

# REAL SPECTRA OF QUANTUM GROUPS

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ABSTRACT. The only noncommutative ring for which the real spectrum has been described so far is the quantum affine ring  $\mathbb{R}_q[x, y]$ , see [21]. The aim of this paper is to describe the real spectra of quantum affine rings  $k_{\mathbf{q}}[x_1, \dots, x_n]$  where  $k$  is a formally real affine  $\mathbb{R}$ -algebra and  $\mathbf{q} \in M_n(\mathbb{R}^+)$ . As a by-product we describe the real spectra of quantized enveloping algebra  $U_q(\mathfrak{sl}_2(\mathbb{R}))$  and quantum special linear group  $\mathcal{O}_q(SL_2(\mathbb{R}))$ . Formal reality and semireality is characterized for the following classes of quantum groups: quantum affine rings, quantized enveloping algebras, quantized function algebras, quantized Weyl algebras.

## 1. INTRODUCTION

Let  $R$  be a ring. A subset  $P \subseteq R$  is an *ordering* if  $P \cdot P \subseteq P$ ,  $P + P \subseteq P$ ,  $P \cup -P = R$  and  $P \cap -P$  is a prime ideal of  $R$ . The set of all orderings of  $R$  is denoted by  $\text{Sper } R$  and called the *real spectrum* of  $R$ . The rings with nonempty real spectrum are called *semireal rings*. The study of real spectra of noncommutative semireal rings is called the *noncommutative real algebraic geometry*. The pioneering work in this field has been done by Murray Marshall and his school, [18, 21, 22].

The mapping  $\text{supp}: \text{Sper } R \rightarrow \text{Spec } R$  defined by  $\text{supp}(P) = P \cap -P$  is called *the support*. Prime ideals in the image of  $\text{supp}$  are called *real prime ideals*. They are always completely prime. If  $J$  is a real prime ideal of  $R$  then the image and preimage of the canonical projection  $R \rightarrow R/J$  give a one-to-one correspondence between orderings of  $R$  with support  $J$  and orderings of  $R/J$  with support zero. If  $R$  is a Noetherian ring then  $R/J$  has a skew field of fractions  $\text{Fract}(R/J)$ . In this case, we also have a one-to-one correspondence between support zero orderings of  $R/J$  and orderings of  $\text{Fract}(R/J)$ . The problem of describing the real spectrum of a Noetherian semireal ring  $R$  therefore consists of two subproblems:

- (1) Describe the real prime ideals of  $R$ .

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- (2) For every real prime ideal  $J$  of  $R$  describe all orderings of  $\text{Fract}(R/J)$ .

A unital ring  $R$  is *formally real* if it has a support zero ordering. Every formally real ring is a domain. A simple ring is formally real if and only if it is semireal.

In Section 2, we study quantum affine rings  $k_{\mathbf{q}}[x_1, \dots, x_n]$ . We characterize their formal reality and semireality. We can reduce the description of their real spectra to the description of the real spectra of polynomial rings, which is known for  $k = \mathbb{R}$  by [18, Theorem 3.3]:

**Example.** Let  $\Gamma$  be a totally ordered group,  $\mathbb{R}((\Gamma))$  its powers series field ordered by the sign of the lowest nonzero coefficient and  $\phi_1, \dots, \phi_l \in \mathbb{R}((\Gamma))$ . Then

$$P_{\phi_1, \dots, \phi_l} = \{f \in \mathbb{R}[x_1, \dots, x_l] : f(\phi_1, \dots, \phi_l) \geq 0\}$$

is an ordering of  $\mathbb{R}[x_1, \dots, x_l]$ . For  $\Gamma = (\mathbb{R}^{2^l-1}, +)$  ordered lexicographically, the mapping  $(\phi_1, \dots, \phi_l) \mapsto P_{\phi_1, \dots, \phi_l}$  gives a one-to-one correspondence between “distinguished  $l$ -tuples” and  $\text{Sper}(\mathbb{R}[x_1, \dots, x_l])$ .

If  $R$  is a formally real affine  $\mathbb{R}$  algebra, then  $R \cong \mathbb{R}[x_1, \dots, x_m]/J$  for a number  $m$  and real prime ideal  $J$ . We can identify  $\text{Sper}(R)$  with  $\{P \in \text{Sper}(\mathbb{R}[x_1, \dots, x_m]) : J \subseteq \text{supp}(P)\}$ .

In Sections 3-5 we discuss quantized enveloping algebras, quantized function algebras and quantized Weyl algebras. Their formal reality and semireality are characterized. When the Gelfand-Kirillov property is satisfied, the description of the real spectrum reduces to the description of the prime spectrum.

## 2. QUANTUM AFFINE RINGS

Let  $k$  be commutative unital ring,  $k^\times$  its set of invertible elements  $n$  a nonnegative integer and  $\mathbf{q} = (q_{ij}) \in M_n(k^\times)$  multiplicatively antisymmetric (i.e.  $q_{ii} = 1$  and  $q_{ij}q_{ji} = 1$  for every  $i, j = 1, \dots, n$ ). The *quantum affine ring*  $k_{\mathbf{q}}[x_1, \dots, x_n]$  is the  $k$ -algebra on  $n$  generators  $x_1, \dots, x_n$  with  $n^2$  relations  $x_i x_j = q_{ij} x_j x_i$ . If  $k$  is a domain, then  $k_{\mathbf{q}}[x_1, \dots, x_n]$  is an Ore domain. Its skew field of quotients is called the *quantum Weyl field*  $k_{\mathbf{q}}(x_1, \dots, x_n)$ . We denote by  $k_{\mathbf{q}}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$  the localization  $k_{\mathbf{q}}[x_1, \dots, x_n]_{x_1, \dots, x_n} \subset k_{\mathbf{q}}(x_1, \dots, x_n)$ . When  $n = 2$  we write  $k_{q_{21}}[x_1, x_2]$  instead of  $k_{\mathbf{q}}[x_1, x_2]$ .

**Proposition 1.** *The quantum affine ring  $k_{\mathbf{q}}[x_1, \dots, x_n]$  is semireal if and only if the ring  $k$  is semireal. It is formally real if and only if  $k$  has a support zero ordering such that  $q_{ij} > 0$  for all  $i, j = 1, \dots, n$ .*

*Proof.* A unital subring of a semireal ring is always semireal. In particular if  $k_{\mathbf{q}}[x_1, \dots, x_n]$  is semireal, then  $k$  is semireal, too. A ring which has a unital homomorphism into a semireal ring is semireal. In particular, sending  $x_1 \rightarrow 0, \dots, x_n \rightarrow 0$  we get a unital ring homomorphism  $\phi: k_{\mathbf{q}}[x_1, \dots, x_n] \rightarrow k$ . If  $k$  is semireal, then  $k_{\mathbf{q}}[x_1, \dots, x_n]$  is semireal, too.

If  $k_{\mathbf{q}}[x_1, \dots, x_n]$  has a support zero ordering, then for every  $i, j = 1, \dots, n$  the element  $x_i x_j$  and  $x_j x_i$  have the same sign. It follows that  $q_{ij} > 0$ . Clearly,  $k$  has a support zero ordering, too.

Assume now that  $k$  has a support zero ordering such that  $q_{ij} > 0$  for all  $i, j = 1, \dots, n$ . Every nonzero element  $z \in k_{\mathbf{q}}[x_1, \dots, x_n]$  can be written uniquely as  $z = \sum_{i=1}^r c_i M_i$  where  $c_i \neq 0$  for  $i = 1, \dots, r$  and  $M_i$  are standard monomials in  $x_1, \dots, x_n$  such that  $M_1 < \dots < M_r$  with respect to lexicographic ordering. Writing  $z > 0$  if and only if  $c_r > 0$  defines a support zero ordering on  $k_{\mathbf{q}}[x_1, \dots, x_n]$ .  $\square$

Proposition 2 collects some well-known facts.

