ON GENERALIZED REAL SEMIPRIME IDEALS

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ABSTRACT. Let R be an associative unital ring. A subset T of R is a P_n-preordering if $-1 \notin T$, $1 \in T$, $T + T \subseteq T$, $T \cdot T \subseteq T$ and $xt_1 \cdots xt_n \in T$ for every $x \in R$ and $t_1, \ldots, t_n \in T$. Our first result is that for every maximal P_n-preordering P on R there exists a (not necessarily abelian) n-torsion group G and a surjective semigroup homomorphism $\sigma\colon R \to G \cup \{0\}$ such that $P = \sigma^{-1}(\{0, 1\})$. If $\mathbb{Q} \subseteq R$ then for every P_n-preordering T the set $T \cap -T$ is a two-sided ideal in R. Our second result is that every such ideal is semiprime (resp. prime) if and only if it is radical (resp. completely prime).

1. Introduction

Orderings are the basic objects of real algebraic geometry because they generalize the notion of a real point. A subset P of a commutative unital ring R is an *ordering* if it is closed for addition and multiplication, $P \cup -P = R$ and $P \cap -P$ is a prime ideal. An alternative definition is that P is closed for addition, does not contain -1 and there exists a semigroup homomorphism $\sigma\colon R \to \{-1, 0, 1\}$ such that $P = \sigma^{-1}(\{0, 1\})$. It is well-known that a commutative ring R has an ordering if and only if $-1 \notin \sum R^2$, i.e. -1 is not a sum of squares. In [10], orderings on associative unital rings are defined using the first definition from above. It is shown that every prime ideal of the form $P \cap -P$ is also completely prime, hence the first and the second definition from above are also equivalent for noncommutative rings. An associative ring R admits an ordering if and only if -1 is not a sum of "permuted" products of squares of elements from R.

We are interested in generalized orderings that are defined by replacing the group $\{-1, 1\}$ from above by an arbitrary group. More

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precisely, if R is an associative unital ring and G a group, then a subset P of R is a *G-ordering* if $P + P \subseteq P$, $-1 \notin P$ and there exists a semigroup homomorphism $\sigma\colon R \to G \cup \{0\}$ such that $P = \sigma^{-1}(\{0, 1\})$.

For every even integer n write $\mu_n = \{z \in \mathbb{C} | \; z^n = 1\}$. A commutative ring R admits a μ_n-ordering if and only if $-1 \notin \sum R^n$, see [2]. It is conjectured that an associative ring R admits a μ_n-ordering if and only if $-1 \notin \sum \Pi_n(R)$, where $\Pi_n(R)$ is the set of all products which are up to a permutation of factors the products of n-th powers of elements from R and $\sum \Pi_n(R)$ is the set of all finite sums of elements from $\Pi_n(R)$. This is true if either R is Noetherian or n is a power of 2, see [3, 4]. The theory is based on earlier papers [1, 7, 11, 12].

The aim of this paper is to characterize rings that admit a G-ordering for G a (not necessarily abelian) torsion group of finite exponent. Let $P_n(R)$ be the smallest subset of R such that $1 \in P_n(R)$ and $xt_1 \cdots xt_n \in P_n(R)$ for every $x \in R$ and $t_1, \ldots, t_n \in P_n(R)$. Write $\sum P_n(R)$ for the set of all finite sums of elements from R. We prove:

Theorem A. *For every associative unital ring R and even n, $-1 \notin \sum P_n(R)$ if and only R admits a G-ordering for some n-torsion group G. In particular, $-1 \notin \sum \Pi_n(R)$ if and only if R admits a G-ordering for some abelian n-torsion group G.*

The second assertion of theorem A is a special case of the first. It can be seen as a contribution to the open problem mentioned above. Since every 2-torsion group is abelian, Theorem A implies that for every ring R, $-1 \notin \sum P_2(R)$ if and only if $-1 \notin \sum \Pi_2(R)$. However, if $n \geq 4$ is even then by the main result of [6], there exists a skew-field D such that $-1 \notin \sum P_n(D)$ and $-1 \in \sum \Pi_n(D)$. The proof of Theorem A for skew-fields is much easier than for general associative rings.

