms}_n(A) < \infty$, then there exist nonzero $a_{i,j} \in A$ such that $0 = \sum_{i=1}^m a_{i,1}^n\cdots a_{i,k_i}^n$. If $a_{i,j} = c_{i,j}\gamma_{i,j}+$ higher terms, then

$$a_{i,1}^n\cdots a_{i,k_i}^n = r_{i,1}^2\cdots r_{i,k_i}^2\gamma_{i,1}^n\cdots\gamma_{i,k_i}^n + \text{higher terms.}$$

Let $\gamma = \min\{\gamma_{i,1}^n\cdots\gamma_{i,k_i}^n : i = 1,\ldots,m\}$ and let i_t, $t = 1,\ldots,l$ be all indices for which $\gamma_{i_t,1}^n\cdots\gamma_{i_t,k_{i_t}}^n = \gamma$. Then

$$0 = \sum_{t=1}^{l} a_{i_t,1}^n\cdots a_{i_t,k_{i_t}}^n = (\sum_{t=1}^{l}\prod_{j=1}^{k_{i_t}} r_{i_t,j}^2)\gamma + \text{higher terms}$$

By comparing the coefficients on both sides, we see that

$$0 = \sum_{t=1}^{l}\prod_{j=1}^{k_{i_t}} r_{i_t,j}^2, \text{ where } r_{i_t,j} \neq 0,$$

contradicting the assumption on R. □

When can we extend an involution from R to $R((\Gamma,\omega))$? We say that $*$ and ω are *compatible* if $\omega_\gamma(r)^* = \omega_{\gamma^{-1}}(r^*)$ for every $r \in R$ and every $\gamma \in \Gamma$. If $*$ and ω are compatible and Γ is abelian then

$$(\sum a_\gamma\gamma)^* = \sum \omega_\gamma(a_\gamma^*)\gamma$$

defines an involution on $R((\Gamma,\omega))$. Note that only relations $(c\gamma)^{**} = c\gamma$ and $((c\gamma)(d\delta))^* = (d\delta)^*(c\gamma)^*$ have to be verified and that these are true by a straightforward computation.

Theorem 2. *Let $A = R((\Gamma, \omega))$, where R is a $*$-domain and $*$ is compatible with ω. If $\mathrm{ms}_2(R, *) = \infty$ and m is a positive integer such that ω_γ has order m in $\mathrm{Aut}(R)$ for every $\gamma \in \Gamma$, then $\mathrm{ms}_{2m}(A, *) = \infty$.*

Proof. If $a = (c\gamma)(c\gamma)^* = c\omega_{\gamma^2}(c^*)\gamma^2$, then

$$\begin{aligned} a^m &= c\omega_{\gamma^2}(c^*)\omega_{\gamma^2}(c)\omega_{\gamma^4}(c^*)\cdots\omega_{\gamma^{2m-2}}(c)\omega_{\gamma^{2m}}(c^*)\gamma^{2m} = \\ &= c\omega_{\gamma^2}(c^*c)\cdots\omega_{\gamma^{2m-2}}(c^*c)\omega_{\gamma^{2m}}(c^*)\gamma^{2m} = rr^*\gamma^{2m}, \end{aligned}$$

where

$$r = \begin{cases} c\omega_{\gamma^2}(c^*c)\cdots\omega_{\gamma^{m-1}}(c^*c) & m \text{ odd} \\ c\omega_{\gamma^2}(c^*c)\cdots\omega_{\gamma^{m-2}}(c^*c)c^* & m \text{ even} \end{cases}$$

In both cases we used that

$$\omega_{\gamma^{2m-2k}}(c^*c) = \omega_{\gamma^{-2k}}(c^*c) = \omega_{\gamma^{2k}}((c^*c)^*) = \omega_{\gamma^{2k}}(c^*c)$$

for every $k = 0, \ldots, m$. In the second case we also used $\omega_{\gamma^m}(c^*c) = c^*c$. The rest of the proof is the same as in Theorem 1. For example, we have that $\prod_{i=1}^k((c_i\gamma_i)(c_i\gamma_i)^*)^m = \prod_{i=1}^k(r_ir_i^*\gamma_i^{2m}) = \prod_{i=1}^k r_ir_i^* \prod \gamma_i^{2m}$. □

3. Skew-fields with finite ps_n and infinite ms_n

Let m be a positive integer, $k = m - 1$ and $n = 2m$. Write $D_n = R((x, \omega))$, where $R = \mathbb{R}(t_1, \ldots, t_k)$ is the field of real rational functions in k variables and ω is an automorphism of R defined by $\omega(f(t_1, \ldots, t_k)) = f(-t_k, t_1 - t_k, \ldots, t_{k-1} - t_k)$. Since R is a field, D_n is a skew field by Lemma 14.17 in [11].