**Proposition 2.** *Let  $k$  be a commutative domain and  $\mathbf{q} = (q_{ij}) \in M_n(k^\times)$  a multiplicatively antisymmetric matrix such that the subgroup  $\langle q_{ij} | i, j = 1, \dots, n \rangle \subset k^\times$  is torsion-free. Write  $S = k_{\mathbf{q}}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ .*

*There exists a natural number  $0 \leq r \leq n$  and integers  $k_{ij}, i, j = 1, \dots, n$  such that the elements  $t_j := x_1^{k_{j1}} \cdots x_n^{k_{jn}}$  have the following properties:*

- (1)  $Z(S) = k[t_1^{\pm 1}, \dots, t_r^{\pm 1}]$ .
- (2) If  $t_{r+1}^{l_{r+1}} \cdots t_n^{l_n} \in Z(S)$  for some  $l_{r+1}, \dots, l_n \in \mathbb{Z}$ , then  $l_{r+1} = \dots = l_n = 0$ .
- (3)  $S \cong Z(S)_{\mathbf{p}}[t_{r+1}^{\pm 1}, \dots, t_n^{\pm 1}]$  where  $\mathbf{p} \in M_{n-r}(k^\times)$  is a multiplicatively antisymmetric matrix.
- (4) Every prime ideal  $I$  of  $S$  is generated by  $I \cap Z(S)$  and  $S/I \cong (Z(S)/I \cap Z(S))_{\mathbf{p}}[t_{r+1}^{\pm 1}, \dots, t_n^{\pm 1}]$ .

*Proof.* Let us define a mapping

$$\Phi: \mathbb{Z}^n \rightarrow (k^\times)^n, \quad \Phi(i_1, \dots, i_n) = (q_{11}^{i_1} \cdots q_{1n}^{i_n}, \dots, q_{n1}^{i_1} \cdots q_{nn}^{i_n}).$$

The assumption that  $\langle q_{ij} | i, j = 1, \dots, n \rangle$  is torsion-free implies that the kernel  $N(\Phi)$  of  $\Phi$  is a pure subgroup of  $\mathbb{Z}^n$ . By [9, Corollary 28.3],  $N(\Phi)$  has a direct complement, say  $N(\Phi)'$ . Let  $\mathbf{k}_1, \dots, \mathbf{k}_r$  be a basis of  $N(\Phi)$  and  $\mathbf{k}_{r+1}, \dots, \mathbf{k}_n$  a basis of  $N(\Phi)'$ . For every  $j = 1, \dots, n$  write  $t_j := x_1^{k_{j1}} \cdots x_n^{k_{jn}}$  where  $(k_{j1}, \dots, k_{jn}) = \mathbf{k}_j$ .

If  $u = x_1^{i_1} \cdots x_n^{i_n}$  then  $(x_1 u x_1^{-1} u^{-1}, \dots, x_n u x_n^{-1} u^{-1}) = \Phi(i_1, \dots, i_n)$ . Therefore, an element  $y = \sum c_k x_1^{i_{1k}} \cdots x_n^{i_{nk}}$  is central if and only if  $x_j y = y x_j$  for  $j = 1, \dots, n$  if and only if  $(i_{1k}, \dots, i_{nk}) \in N(\Phi)$  for every

$k$  such that  $c_k \neq 0$ . Assertion (1) now follows from the definition of  $t_1, \dots, t_r$ . Assertion (3) of the proposition follows from assertion (1) and the fact that  $\mathbf{k}_1, \dots, \mathbf{k}_n$  is a basis of  $\mathbb{Z}^n$ . Since  $t_{r+1}^{l_{r+1}} \cdots t_n^{l_n}$  is colinear to  $x_1^{k_{r+1,1}l_{r+1}+\dots+k_{n,1}l_n} \cdots x_n^{k_{r+1,n}l_{r+1}+\dots+k_{n,n}l_n}$ , it belongs to the center if and only if  $l_{r+1}\mathbf{k}_{r+1} + \dots + l_n\mathbf{k}_n \in N(\Phi)$ . Since  $N(\Phi) \cap N(\Phi)' = \{0\}$ , it follows that  $l_{r+1} = \dots = l_n = 0$ . This proves assertion (2).

Let  $I$  be a prime ideal of  $S$ . For any  $s = r+1, \dots, n$  and any polynomials  $c_i$  in  $t_1, \dots, t_{s-1}$  we have  $\sum_i c_i t_s^i \in I$  if and only if all  $c_i \in I$ . Namely, pick  $r$  such that  $x_r t_s x_r^{-1} t_s^{-1} = q \neq 1$  and note that  $\sum_i c_i q^{ij} t_s^i = x_s^j (\sum_i c_i t_s^i) x_r^{-j} \in I$  for every  $j$ . Since  $(q^{ij})$  is an invertible matrix it follows that  $c_i t_s^i \in I$  for every  $i$ . Assertion (4) follows by induction on  $s$ .  $\square$

**Remark.** The assumption that  $\langle q_{ij} | i, j = 1, \dots, n \rangle \subseteq k^\times$  is torsion-free is fulfilled if the quantum affine ring  $k_{\mathbf{q}}[x_1, \dots, x_n]$  is formally real. Since  $\langle q_{ij} | i, j = 1, \dots, n \rangle \subset P$  for an ordering  $P$  of  $k$  (Proposition 1), it suffices to show that  $P^\times = P \cap k^\times$  is a torsion-free subgroup of  $k^\times$ . Take any  $x \in P^\times$  which is a root of 1. If  $x^2 = 1$  and  $x \neq 1$  then  $x = -1$ , a contradiction with  $1 \in P$  and  $P \cap -P = \{0\}$ . If  $x^{2m+1} = 1$  and  $x \neq 1$ , then  $1 + x + \dots + x^{2m} = 0$ . Hence  $-1 = (1+x)^2 + (x+x^2)^2 + \dots + (x^{m-1} + x^m)^2 + x^{2m} \in P$ , a contradiction.

Let  $R = k_{\mathbf{q}}[x_1, \dots, x_m]$  be a formally real quantum affine ring. For every  $n = 0, \dots, m$  write  $S_n = k_{\mathbf{q}_n}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$  where  $\mathbf{q}_n$  is obtained from  $\mathbf{q}$  by deleting last  $m-n$  rows and columns. By Proposition 2, we can find  $t_1, \dots, t_n$  and  $\mathbf{p}_n$  such that  $Z(S_n) = k[t_1^{\pm 1}, \dots, t_r^{\pm 1}]$  and  $S_n = Z(S_n)_{\mathbf{p}_n}[t_{r+1}^{\pm 1}, \dots, t_n^{\pm 1}]$ .

**Proposition 3.** *With the notation from above there is a natural one-to-one correspondence between:*

- *real prime ideals of  $R = k_{\mathbf{q}}[x_1, \dots, x_m]$  which avoid  $x_1, \dots, x_n$  and contain  $x_{n+1}, \dots, x_m$ ,*
- *real prime ideals of  $Z(S_n) = k[t_1^{\pm 1}, \dots, t_r^{\pm 1}]$ .*

*Moreover, if a real prime ideal  $I$  of  $R$  corresponds to a real prime ideal  $I'$  of  $Z(S_n)$ , then  $(R/I)_{\bar{x}_1, \dots, \bar{x}_n} \cong (Z(S_n)/I')_{\mathbf{p}_n}[t_{r+1}^{\pm 1}, \dots, t_n^{\pm 1}]$ .*

*Proof.* Write  $R_n = k_{\mathbf{q}_n}[x_1, \dots, x_n]$ . The mappings  $I \mapsto I_n = I \cap R_n$  and  $I_n \mapsto I = I_n + (x_{n+1}, \dots, x_m)$  give a one-to-one correspondence between the set of all ideals of  $R$  which avoid  $x_1, \dots, x_n$  and contain  $x_{n+1}, \dots, x_m$  and the set of all ideals of  $R_n$  which avoid  $x_1, \dots, x_n$ . Clearly,  $R/I \cong R_n/I_n$ . Hence, the correspondence preserves real primes.