A subset M of R is a *P_n-module* if $1 \in M$, $M + M \subseteq M$ and $P_n(R) \cdot M \cdot P_n(R) \subseteq M$. A two-sided ideal J of R is *P_n-real* if there exists a proper P_n-module M such that $J = M \cap -M$. Replacing $P_n(R)$ with $\Pi_n(R)$ we obtain the definition of a Π_n-module and Π_n-real ideal. We prove a generalization of a result from [10] mentioned above.

Theorem B. *Let R be an associative unital ring and n an even integer. Every P_n-real ideal R on is prime if and only if it is completely prime. If also $\mathbb{Q} \subseteq R$ then every P_n-real ideal on R is semiprime if and only if it is radical.*

Since $P_n(R) \subseteq \Pi_n(R)$, every Π_n-real ideal is P_n-real. We will also show that every prime P_2-real ideal is Π_2-real. However, if $n \geq 4$ and D is a skew-field such that $-1 \notin \sum P_n(D)$ and $-1 \in \sum \Pi_n(D)$ (see above), then $\{0\}$ is a P_n-real prime ideal of D which is not Π_n-real.

2. The main technical result

From now on R will be a fixed associative unital ring and n an even integer. We will write $P_n = P_n(R)$. A subset $U \subseteq P_n$ is *a P_n-system* if $1 \in U$ and for every elements $x, y \in U$ there exists some $c \in P_n$ such that $xcy \in U$. Every multiplicative subset of P_n containing 1 is a P_n-system.

Theorem 1 is motivated by [13, Theorem 1.1.4]. It will be used in the proofs of Theorems A and B.

Theorem 1. *Let U be a P_n-system such that the family $\mathbf{M}$ of all P_n-modules avoiding $-U$ is nonempty. Every element of $\mathbf{M}$ is contained in some maximal element of $\mathbf{M}$. For every maximal element S of $\mathbf{M}$ we have that $S \cup -S = R$ and $S \cap -S$ is a completely prime ideal.*

Proof. The first assertion follows from the Zorn's Lemma. Let S be a maximal element of the family $\mathbf{M}$.

Claim 1: If $kr \in S$ for some $k \in \mathbb{N}$ and $r \in R$ then $r \in S$.

The set $S^e := \{r \in R|\ kr \in S \text{ for some } k \in \mathbb{N}\}$ is clearly a P_n-module. If $-u \in S^e$ for some $u \in U$, then $-ku \in S$ for some $k \in \mathbb{N}$. Since $u \in P_n \subseteq S$, it follows that $-u = -ku + (k-1)u \in S$, a contradiction. It follows that $S^e \in \mathbf{M}$. By the maximality of S, we have that $S^e = S$. The claim follows.

Claim 2: The set $J := S \cap -S$ is a two-sided ideal.

To prove that the set J is a left ideal take any $b \in J$ and $a \in R$. The identity $n!a = \sum_{i=0}^{n-1} (-1)^{n-i-1}\binom{n-1}{i}((a+i)^n - i^n)$ implies that there exist $s, t \in \Sigma P_n$ such that $n!a = s - t$. Clearly, the set J is an additive subgroup of R containing sb and tb. It follows that $n!ab \in J$. Since $S^e = S$ by Claim 1, it follows that $ab \in J$. The proof that J is a right ideal is similar.

Claim 3: If $a \in R$ and $aP_na \subseteq J$, then $a \in J$.

If Claim 3 is false, then there exists an element $a \in R$ such that $aP_na \subseteq J$ and $a \notin S$. Since the P_n-module $S + \Sigma(P_naP_n)$ properly contains S, it must meet $-U$. Hence, there exist elements $v \in U$, $m \in S$ and $z \in \Sigma(P_naP_n)$ such that $-v = m + z$. Since U is a P_n-system, there exist elements $c_1, \ldots, c_{n-1} \in P_n$ such that $u := vc_1vc_2v \cdots c_{n-1}v \in U$. It follows that $(v+z)c_1(v+z)c_2 \cdots c_{n-1}(v+z) = u + x + y$, where $x = zc_1vc_2 \cdots v + vc_1zc_2 \cdots v + \ldots + vc_1vc_2 \cdots z$ and $y \in \Sigma(RzP_nzR) \subseteq \Sigma(RaP_naR) \subseteq J$. We have that $x = (-m-v)c_1vc_2 \cdots v + vc_1(-m-v)c_2 \cdots v + \ldots + vc_1vc_2 \cdots (-m-v) = -nu - m'$ where $m' = mc_1vc_2v \cdots c_{n-1}v + vc_1mc_2v \cdots c_{n-1}v + \ldots + vc_1vc_2v \cdots c_{n-1}m \in S$. This implies a contradiction $-u = (u+x+y) + (n-2)u + m' - y \in P_n + \Sigma P_n + S + J \subseteq S$ with the assumption $-U \cap S = \emptyset$.