Theorem 3. *With the notation above, $\mathrm{ps}_n(D_n) \le k$ and $\mathrm{ms}_n(D_n) = \infty$.*

Proof. To prove that $\mathrm{ms}_n(D_n) = \infty$ we must check the assumptions of Theorem 1. The fact that $m_2(\mathbb{R}) = \infty$ implies that $m_2(R) = \infty$. Define an $m \times m$ matrix

$$A = \begin{bmatrix} 0 & 0 & \cdots & 0 & -1 \\ 1 & 0 & \cdots & 0 & -1 \\ 0 & 1 & \cdots & 0 & -1 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -1 \end{bmatrix}^T.$$

A simple computation shows that $I + A + \ldots + A^{m-1} = 0$. Since $\omega(f(t_1, \ldots, t_k)) = f((t_1, \ldots, t_k)A^i)$, and $A^m = I$, we have that ω^m is the identity.

Write x^i for the image of $i \in \Gamma$ in D_n and check that $[cx^i, dx^j] = (cx^i)(dx^j)(cx^i)^{-1}(dx^j)^{-1} = \frac{c\omega^i(d)}{d\omega^j(c)}$. It follows that

$$\sum_{i=1}^k [x^i, t_k] = \sum_{i=1}^k [1 \cdot x^i, t_k \cdot x^0] = \sum_{i=1}^k \frac{\omega^i(t_k)}{t_k} = \sum_{i=1}^k \frac{\mathrm{pr}_k((t_1,\ldots,t_k)A^i)}{t_k} = \\ = \frac{\mathrm{pr}_k((t_1,\ldots,t_k)\sum_{i=1}^k A^i)}{t_k} = \sum_{i=1}^k \frac{\mathrm{pr}_k((t_1,\ldots,t_k)(-I))}{t_k} = \frac{-t_k}{t_k} = -1,$$

so that $\mathrm{ps}_n(D_n) \leq k$. □

Let $m \geq 3$ be a positive integer, $n = 2m$ and ξ a primitive m-th root of 1. Write $K_n = R((x, \omega))$, where $R = \mathbb{C}(z)$ and $\omega(f(z)) = f(\xi z)$. The field R has a natural involution given by

$$\left(\frac{a_0 z^k + \ldots + a_k}{b_0 z^l + \ldots + b_l}\right)^* = \frac{\bar{a}_0 z^k + \ldots + \bar{a}_k}{\bar{b}_0 z^l + \ldots + \bar{b}_l}.$$

This involution is clearly compatible with ω, hence it extends to an involution on K_n by $(\sum f_i(z)x^i)^* = \sum f_i^*(\xi^i z)x^i$. By Lemma 14.17 from [11], $(K_n, *)$ is a $*$-field.

Theorem 4. *With the notation above,* $\mathrm{ps}'_n(K_n, *) \leq m - 1$ *and* $\mathrm{ms}_n(K_n, *) = \infty$.

Proof. Since $\mathrm{ms}_2(\mathbb{C}, *) = \infty$, it follows that $\mathrm{ms}_2(\mathbb{C}(z), *) = \infty$ and since $\xi^m = 1$, we have that ω has order m. Therefore, $\mathrm{ms}_n(K_n, *) = \infty$ by Theorem 2.

If $a = zz^* = z^2 \cdot x^0$ and $b = xx^* = 1 \cdot x^2$, then $ab = z^2x^2$, $ba = (\xi^2 z)^2 x^2 = \xi^4 z^2 x^2$ and $[a, b] = aba^{-1}b^{-1} = \xi^{-4}$. We assumed that $m \neq 1, 2$, hence $[a, b] \neq 1$. If l is the order of $[a, b]$, then $-1 = [a, b] + [a, b]^2 + \ldots + [a, b]^{l-1}$ and $l | m$. Therefore, $\mathrm{ps}'_n(K_n, *) \leq l - 1 \leq m - 1$. □

Open problem. Fix a positive integer m. Compute the minimum of $\mathrm{ps}_{2m}(D)$ where D runs through all skew-fields such that $\mathrm{ms}_{2m}(D) = \infty$. Compute the minimum of $\mathrm{ps}'_{2m}(D, *)$ where $(D, *)$ runs through all skew-fields with involution such that $\mathrm{ms}_{2m}(D, *) = \infty$. We conjecture that both numbers are equal to 1.

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