The subset  $U_n$  of  $R_n$  multiplicatively generated by  $x_1, \dots, x_n$  is a Ore set and  $S_n = (R_n)_{U_n}$ . Every completely prime ideal  $J$  of  $R_n$  which

avoids  $U_n$  extends uniquely to a completely prime ideal  $J' = J \cdot S_n$  of  $S_n$ . Since  $R_n/J$  is formally real if and only if  $S_n/J' = (R_n/J)_{\overline{U}_n}$  is formally real,  $J$  is a real prime if and only if  $J'$  is a real prime. Hence, the extension and restriction give a one-to-one correspondence between the set of all real primes of  $R_n$  which avoid  $x_1, \dots, x_n$  and the set of all real primes of  $S_n$ .

By assertion (4) of Proposition 2, the restriction and extension give a one-to-one correspondence between the prime ideals of  $S_n$  and prime ideals of  $Z(S_n)$ . Since  $S/I \cong (Z(S)/I \cap Z(S))_{\mathbf{p}[t_{r+1}^{\pm 1}, \dots, t_n^{\pm 1}]}$ , the correspondence preserves real primes.  $\square$

**Example.** Let  $A = \mathbb{R}[t]_q[x_1, x_2]$  with  $q \in \mathbb{R}^+ \setminus \{1\}$ . Proposition 3 gives the complete list of real prime ideals  $J$  of  $A$ . We also list their factor domains  $A/J$  for later reference.

- (1) Real primes which avoid  $x_1, x_2$  are in one-to-one correspondence with the real prime ideals of  $Z(\mathbb{R}[t]_q[x_1^{\pm 1}, x_2^{\pm 1}]) = \mathbb{R}[t]$ . We have two types:
  - $J = (0)$ ,  $A/J \cong A$ .
  - $J = (t - \alpha)$  ( $\alpha \in \mathbb{R}$ ),  $A/J \cong \mathbb{R}_q[x_1, x_2]$ .
- (2) Real primes which avoid  $x_1$  and contain  $x_2$  are in one-to-one correspondence with the real prime ideals of  $\mathbb{R}[t, x_1]$  which avoid  $x_1$ . We have two types:
  - $J = (x_2)$ ,  $A/J \cong \mathbb{R}[t, x_1]$ .
  - $J = (x_2, f(t, x_1))$  ( $f(t, x_1) \neq x_1$  irreducible with real zero),  $A/J \cong \mathbb{R}[t, x_1]/(f(t, x_1))$ . If  $f$  has degree zero in  $x_1$ , then  $f(t, x_1) = t - \alpha$  for some  $\alpha \in \mathbb{R}$  and  $A/J \cong \mathbb{R}[x_1]$ . If  $f$  has degree  $\geq 1$  in  $x_1$ , then  $A/J$  is an algebraic extension of  $\mathbb{R}[t]$ .
- (3) Real primes which avoid  $x_2$  and contain  $x_1$ . Interchange  $x_1$  and  $x_2$  in case (2).
- (4) Real primes which contain both  $x_1$  and  $x_2$  are in one-to-one correspondence with the real prime ideals of  $\mathbb{R}[t]$ . We have two types:
  - $J = (x_1, x_2)$ ,  $A/J \cong \mathbb{R}[t]$ .
  - $J = (t - \alpha, x_1, x_2)$  ( $\alpha \in \mathbb{R}$ ),  $A/J \cong \mathbb{R}$ .

**Remark.** Recall that extension and restriction give a one-to-one correspondence between the set of all support zero orderings of an Ore domain and the set of all orderings of its skew field of fractions. It follows that for every real prime ideal  $I$  of  $R$  which avoids  $x_1, \dots, x_n$  and contains  $x_{n+1}, \dots, x_m$ , there are one-to-one correspondences between:

- orderings of  $R$  with support  $I$ ,

- support zero orderings of  $R/I$ ,
- support zero orderings of  $(R/I)_{\bar{x}_1, \dots, \bar{x}_n}$ ,
- orderings of  $\text{Fract}(R/I)$ ,
- support zero orderings of  $(Z(S_n)/I')_{\mathbf{p}_n}[t_{r+1}^{\pm 1}, \dots, t_n^{\pm 1}]$ ,
- orderings of  $\text{Fract}(Z(S_n)/I')_{\mathbf{p}_n}(t_{r+1}^{\pm 1}, \dots, t_n^{\pm 1})$ .

The fact that every prime ideal of a quantum affine ring is generated by its intersection with the center implies the following observation: If  $t_{r+1}^{l_{r+1}} \cdots t_n^{l_n}$  belongs to the center of  $(Z(S_n)/I')_{\mathbf{p}_n}(t_{r+1}^{\pm 1}, \dots, t_n^{\pm 1})$  for some  $l_{r+1}, \dots, l_n \in \mathbb{Z}$  then  $l_{r+1} = \dots = l_n = 0$ . This observation, together with Theorem 4, gives a description of support zero orderings on  $(Z(S_n)/I')_{\mathbf{p}_n}[t_{r+1}^{\pm 1}, \dots, t_n^{\pm 1}]$ .

Let  $P$  be a support zero ordering on a domain  $A$ . We recall the construction of the natural valuation  $v_P$  from [21]. For any  $a \in A$  write  $|a| = a$  if  $a \in P$  and  $|a| = -a$  otherwise. For any  $a, b \in \dot{A} := A \setminus \{0\}$  write  $aLb$  if there exist  $r \in \mathbb{N}$  such that  $r|a| - |b| \in P$ . Clearly,

- $L$  is transitive and reflexive,
- if  $aLb$  for some  $a, b \in \dot{A}$ , then  $acLbc$  and  $caLcb$  for every  $c \in \dot{A}$ ,
- if  $acLbc$  or  $caLcb$  for some  $a, b, c \in \dot{A}$ , then  $aLb$ ,
- for any  $a, b \in \dot{A}$ , either  $aLb$  or  $bLa$ .

Let  $\sim$  be a relation, defined by  $a \sim b$  if and only if  $aLb$  and  $bLa$ . The properties of  $L$  imply that  $L$  is a congruence relation and that the factor semigroup  $\Gamma_P = \dot{A}/\sim$  is cancellative. The natural projection  $v_P: \dot{A} \rightarrow \Gamma_P$  is called *the natural valuation* of the ordering  $P$ . The semigroup  $\Gamma_P$  is totally ordered by  $v_P(a) \leq v_P(b)$  if and only if  $aLb$ . We call  $\leq$  *the natural ordering* of  $\Gamma_P$ . Note that  $v_P(a+b) \geq \min\{v_P(a), v_P(b)\}$  for every  $a, b \in \dot{A}$  such that  $a+b \neq 0$ . Moreover, if  $v_P(a) < v_P(b)$ , then  $v_P(a+b) = v_P(a)$  and  $a, a+b$  have the same sign with respect to  $P$ .

Theorem 4 completes the description of orderings of quantum affine rings over formally real affine real algebras and with  $q_{ij} \in \mathbb{R}^+$ .

**Theorem 4.** *Let  $K$  be a formally real affine  $\mathbb{R}$ -algebra,  $\mathbf{p} \in M_s(\mathbb{R}^+)$  a matrix such that  $p_{ii} = 1$  and  $p_{ij}p_{ji} = 1$  for every  $i, j = 1, \dots, s$  and let  $R = K_{\mathbf{p}}[z_1^{\pm 1}, \dots, z_s^{\pm 1}]$  be such that  $z_1^{l_1} \cdots z_s^{l_s}$  is central if and only if  $l_1 = \dots = l_s = 0$ .*

*For every support zero ordering  $Q$  of  $K$  there exists a natural one-to-one correspondence between the set  $\text{Ord}_Q(R)$  of all support zero orderings of  $R$  which extend  $Q$  and the set  $\text{Tot}(\Gamma_Q \times \mathbb{Z}^s) \times \{-1, 1\}^s$  where  $\text{Tot}(\Gamma_Q \times \mathbb{Z}^s)$  is the set of all total orderings of the commutative semigroup  $\Gamma_Q \times \mathbb{Z}^s$  which extend the natural ordering of  $\Gamma_Q$ .*

*Proof.* We will divide the proof into three claims.