Claim 4: $S \cup -S = R$.

If the Claim 4 is false, then there exists an element $a \notin S \cup -S$. Since the P_n-modules $S + \Sigma(P_n a P_n)$ and $S - \Sigma(P_n a P_n)$ properly contain S, they must meet $-U$. Hence, there exist elements $u, v \in U$, $m_1, m_2 \in S$ and $z, w \in \Sigma(P_n a P_n)$ such that $-u = m_1 + z$ and $-v = m_2 - w$. Since U is a P_n-system, there exists an element $c \in P_n$ such that $ucv \in U$. Write $z' := zcv \in \Sigma(P_n a P_n)$. Pick any $c_1, \ldots, c_{n-1} \in P_n$ and write $t_1 := vc_1 z' c_2 z' \cdots z' c_{n-1} zcw$ and $t_2 := z' c_1 z' c_2 \cdots z' c_{n-1} z$. Since $t_1, t_2 \in (\Sigma(P_n a P_n))^n \subseteq \Sigma P_n$ and $zct_1 = t_2 cw$, we have that $-uct_1 - t_2 cv = (m_1 + z)ct_1 + t_2 c(m_2 - w) = m_1 ct_1 + t_2 cm_2 \subseteq S$. Consequently, $z' c_1 z' c_2 \cdots z' c_{n-1} z' = t_2 cv \in J$ for any $c_1, \ldots, c_{n-1} \in P_n$. Since J is an ideal, it follows that $z' c_1 z' c_2 \cdots z' c_{2^n - 1} z' \in J$ for any $c_1, \ldots, c_{2^n-1} \in P_n$. Applying Claim 3 n-times, we see that $z' \in J$. This implies a contradiction $-ucv = m_1 cv + z' \in -U \cap S = \emptyset$.

Claim 5: If $a \in R$ and $a^2 \in J$, then $a \in J$. (i.e. J is radical)

By Claim 4, we may assume that $a \in S$. Pick any $x \in P_n$ and note that either $ax - xa \in S$ or $xa - ax \in S$. If $ax - xa \in S$, then $(xa)^n(ax - xa) \in S$. Since $(xa)^n ax \in Ra^2 R \subseteq J$ and $(xa)^{n+1} \in P_n \cdot S \subseteq S$, it follows that $(xa)^{n+1} \in J$. If $xa - ax \in S$, then $(xa - ax)(ax)^n \in S$. Since $xa(ax)^n \in Ra^2 R \subseteq J$ and $(ax)^{n+1} \in S$, it follows that $(ax)^{n+1} \in J$. Since J is an ideal and $2n > n + 1$, we have in both cases that $(ax)^{2n-1} a \in J$. If we put $x = c_1 + \ldots + c_n$, where $c_1, \ldots, c_{2n-1} \in P_n$, we get $\sum_{1 \le j_1, \ldots, j_{2n-1} \le 2n-1} ac_{j_1} a \cdots c_{j_{2n-1}} a \in J$. Since every term of this sum belongs to P_n, it follows that all terms belong to J, in particular $ac_1 a \cdots c_{2n-1} a \in J$. Claim 3 implies that $a \in J$.

Claim 6: The ideal J is completely prime.

For any $a, b \in R$ such that $ab \in J$ we have that $a^n b^n \in J$. By Claim 4 either $a^n - b^n \in S$ or $b^n - a^n \in S$. If $a^n - b^n \in S$, then $-b^{2n} = (a^n - b^n)b^n - a^n b^n \in S$. If $b^n - a^n \in S$, then $-a^{2n} = a^n(b^n - a^n) - a^n b^n \in S$. It follows that either $a^{2n} \in J$ or $b^{2n} \in J$. Claim 5 implies that either $a \in J$ or $b \in J$. □

3. The proof of Theorem A

A subset T of R is a P_n-*preordering* if $-1 \notin T$, $T + T \subseteq T$, $T \cdot T \subseteq T$ and $(xT)^n \subseteq T$ for every $x \in R$. The set $T^0 := T \cap -T$ is called *the support* of T, write also $T^+ = T \setminus T^0$. If $-1 \notin \Sigma P_n(R)$ then $\Sigma P_n(R)$ is the smallest P_n-preordering on R.