**Claim (1).** For every ordering  $P \in \text{Ord}_Q(R)$ , any  $c, d \in K$  and any  $i_1, \dots, i_s, j_1, \dots, j_s \in \mathbb{Z}$  we have  $v_P(cz_1^{i_1} \cdots z_s^{i_s}) = v_P(dz_1^{j_1} \cdots z_s^{j_s})$  if and only if  $v_P(c) = v_P(d)$  and  $(i_1, \dots, i_s) = (j_1, \dots, j_s)$ .

The only if part is trivial. Write  $y_1 = cz_1^{i_1} \cdots z_s^{i_s}$  and  $y_2 = dz_1^{j_1} \cdots z_s^{j_s}$ . Write  $z := y_2y_1^{-1} = pcd^{-1}z_1^{j_1-i_1} \cdots z_s^{j_s-i_s}$  where  $p \in \mathbb{R}^+$ . If  $z \in Z(A)$ , then by the assumption on  $R$  we have that  $i_1 = j_1, \dots, i_s = j_s$ . Since  $v_P(z) = 0$  and  $v_P(p) = 0$ , it follows that  $v_P(c) = v_P(d)$ . If  $z$  is not central, then there exists  $t \in \{z_1, \dots, z_s\}$  such that  $tz \neq zt$ . We know that  $txt^{-1} = qz$  for some  $q \in \mathbb{R}^+$ ,  $q \neq 1$ . Replacing  $t$  by  $t^{-1}$  if necessary we may assume that  $q < 1$ . Since  $v_P(q) = 0$ , it follows that  $v_P(z) = v_P(txt^{-1})$ . Since  $v_P(z) = v_P(1) = 0$ , there exists  $r \in \mathbb{Q}$  such that  $|z| < r$ . It follows that  $|z| = q^i |t^{-i} z t^i| \leq q^i r$  for every  $i \in \mathbb{N}$ . Hence  $|z| < \epsilon$  for every  $\epsilon \in \mathbb{Q}^+$ . In other words, we get  $v_P(z) > 0$ , a contradiction.

**Claim (2).** There is a one-to-one correspondence between the set  $V_Q$  of equivalence classes of natural valuations of orderings from  $\text{Ord}_Q(R)$  and the set  $\text{Tot}(\Gamma_Q \times \mathbb{Z}^s)$ .

Every element  $a \in \dot{R}$  can be expressed uniquely as  $a = \sum_i c_i z_1^{m_{i1}} \cdots z_s^{m_{is}}$ . Claim (1) implies that  $v_P(a) = \min_i v_P(c_i z_1^{m_{i1}} \cdots z_s^{m_{is}})$ . In particular  $v_P(K), v_P(z_1), \dots, v_P(z_s)$  are  $\mathbb{Z}$ -linearly independent and they span  $\Gamma_P$ . The natural embedding of  $\Gamma_Q$  into  $\Gamma_P$  identifies  $\Gamma_Q$  with its image  $v_P(\dot{K})$ . Hence there exists an isomorphism  $\phi: \Gamma_P \rightarrow \Gamma_Q \times \mathbb{Z}^s$  such that  $\phi(v_P(cz_1^{j_1} \cdots z_s^{j_s})) = (v_Q(c), j_1, \dots, j_s)$ . The natural ordering of  $\Gamma_P$  defines via  $\phi$  a total ordering  $F(v_P)$  of  $\Gamma_Q \times \mathbb{Z}^s$  which extends the natural ordering of  $\Gamma_Q$ . If  $P' \in \text{Ord}_Q(R)$  is such that  $v_{P'}$  is equivalent to  $v_P$ , then  $\Gamma_{P'} = \Gamma_P$  and  $v_{P'} = v_P$ . Hence,  $v_P \rightarrow F(v_P)$  is a well defined mapping from  $V_P$  to  $\text{Tot}(\Gamma_Q \times \mathbb{Z}^s)$ .

Conversely, take any  $O \in \text{Tot}(\Gamma_Q \times \mathbb{Z}^s)$  and define a valuation  $G(O)$  from  $\dot{R}$  to the ordered group  $(\Gamma_Q \times \mathbb{Z}^s, O)$  by  $G(O)(\sum_{i=1}^l c_i z_1^{m_{i1}} \cdots z_s^{m_{is}}) = \min_O \{(v_Q(c_i), m_{i1}, \dots, m_{is}), i = 1, \dots, l\}$ . Note that  $G(O)$  is the natural valuation of the ordering  $P_O := \{0\} \cup \{cz_1^{i_1} \cdots z_s^{i_s} + h \mid c \in \dot{Q} \text{ and } G(O)(cz_1^{i_1} \cdots z_s^{i_s}) < G(O)(h)\}$ . Hence  $O \rightarrow G(O)$  defines a mapping from  $\text{Tot}(\Gamma_Q \times \mathbb{Z}^s)$  to  $V_P$ .

Clearly,  $F(G(O)) = O$  for every  $O \in \text{Tot}(\Gamma_Q \times \mathbb{Z}^s)$ . For every  $P \in \text{Ord}_Q(R)$  we have  $G(F(v_P)) = \phi \circ v_P$ , where  $\phi: \Gamma_P \rightarrow \Gamma_Q \times \mathbb{Z}^s$  is the isomorphism from above. Hence, the valuation  $G(F(v_P))$  is equivalent to  $v_P$ . Therefore,  $F$  and  $G$  give a one-to-one correspondence between  $V_Q$  and  $\text{Tot}(\Gamma_Q \times \mathbb{Z}^s)$ .

**Claim (3).** For every  $v \in V_Q$ , there is a one-to-one correspondence between the orderings  $P \in \text{Ord}_Q(R)$  such that  $v_P = v$  and the set  $\{-1, 1\}^s$ .

By claim (1), every element  $z \in \dot{R}$  has the lowest monomial with respect to  $v_P$ , we denote it  $l_P(z)$ . We have  $v_P(z - l_P(z)) > v_P(l_P(z))$ , since every monomial of  $z - l_P(z)$  has strictly larger  $v_P$  than  $l_P(z)$ . Therefore,  $v_P(z) = v_P(l_P(z))$  and  $z, l_P(z)$  have the same sign with respect to  $P$ . Consequently,  $P$  is uniquely determined by  $v_P$  and the signs of  $z_1, \dots, z_s$ . The one-to-one correspondence is given explicitly by  $\text{Tot}(\Gamma_Q \times \mathbb{Z}^s) \times \{-1, 1\}^s \rightarrow \text{Ord}_Q(R)$ ,  $(O, \sigma_1, \dots, \sigma_s) \mapsto P_{O, \sigma_1, \dots, \sigma_s} := \{cz_1^{i_1} \cdots z_s^{i_s} + h \mid c\sigma_1^{i_1} \cdots \sigma_s^{i_s} \in \dot{Q} \text{ and } G(O)(cz_1^{i_1} \cdots z_s^{i_s}) < G(O)(h)\}$ .  $\square$

**Remark.** Every total ordering of  $\Gamma_Q \oplus \mathbb{Z}^s$  extends uniquely to a total ordering of the group of differences  $D(\Gamma_Q \oplus \mathbb{Z}^s) = D(\Gamma_Q) \oplus \mathbb{Z}^s$  which extends uniquely to a total ordering of the  $\mathbb{Q}$ -vector space  $(D(\Gamma_Q) \oplus \mathbb{Z}^s) \otimes \mathbb{Q} = (D(\Gamma_Q) \otimes \mathbb{Q}) \oplus \mathbb{Q}^s$ . The dimension of  $D(\Gamma_Q) \otimes \mathbb{Q}$  over  $\mathbb{Q}$  is bounded by the transcendence degree of  $K$  over  $\mathbb{R}$ .

**Example.** Let  $A$  be as in the previous example. We want to describe the real spectrum of  $A$ . Note that the description of orderings on  $\text{Fract}(A/J)$  is known for all real prime ideals  $J$  except for  $J = (0)$ . (See [17] for  $\mathbb{R}(t, x)$ , [21] or our Theorem 4 for  $\mathbb{R}_q(x_1, x_2)$ , [16] or our comments below for  $\mathbb{R}(t)$ . The description of orderings on an algebraic extension of  $\mathbb{R}(x)$  can be obtained in principle by the extension theory for valuations. Finally,  $\mathbb{R}$  has exactly one ordering.)

It remains to describe orderings with zero support. For each  $a \in \mathbb{R} \cup \{\infty\}$  we define a valuation  $v_a: \mathbb{R}[t] \setminus \{0\} \rightarrow \mathbb{Z}$ :  $v_\infty = -\deg$  and  $v_a(f(t)) = m$  if  $f^{(i)}(a) = 0$  for  $i = 0, 1, \dots, m-1$  and  $f^{(m)}(a) \neq 0$ . The natural valuation of every support zero ordering on  $\mathbb{R}[t]$  is equal to one of  $v_a$ . For every  $v_a$ , there exist exactly two orderings of  $\mathbb{R}[t]$  with  $v_P = v_a$ . Let  $O$  be an ordering on  $\mathbb{R}^3$ , which extends the natural ordering on the first factor (this means that  $(1, 0, 0) \in O$ ) and let  $a \in \mathbb{R} \cup \infty$ . The valuation  $v_{a,O}$  is defined by  $v_{a,O}(\sum_{(i,j) \in \Lambda} r_{ij}(t)x_1^i x_2^j) = \min_O\{(v_a(r_{ij}(t)), i, j), (i, j) \in \Lambda\}$  where  $r_{ij}(t) \neq 0$  for all  $(i, j) \in \Lambda$ . For each  $v_{a,O}$ , there are exactly eight orderings of  $A$  with  $v_P = v_{a,O}$ .

### 3. QUANTIZED ENVELOPING ALGEBRAS

Let  $\mathfrak{g}$  be a complex semisimple Lie algebra. Let  $\Phi$  be its root system and let  $\Delta = \{\alpha_1, \dots, \alpha_n\}$  be a system of simple roots in  $\Phi$ . Write  $d_i = (\alpha_i, \alpha_i)/2 \in \{1, 2, 3\}$  and  $a_{ij} = (\alpha_i, \alpha_j)/d_i \in \mathbb{Z}$  for  $i, j = 1, \dots, n$ .

Let  $k$  be a field and  $q$  a nonzero element of  $k$  which is not a root of 1. Write  $q_i = q^{d_i}$ ,  $[n]_i = q_i^{n-1} + q_i^{n-3} + \dots + q_i^{-n+1}$ ,  $[n]_i! = [1]_i [2]_i \cdots [n]_i$ ,



$E_i^{(s)} = E_i^s/[s]_i!$ ,  $F_i^{(s)} = F_i^s/[s]_i!$ . Then  $U_q(\mathfrak{g})$  is the associative unital  $k$  algebra with  $4n$  generators  $E_i, F_i, K_i, K_i^{-1}$  subjected to relations

$$\begin{aligned} K_i K_i^{-1} &= K_i^{-1} K_i = 1, & K_i K_j &= K_j K_i, \\ E_j K_i &= q_i^{-a_{ij}} K_i E_j, & K_i F_j &= q_i^{-a_{ij}} F_j K_i, \\ E_i F_j - F_j E_i &= \delta_{ij} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}}, \\ \sum_{r=0}^{1-a_{ij}} (-1)^r E_i^{(1-a_{ij}-r)} E_j E_i^{(r)} &= 0, & (i \neq j), \\ \sum_{r=0}^{1-a_{ij}} (-1)^r F_i^{(1-a_{ij}-r)} F_j F_i^{(r)} &= 0, & (i \neq j). \end{aligned}$$

The Hopf algebra structure of  $U_q(\mathfrak{g})$  is defined by

$$\begin{aligned} \Delta(K_i) &= K_i \otimes K_i & \epsilon(K_i) &= 1 & S(K_i) &= K_i^{-1} \\ \Delta(E_i) &= E_i \otimes 1 + K_i \otimes E_i & \epsilon(E_i) &= 0 & S(E_i) &= -K_i^{-1} E_i \\ \Delta(F_i) &= F_i \otimes K_i^{-1} + 1 \otimes F_i & \epsilon(F_i) &= 0 & S(F_i) &= -F_i K_i \end{aligned}$$

Let  $U^+$  be a unital subalgebra of  $U_q(\mathfrak{g})$  generated by  $E_i, i = 1, \dots, n$ ,  $U^-$  a unital subalgebra generated by  $F_i, i = 1, \dots, n$  and  $U^0$  a unital subalgebra generated by  $K_i$  and  $K_i^{-1}, i = 1, \dots, n$ . There is a unique automorphism  $\omega$  of  $U_q(\mathfrak{g})$  with  $\omega(E_i) = F_i, \omega(K_i) = K_i^{-1}$  and  $\omega(F_i) = E_i$ . Since  $\omega(U^+) = U^-, U^+$  and  $U^-$  are isomorphic.

**Proposition 5.** *For any  $k, q, \mathfrak{g}$ , the following are equivalent:*

- (1)  $U_q(\mathfrak{g})$  is formally semireal,
- (2)  $U^+$  is formally semireal,
- (3)  $k$  is formally real.

*Proof.*  $k$  is a unital subring of  $U^+$ ,  $U^+$  is a unital subring of  $U_q(\mathfrak{g})$  and there exists a unital homomorphism  $\phi: U_q(\mathfrak{g}) \rightarrow k$  defined by  $\phi(E_i) = 0, \phi(F_i) = 0, \phi(K_i) = 1$  for  $i = 1, \dots, n$ .  $\square$

Let  $A$  be an associative ring and  $(\Gamma, +)$  a totally ordered cancellative semigroup. We can extend the operation and ordering to  $\Gamma \cup \infty$  by  $\gamma + \infty = \infty + \gamma = \infty + \infty = \infty$  and  $\gamma < \infty$  for every  $\gamma \in \Gamma$ . A mapping  $v: A \rightarrow \Gamma \cup \infty$  is a *filtration* if for any  $a, b \in A$

- (1)  $v(a) = \infty$  if and only if  $a = 0$ .
- (2)  $v(ab) \geq v(a) + v(b)$ ,
- (3)  $v(a + b) \geq \min\{v(a), v(b)\}$ .

For every filtration  $v: A \rightarrow \Gamma \cup \infty$  and every  $\gamma \in \Gamma$ , the sets  $B_\gamma = \{a \in A: v(a) \geq \gamma\}$  and  $C_\gamma = \{a \in A: v(a) > \gamma\}$  are closed for addition. Write  $A_\gamma = B_\gamma/C_\gamma$ , the canonical projection from  $B_\gamma$  to  $A_\gamma$  will be denoted by  $a \mapsto \bar{a}$ . Note that the mapping  $A_\gamma \times A_\delta \rightarrow A_{\gamma+\delta}, (\bar{a}, \bar{b}) \mapsto \overline{ab}$  is well-defined. We call  $\text{gr}(A, v) = \bigoplus_{\gamma \in \Gamma} A_\gamma$  the *graded ring* of  $A$  with respect to  $v$ . Let  $\text{gr}(v)$  be the natural filtration of  $\text{gr}(A, v)$ .

For every filtration  $v: A \rightarrow \Gamma \cup \infty$ , the following are equivalent

- $v(ab) = v(a) + v(b)$  for every nonzero  $a, b \in A$ ,
- $\text{gr}(A, v)$  has no zero divisors.