Proposition 2. *Every P_n-preordering is contained in a preordering whose support is a completely prime ideal.*

Proof. Let T be a P_n-preordering. Clearly, T is also a proper P_n-module. Applying Theorem 1 with $U = \{1\}$, we get a proper P_n-module S containing T such that $S \cup -S = R$ and $J := S \cap -S$ is a completely prime ideal. Write $T' = T + J$. It is easy to verify that T' is a P_n-preordering. We claim that $T' \cap -T' = J$. One direction is clear. Now, pick $x \in T' \cap -T'$. There exist $t_1, t_2 \in T$ and $z_1, z_2 \in J$ such that $x = t_1 + z_1$ and $-x = t_2 + z_2$. It follows that $t_1 = -z_1 - z_2 - t_2$. Since $t \in S$ and $-z_1 - z_2 - t_2 \in -S$, it follows that $t_1 \in J$. So $x \in J$. □

For every P_n-preordering T whose support in a completely prime ideal we define its *division closure* $\overline{T} := \{x \in R | xT^+ \cap T \neq \emptyset\}$. We say that T is *divisible* if $T = \overline{T}$.

Proposition 3. *If T is a P_n-preordering with a completely prime support then $\overline{T}$ is the smallest divisible P_n-preordering containing T. Moreover, $\overline{T}^0 = T^0$ and $ab \in \overline{T}$ is equivalent to $ba \in \overline{T}$ for every $a, b \in T$.*

Proof. Clearly, $T \subseteq \overline{T}$ and $-1 \notin \overline{T}$.

We start by showing that $\overline{T} = \{x \in R | T^+x \cap T \neq \emptyset\}$. Suppose that $xt \in T$ for some $t \in T^+$. If $x \in T^0$, then $1 \cdot x \in T$. If $x \notin T^0$, then $x^n t \in T^+$ and $(x^n t)x = x^{n-1}(xt)x \in (xT)^n \subseteq T$. Hence, in both cases $t'x \in T$ for some $t' \in T^+$. The same argument proves the opposite direction. Manipulating this equality it can be easily seen that $\overline{T} + \overline{T} \subseteq \overline{T}$, $\overline{T} \cdot \overline{T} \subseteq \overline{T}$, $\overline{T}^0 = T^0$ and $\overline{T}$ is divisible.

If $ab \in \overline{T}$ some $a, b \in R$ then $abt \in T$ for some $t \in T^+$. It follows that $b(abt)b^{n-1} \in T$ and $btb^{n-1} \in T$. We distinguish two cases. If $btb^{n-1} \in T^+$ then $ba \in \overline{T}$ since $ba(btb^{n-1}) \in T$. If $btb^{n-1} \in T^0$ then $b \in T^0$ so that $ba \in T^0 \subseteq \overline{T}$ as well. Finally, suppose that $x \in R$ and $t_1, \ldots, t_n \in \overline{T}$. Then $x^n \in \overline{T}$, $x^n t_1 \in \overline{T}$, $x^{n-1}t_1 x \in \overline{T}$, $x^{n-1}t_1 x t_2 \in \overline{T}$, $x^{n-2}t_1 x t_2 x \in \overline{T}$, … , $xt_1 \cdots xt_n \in \overline{T}$. Hence $(x\overline{T})^n \subseteq \overline{T}$. □

Now we are able to prove Theorem A.

Proof. By Proposition 2, there exists a P_n-preordering T on R such that T^0 is a completely prime ideal. Let $\overline{T}$ be the division closure of T, cf Proposition 3. Let S be the multiplicative semigroup $R \backslash T^0 = R \backslash \overline{T}^0$. For any $x, y \in S$ write $x \sim y$ if and only if there exists $z \in S$ such that $xz, yz \in \overline{T}$. Applying the property $ab \in \overline{T} \Rightarrow ba \in \overline{T}$ several times, we can see that $\sim$ is a congruence relation and that the factor semigroup $G := S/\sim$ is an n-torsion group. The canonical projection $S \to G$ can be extended to a semigroup homomorphism $\sigma \colon R \to G^0$ by sending the elements of T^0 to zero. Clearly, $\sigma^{-1}(\{0, 1\}) = \overline{T}$, hence $\overline{T}$ is a G-ordering. If $\Pi_n \subseteq T$, then G is abelian. Hence the second assertion follows from the first. □

4. The proof of Theorem B

We will split the proof of Theorem B into Propositions 4 and 5. Recall from the proof of Claim 2 of Theorem 1 that $n!R \subseteq \Sigma P_n - \Sigma P_n$.

Proposition 4. *If J is a P_n-real ideal then the following are equivalent:*

(1) *J is a completely prime ideal.*
(2) *J is a prime ideal.*
(3) *If $a, b \in R$ and $aP_nb \subseteq J$, then either $a \in J$ or $b \in J$.*

Proof. Clearly (1) implies (2). If (2) is true and $aP_nb \in J$ for some $a, b \in R$, then the relation $n!R \subseteq \sum P_n - \sum P_n$ implies that $n!(aRb) \subseteq J$. Since J is an ideal, we have that $(n!)RaRb \subseteq J$. Note that $n! \notin J$ because J is Π_n-real and $1 \notin J$. Since the ideal J is prime it follows that either $a \in J$ or $b \in J$. Hence, (3) is true.

If (3) is true, then the set $U := P_n \setminus J$ is a P_n-system. Clearly, $-U \cap \sum P_n = \emptyset$. By Theorem 1, with $M = \sum P_n$, there exists a P_n-module S avoiding $-U$ such that the set $I = S \cap -S$ is a completely prime ideal. We claim that $J = I$. Clearly, $I \cap P_n = J \cap P_n$. If $x \in I$, then $(xP_n)^{n-1}x \subseteq I \cap P_n = J \cap P_n \subseteq J$. By (3), we have that $x \in J$. Hence, (1) is true. □

Proposition 5. *If $\mathbb{Q} \subset R$ then the following assertions are equivalent for every P_n-real ideal I on R:*

(1) *I is semiprime.*
(2) *If $a \in R$ and $aP_na \subset I$ then $a \in I$.*
(3) *I is equal to an intersection of P_n-real prime ideals.*
(4) *I is radical.*

Proof. Clearly, (3) implies (4) and (4) implies (1). The equivalence between (1) and (2) follows from the identity $n!R \subseteq \sum P_n - \sum P_n$ and the invertibility of $n!$.

To show that (2) implies (3) recall that every semiprime ideal is the intersection of all prime ideals containing it. Therefore, it suffices to show that for every prime ideal $J \supset I$ there exists a real prime ideal J' such that $J \supset J' \supset I$. Note that the set $U = \Pi_n \setminus J$ is a Π_n-system. If not, we can find $x, y \in U$ such that $x\Pi_n y \subseteq J$. Since $n!R \subseteq \sum P_n - \sum P_n$ and $n!$ is invertible, we get $xRy \subseteq J$. Hence $x \in J$ or $y \in J$, a contradiction. Since I is P_n-real, there exists a P_n-module M such that $I = M \cap -M$. Since $J \supset I$, we have that $-U \cap M = \emptyset$. By Theorem 1, there exists a P_n module $S \supset M$ such that $-U \cap S = \emptyset$ and $J' := S \cap -S$ is a completely prime ideal. Clearly J' is a P_n-real prime ideal such that $J' \supset I$. Since $U \cap J' = \emptyset$, it follows that $J' \cap P_n \subseteq P_n \setminus U = J \cap P_n$. If $a \in J'$ then $(aP_n)^n \in J' \cap P_n$, so that $(aP_n)^n \in J \cap P_n$. Since J is an ideal and $2^n > n$, it follows that $(aP_n)^{2^n} \in J$. Applying (2) n-times, we get that $a \in J$ as required. □

5. Further comments

It would be interesting to know whether the assumption $\mathbb{Q} \subset R$ in Corollary 5 can be omitted. Clearly, for any unital ring R there exists a bijective correspondence between the ideals on $R \otimes_{\mathbb{Z}} \mathbb{Q}$ and ideals on R such that $I^e = I$ (i.e. $kx \in I$ for $k \in \mathbb{N}$ and $x \in R$ implies that $x \in I$). The correspondence preserves prime, semiprime and P_n-real ideals. Therefore, it suffices to prove that $I^e = I$ for every P_n-real semiprime ideal I. I can only show this for Π_n-real ideals. Thus every Π_n-real ideal on every unital ring R is semiprime if and only if it is radical.