A filtration which satisfies one of these equivalent properties is called a *valuation*. Each property implies that  $A$  has no zero divisors.

**Example.** In [8, Section 1], De Concini and Kac construct a total degree  $d: U_q(\mathfrak{g}) \rightarrow \Gamma$ . They show that  $v = -d$  is a filtration and that  $\text{gr}(U_q(\mathfrak{g}), v)$  has a presentation with generators  $E_\alpha, F_\alpha$  ( $\alpha \in \Phi^+$ ),  $K_i^{\pm 1}$  ( $i = 1, \dots, n$ ) and relations

$$\begin{aligned} K_i K_j &= K_j K_i, K_i K_i^{-1} = 1, E_\alpha F_\beta = F_\beta E_\alpha, \\ K_i E_\alpha &= q^{(\alpha, \alpha_i)} E_\alpha K_i, K_i F_\alpha = q^{-(\alpha, \alpha_i)} F_\alpha K_i, \\ E_\alpha E_\beta &= q^{(\alpha, \beta)} E_\beta E_\alpha, F_\alpha F_\beta = q^{(\alpha, \beta)} F_\beta F_\alpha. \end{aligned}$$

We will describe  $\text{gr}(v)$  for later reference. If  $s_{i_1} \cdots s_{i_N}$  is the longest reduced expression in the Weyl group  $W(\Phi)$ , then the elements  $\beta_m = s_{i_1} \cdots s_{i_{m-1}}(\alpha_{i_m})$  ( $m = 1, \dots, N$ ) are different and  $\Phi^+ = \{\beta_1, \dots, \beta_N\}$ . For every  $\beta = \sum_{i=1}^n k_i \alpha_i \in \Phi^+$  we define its *height* by  $\text{ht}(\beta) = \sum_{i=1}^n k_i$ . For any  $\mathbf{a} = (a_1, \dots, a_N) \in \mathbb{Z}_{\geq 0}^N$  and  $\mathbf{b} = (b_1, \dots, b_N) \in \mathbb{Z}_{\geq 0}^N$  we write  $E^{\mathbf{a}} F^{\mathbf{b}} = E_1^{a_1} \cdots E_N^{a_N} F_1^{b_1} \cdots F_N^{b_N}$  and  $\text{ht}(E^{\mathbf{a}} F^{\mathbf{b}}) = \sum_{i=1}^N (a_i + b_i) \text{ht}(\beta_i)$ . Every nonzero element  $z \in \text{gr}(A, v)$  can be expressed uniquely as  $z = \sum_{i=1}^r c_i E^{\mathbf{a}_i} F^{\mathbf{b}_i}$  where  $c_i$  are nonzero polynomials in  $K_i^{\pm 1}$  ( $i = 1, \dots, r$ ) and  $2N$ -tuples  $(\mathbf{a}_i, \mathbf{b}_i)$ ,  $i = 1, \dots, r$  are pairwise different. Then

$$\text{gr}(v)(z) = \min_i(\mathbf{a}_i, \mathbf{b}_i, \text{ht}(E^{\mathbf{a}_i} F^{\mathbf{b}_i})) \in \mathbb{Z}_{\geq 0}^{2N+1}$$

where  $\mathbb{Z}_{\geq 0}^{2N+1}$  has the reverse lexicographic ordering.

Let  $v$  be a valuation on a domain  $A$ . We say that a support zero ordering  $P$  of  $A$  is *compatible* with  $v$  if  $a + b \in P$  for every  $a \in P$  and every  $b \in A$  such that  $v(b) > v(a)$ . (e.g.  $P$  is always compatible with  $v_P$ , we used this already.) The same argument as in [23, Proposition 2.5] shows that there is a natural one-to-one correspondence between

- support zero orderings of  $A$  compatible with  $v$  and
- support zero orderings of  $\text{gr}(A, v)$  compatible with  $\text{gr}(v)$ ,

given by  $P \mapsto \{\bar{a} + \xi : a \in P, \text{gr}(v)(\xi) > \text{gr}(v)(\bar{a})\}$  and  $\bar{P} \mapsto \{a : \bar{a} \in \bar{P}\}$ .

**Example.** Let  $A$  and  $v$  be as in the previous example. Suppose that  $k$  has an ordering  $>$  such that  $q^{(\alpha_i, \alpha_j)} > 0$  for every  $i, j = 1, \dots, n$ . Then  $\text{gr}(A, v)$  is a (localization of) quantum affine ring with positive parameters, hence it is formally real by Proposition 1. However, this is not enough to show that  $A$  is formally real. We must construct a support zero ordering of  $\text{gr}(A, v)$  which is compatible with  $\text{gr}(v)$ . It will follow then that  $A$  has a support zero ordering compatible with  $v$ .

The elements  $K^{\mathbf{m}}E^{\mathbf{a}}F^{\mathbf{b}}$  where  $\mathbf{m} \in \mathbb{Z}^n$  and  $\mathbf{a}, \mathbf{b} \in \mathbb{Z}_{\geq 0}^{2N+1}$  form a  $k$ -basis of  $\text{gr}(A, v)$ . For every total ordering  $O$  of  $(\mathbb{Z}^n, +)$  we define a monomial ordering  $\succ_O$  by  $K^{\mathbf{m}}E^{\mathbf{a}}F^{\mathbf{b}} \succ_O K^{\mathbf{m}'}E^{\mathbf{a}'}F^{\mathbf{b}'}$  if and only if either  $\text{gr}(v)(E^{\mathbf{a}}F^{\mathbf{b}}) \succ_{\text{rlex}} \text{gr}(v)(E^{\mathbf{a}'}F^{\mathbf{b}'})$  or  $\text{gr}(v)(E^{\mathbf{a}}F^{\mathbf{b}}) = \text{gr}(v)(E^{\mathbf{a}'}F^{\mathbf{b}'})$  and  $\mathbf{m} - \mathbf{m}' \in O$ . For every element  $z = \sum_{i=1}^r d_i K^{\mathbf{m}_i} E^{\mathbf{a}_i} F^{\mathbf{b}_i}$  where  $d_1, \dots, d_r \in k \setminus \{0\}$  and  $K^{\mathbf{m}_1} E^{\mathbf{a}_1} F^{\mathbf{b}_1} \succ_O \dots \succ_O K^{\mathbf{m}_r} E^{\mathbf{a}_r} F^{\mathbf{b}_r}$  we write  $l_O(z) = d_r$ . Then  $P_O = \{z \in \text{gr}(A, v) : z = 0 \text{ or } l_O(z) > 0\}$  is a support zero ordering compatible with  $\text{gr}(v)$ .

**Proposition 6.** *For any  $k, q, \mathfrak{g}$ , the following are equivalent:*

- (1)  $U_q(\mathfrak{g})$  is formally real,
- (2)  $U^+$  is formally real,
- (3)  $k$  is formally real and  $q^{(\alpha_i, \alpha_j)} > 0$  for any  $i, j = 1, \dots, n$ .