Proposition 6. *Every Π_n-real semiprime ideal I satisfies $I^e = I$.*

Proof. By modding out I, we may assume that 0 is a Π_n-real ideal. We want to prove that $0^e = 0$.

Pick any $n \in \mathbb{N}$ such that $\sum \Pi_n \cap - \sum \Pi_n = 0$. Let $n = 2^{m_0} + 2^{m_1} + \ldots + 2^{m_r}$ be a binary representation of n (i.e. $m_0 > m_1 > \ldots > m_r$). Since n is even, we have that $m_r > 0$.

For every x in the torsion of $(R, +)$ and any elements $c_0, \ldots, c_{m_0} \in R$ form a sequence $x_0, \ldots, x_{m_0+1}$ by setting $x_0 = x$ and

$$x_{i+1} = \begin{cases} x_i c_i x_i c_i^{n-1} & \text{if} i \notin \{m_0, \ldots, m_r\} \\ c_i^{n-1} x_i c_i x_i & \text{if} i \in \{m_0, \ldots, m_r\} \end{cases} \quad (i = 0, \ldots, m_0).$$

Note that there exist elements $w_1, \ldots, w_r \in R$ such that $x_{m_0} = x_{m_1+1} w_1, \ldots, x_{m_{r-1}} = x_{m_r+1} w_r$. Write $z = \prod_{j=0}^{r} c_{m_j}^{n-1} x_{m_j} c_{m_j}$ and $w = x_{m_r} w_r \cdots w_1$. A short computation shows that $x_{m_0+1} = zw$.

From the binary representation of n we see that $z \in \Pi_n(R)$. Since x is a torsion element, we have that $kx = 0$ for some $k \in \mathbb{N}$. It follows that $kz = 0$. The relation $-z = (k-1)z$ implies that $z \in \sum \Pi_n \cap - \sum \Pi_n = \{0\}$. Hence, $x_{m_0+1} = zw = 0$. We claim that if $x_{s+1} = 0$ for any $c_1, \ldots, c_s$ then $x_s = 0$ for any $c_1, \ldots, c_{s-1}$. It follows by induction on s that $x = x_0 = 0$.

To prove the claim assume that $0 = x_{s+1} = x_s c_s x_s c_s^{n-1}$ for every c_s. (Similarly if $0 = x_{s+1} = c_s^{n-1} x_s c_s x_s$ for every c_s.) For $c_s = 1$, we get $x_s^2 = 0$. For every $k \in \mathbb{N}$ such that $x_s c_s x_s c_s^k = 0$ for every c_s we have that $0 = (x_s(1+c_s)x_s(1+c_s)^k)c_s^{k-1} = x_s c_s x_s (1+c_s)^k c_s^{k-1} = x_s c_s x_s c_s^{k-1} + \binom{k}{1} x_s c_s x_s c_s^k + \ldots + \binom{k}{k-1} x_s c_s x_s c_s^{2k-2} + x_s c_s x_s c_s^{2k-1} = x_s c_s y_s c_s^{k-1}$. The claim follows by induction on k. □

A similar argument shows that every P_2-real semiprime ideal I satisfies $I^e = I$. Hence, every P_2-real semiprime ideal is radical.

The following observation is also interesting.

Proposition 7. *If a prime ideal is P_2-real then it is Π_2-real. (The converse is clear.)*

Proof. Let J be a P_2-real prime ideal. Then 0 is a P_2-real and $D = R/J$ is a domain. Write $T = \sum P_2(D)$ and $P = \sum \Pi_2(D)$. For any $x, y, z \in R$ we have that $(xyz)(yxz)(yx)^2 = xy(zyx)^2yx \in T$ so that $xyz \in \overline{T}$ if and only if $yxz \in \overline{T}$. Since every permutation is a product of transpositions of neighbouring elements, it follows that $P \subset \overline{T}$. Since $T^0 = 0$, it follows that $\overline{T}^0 = 0$, so that $P^0 = 0$. Hence, 0 is a Π_2-real ideal on D as required. □

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