*Proof.* If  $P$  is a support zero ordering of  $U_q(\mathfrak{g})$ , then  $P \cap U^+$  is a support zero ordering of  $U^+$ . Hence (1) implies (2). If  $Q$  is a support zero ordering of  $U^+$ , then  $Q \cap k$  is a support zero ordering of  $k$ . Since  $E_j K_i$  and  $K_i E_j$  have the same sign with respect to  $Q$ , it follows that  $q_i^{-\alpha_{ij}} > 0$  for every  $i, j = 1, \dots, n$ . Hence (2) implies (3). The last example shows that (3) implies (1).  $\square$

**Example.** Let  $q \in \mathbb{R} \setminus \{0, 1\}$  and let  $A = U_q(\mathfrak{sl}_2(\mathbb{R}))$  be the  $\mathbb{R}$ -algebra with generators  $E, F, K, K^{-1}$  and relations:

$$\begin{aligned} KK^{-1} &= K^{-1}K = 1, \\ KEK^{-1} &= q^2E, \quad KFK^{-1} = q^{-2}F, \\ EF - FE &= \frac{K - K^{-1}}{q - q^{-1}}. \end{aligned}$$

Let  $J$  be a real prime ideal of  $A$ . If  $E \notin J$ , then  $J$  extends to a prime ideal of  $A_E$ . Note that  $A_E \cong \mathbb{R}[C]_{q^2}[K^{\pm 1}, E^{\pm 1}]$  where  $C = EF + \frac{q^{-1}K + qK^{-1}}{(q - q^{-1})^2} = FE + \frac{qK + q^{-1}K^{-1}}{(q - q^{-1})^2}$  is the *quantum Casimir element*. By the example after Proposition 3, it follows that either  $J = (0)$  or  $J = (C - \lambda)$ ,  $(\lambda \in \mathbb{R})$ . If  $E \in J$ , then  $K - K^{-1} = (q - q^{-1})(EF - FE) \in J$ . It follows that either  $K - 1 \in J$  or  $K + 1 \in J$ . Since  $(q^2 - 1)KF = FK - KF = F(K \pm 1) - (K \pm 1)F \in J$  and  $K \notin J$ , it follows that  $F \in J$ . Hence,  $J = (E, F, K + 1)$  or  $J = (E, F, K - 1)$ .

The description of  $\text{Sper}(A)$  consists of a complete list of real prime ideals and a complete list of orderings of the skew field of fractions of each factor domain:

- If  $J = (E, F, K + 1)$  or  $J = (E, F, K - 1)$  then  $\text{Fract}(A/J) \cong \mathbb{R}$  has exactly one ordering.
- If  $J = (C - \lambda)$  where  $\lambda \in \mathbb{R}$  then  $\text{Fract}(A/J) \cong \mathbb{R}_{q^2}(K, E)$  and we have a four-to-one correspondence between the orderings of  $\mathbb{R}_{q^2}(K, E)$  and the orderings of the abelian group  $\mathbb{Z} \times \mathbb{Z}$ .

- If  $J = (0)$  then  $\text{Fract}(A/J) \cong \mathbb{R}(C)_{q^2}(K, E)$  and we have an eight-to-one correspondence between the orderings of  $\mathbb{R}(C)_{q^2}(K, E)$  and the cartesian product of the set  $\mathbb{R} \cup \{\infty\}$  and the set of all orderings of the abelian group  $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$  which contain  $(1, 0, 0)$ .

**Remark.** It is conjectured that  $\text{Fract}(U_q(\mathfrak{g}))$  is a quantum Weyl field for every  $\mathfrak{g}$ . By [13], this is true if  $\mathfrak{g} = \mathfrak{sl}_n$ . Not much is known about the prime spectrum of  $U_q(\mathfrak{g})$ .

**Remark.** By [4, Theoreme 6.2.2], one can find a quantum affine ring  $\overline{U^+}$  and a natural embedding  $\phi: \text{Spec}(U^+) \rightarrow \text{Spec}(\overline{U^+})$  such that  $\text{Fract}(U^+/P) \cong \text{Fract}(\overline{U^+}/\phi(P))$  for every  $P \in \text{Spec} U^+$ . It follows that  $\phi$  maps real primes into real primes. Moreover, there is a one-to-one correspondence between orderings of  $U^+$  with support  $P$  and support zero orderings of the ring  $\overline{U^+}/\phi(P)$ . By Proposition 2,  $\overline{U^+}/\phi(P)$  is a quantum affine ring and its center  $Z$  is a formally real affine algebra over  $k$ . If  $k = \mathbb{R}$  then, it is possible in principle to describe all support zero orderings of  $Z$  by the method sketched in the introduction. Their extensions to  $\overline{U^+}/\phi(P)$  are described by Theorem 4. An explicit presentation of  $\text{Fract} U^+$  is given in [2].

#### 4. QUANTIZED FUNCTION ALGEBRAS

Let  $\mathfrak{g}$  be a semisimple Lie algebra,  $k$  a field and  $q$  a nonzero element of  $k$ . For every dominant weight  $\lambda$ , write  $L_q(\lambda)$  for the unique simple left  $U_q(\mathfrak{g})$ -module with highest weight  $\lambda$  and let  $L_q(\lambda)^*$  be its vector space dual considered as a right  $U_q(\mathfrak{g})$ -module. For every dominant weight  $\lambda$ , every  $\xi \in L_q(\lambda)^*$  and every  $m \in L_q(\lambda)$  we define an element  $c_{\xi, m}^\lambda \in U_q(\mathfrak{g})^*$  by

$$c_{\xi, m}^\lambda(a) = \xi(am), \quad a \in U_q(\mathfrak{g}).$$

The  $k$ -subspace of  $U_q(\mathfrak{g})^*$  spanned by all  $c_{\xi, m}^\lambda$  is called the *quantized function algebra*  $k_q[G]$  ( $G$  is the simply connected Lie group of  $\mathfrak{g}$ .) Many authors write  $\mathcal{O}_q(G)$  instead of  $k_q[G]$ . The dual  $U_q(\mathfrak{g})^*$  has an algebra structure defined by

$$cc'(a) = \sum c(a_{(1)})c'(a_{(2)}) \quad \text{if } \Delta(a) = \sum a_{(1)} \otimes a_{(2)},$$

where  $\Delta: U_q(\mathfrak{g}) \rightarrow U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})$  is the comultiplication of  $U_q(\mathfrak{g})$ . The counit  $\epsilon: U_q(\mathfrak{g}) \rightarrow k$  plays the role of 1 in  $U_q(\mathfrak{g})^*$ . It turns out that  $k_q[G]$  is a unital subalgebra of  $U_q(\mathfrak{g})^*$ .

**Proposition 7.** *Let  $k$  be a field,  $q \in k$  and  $G$  a simply connected Lie group. The ring  $k_q[G]$  is semireal if and only if  $k$  is a formally real*

field. The ring  $k_q[G]$  is formally real if and only if  $k$  has an ordering such that  $q > 0$ .

*Proof.* The claim about formal semireality is proved as usual. The claim about formal reality follows from the fact that  $\text{Fract}(k_q[G])$  is a quantum Weyl field [2, Section 3.3] and from our computation of its parameters, see below.

Let  $\alpha_1, \dots, \alpha_n$  be a base of the root system  $\Phi$ ,  $s_1, \dots, s_n$  the corresponding reflections ( $s_i(\beta) = \beta - \frac{2(\beta, \alpha_i)}{(\alpha_i, \alpha_i)}\alpha_i$ ), and  $\lambda_1, \dots, \lambda_n$  the corresponding fundamental weights ( $(\lambda_i, \frac{2\alpha_j}{(\alpha_j, \alpha_j)}) = \delta_{ij}$ ). Let  $w_0 = s_{i_1} \cdots s_{i_N}$  be the longest reduced expression in  $W$ . Write  $y_0 = \text{Id}$  and  $y_k = s_{i_1} \cdots s_{i_k}$  for every  $k = 1, \dots, N$ . The elements  $\beta_1 = y_0(\alpha_{i_1}), \beta_2 = y_1(\alpha_{i_2}), \dots, \beta_N = y_{N-1}(\alpha_{i_N})$  are distinct positive roots and every positive root is one of them. Write  $\rho = \sum_{i=1}^n \lambda_i = \frac{1}{2} \sum_{j=1}^N \beta_j$ .

In section 3.3 of [2], Caldero defines elements

$$\begin{aligned} c_i &= c_{w_0 \lambda_i, \lambda_i}^{\lambda_i} & i &= 1, \dots, n, \\ d_i &= c_{y_{i-1} \rho, -y_i \rho}^{\rho} & i &= 1, \dots, N \\ d'_i &= c_{y_{i-1} \rho, -y_{i-1} \rho}^{\rho} & i &= 1, \dots, N \end{aligned}$$

and proves that they generate a quantum affine ring whose skew field of fractions is isomorphic to  $\text{Fract}(k_q[G])$ . One can obtain explicit  $q$ -commutation relations between the elements  $c_i, d_i, d'_i$ . By 9.1.6(\*\*) in [14], we have  $c_k c_l = c_l c_k$  for all  $k, l = 1, \dots, n$ . By 9.1.4(ii) in [14], we have

$$\begin{aligned} d_k c_l &= q^{-(y_k \rho, \lambda_l) - (y_{k-1} \rho, w_0 \lambda_l)} c_l d_k, \\ d'_k c_l &= q^{-(y_{k-1} \rho, \lambda_l) - (y_{k-1} \rho, w_0 \lambda_l)} c_l d_k \end{aligned}$$

From (1.5.1) in [2] we obtain

$$\begin{aligned} d_k d'_l &= q^{(y_{k-1} \rho, y_{l-1} \rho) - (y_k \rho, y_{l-1} \rho)} d'_l d_k & \text{if } k \geq l, \\ d_k d'_l &= q^{-(y_{k-1} \rho, y_{l-1} \rho) + (y_k \rho, y_{l-1} \rho)} d'_l d_k & \text{if } k < l, \\ d_k d_l &= q^{(y_{k-1} \rho, y_{l-1} \rho) - (y_k \rho, y_l \rho)} d_l d_k & \text{if } k \geq l, \\ d'_k d'_l &= d'_l d'_k. \end{aligned}$$

All exponents of  $q$  are integers. We claim that at least one of them is odd. Let  $d_k c_l = q^{m(k,l)} c_l d_k$  where  $m(k, l) = -(y_k \rho, \lambda_l) - (y_{k-1} \rho, w_0 \lambda_l)$ . For every  $k = 1, \dots, N$  we have  $\sum_{l=1}^n m(k, l) = -(y_k \rho, \rho) - (y_{k-1} \rho, w_0 \rho)$ . Since  $y_k \rho = y_{k-1} \rho - \beta_k$  and  $w_0 \rho = -\rho$ , it follows that  $\sum_{l=1}^n m(k, l) = (\beta_k, \rho)$ . If  $k$  is such that  $\beta_k$  is a short simple root, then  $\sum_{l=1}^n m(k, l) = 1$ . It follows that that at least one  $m(k, l)$  is odd.  $\square$

**Example.** If  $k = \mathbb{R}$ ,  $q > 0$ ,  $G = SL_2(\mathbb{R})$  and  $R = k_q[G](= \mathcal{O}_q(SL_2(\mathbb{R})))$ , then  $\text{Sper}(R)$  can be completely described. The ring  $R$  has generators

$a, b, c, d$  and relations

$$\begin{aligned} ab &= qba, \quad ac = qca, \quad bd = qdb, \quad cd = qdc, \quad bc = cb, \\ ad - qbc &= da - q^{-1}bc = 1 \end{aligned}$$

A prime ideal which contains  $a$ , contains also  $b$  or  $c$ , therefore it contains 1. So there is a one-to-one correspondence between  $\text{Sper}R$  and  $\text{Sper}R_a$ . But  $R_a$  has generators  $a, b, c$  and relations  $ab = qba, ac = qca, bc = cb$ , hence it is a quantum affine ring over  $\mathbb{R}$ . If a prime ideal  $J$  contains  $b$ , then  $R_a/J$  is a factor domain of  $\mathbb{R}_q[a^{\pm 1}, c]$ . If  $b \notin J$ , every ordering with support  $J$  extends uniquely to  $R_{a,b} \cong \mathbb{R}[t]_q[a^{\pm 1}, b^{\pm 1}]$  ( $t = cb^{-1}$ ). The orderings  $\mathbb{R}_q[a, c]$  are known and the orderings of  $\mathbb{R}[t]_q[a, b]$  were described in section 2.

**Remark.** The structure of  $\text{Spec}(k_q[G])$  is described in [1, Section II.4.]. If  $G = M_n(k)$  then  $\text{Fract}(k_q[G]/J)$  is a quantum Weyl field for every  $J \in \text{Spec}(k_q[G])$ , see [5].

## 5. QUANTIZED WEYL ALGEBRAS

Let  $k$  be a field  $Q = (q_1, \dots, q_n) \in (k^\times)^n$  and let  $\Gamma = (\gamma_{ij})$  be a multiplicatively antisymmetric  $n \times n$  matrix over  $k$ . The *multiparameter quantized Weyl algebra of degree  $n$  over  $k$*  is the  $k$ -algebra  $A_n^{Q, \Gamma}$  generated by elements  $x_1, y_1, \dots, x_n, y_n$  subject to the following relations

$$\begin{aligned} y_i y_j &= \gamma_{ij} y_j y_i && (\text{all } i, j) \\ x_i x_j &= q_i \gamma_{ij} x_j x_i && (i < j) \\ x_i y_j &= \gamma_{ji} y_j x_i && (i < j) \\ x_i y_j &= q_j \gamma_{ji} y_j x_i && (i > j) \\ x_j y_j &= 1 + q_j y_j x_j + \sum_{l < j} (q_l - 1) y_l x_l && (\text{all } j) \end{aligned}$$

**Proposition 8.** *The  $k$ -algebra  $R = A_n^{Q, \Gamma}$  has a support zero ordering if and only if  $k$  has an ordering such that  $q_i > 0$  and  $\gamma_{ij} > 0$  for every  $i, j = 1, \dots, n$ .*

*The  $k$ -algebra  $R$  is semireal if and only if  $k$  is semireal and for every  $m \in \{1, \dots, n\}$  such that  $q_1 = q_2 = \dots = q_{m-1} = 1$  we have  $\gamma_{ij} > 0$  for all  $i, j = 1, \dots, m$ .*

*Proof.* If  $R$  has a support zero ordering  $P$  then  $\gamma_{ij} \in P \cap k^\times$  for  $i, j = 1, \dots, n$  since  $y_i y_j$  and  $y_j y_i$  have the same sign. Write  $z_i = x_i y_i - y_i x_i$  for  $i = 1, \dots, n$  and note that  $z_i y_i = q_i y_i z_i$ . Since  $y_i z_i$  has the same sign as  $z_i y_i$ , it follows that  $q_i \in P \cap k^\times$  for  $i = 1, \dots, n$ .

If  $k$  is formally real,  $q_i > 0$  and  $\gamma_{ij} > 0$  for all  $i, j$ , then

$$P := \{0\} \cup \left\{ \sum_{r=1}^s c_r y_1^{i_{r1}} \dots y_n^{i_{rn}} x_1^{j_{r1}} \dots x_n^{j_{rn}} \mid (i_{11}, \dots, j_{1n}) <_{\text{lex}} \dots <_{\text{lex}} (i_{s1}, \dots, j_{sn}) \text{ and } c_s > 0 \right\}$$

is a support zero ordering of  $R$ .

Assume now that  $R$  is semireal. Clearly,  $k$  is semireal, too. For every  $m$  such that  $q_1 = \dots = q_{m-1} = 1$ , we have  $x_j y_j = 1 + q_j y_j x_j$  for  $j = 1, \dots, m$ . By definition,  $R$  has a proper real prime ideal  $J$ . If  $\gamma_{ij} < 0$  for some  $i, j = 1, \dots, m$  then it follows from  $y_i y_j = \gamma_{ij} y_j y_i$  that either  $y_i \in J$  or  $y_j \in J$ , a contradiction with  $x_i y_i = 1 + q_i y_i x_i$  or  $x_j y_j = 1 + q_j y_j x_j$ . Therefore,  $\gamma_{ij} > 0$  for  $i, j = 1, \dots, m$ .

To prove the converse, we distinguish two cases. If  $q_1 = \dots = q_n = 1$ , then  $\gamma_{ij} > 0$  for all  $i, j = 1, \dots, n$  by the assumption of the proposition. Then  $R$  is formally real by the first paragraph. Otherwise, there is  $m \in \{1, \dots, n\}$  such that  $q_1 = q_2 = \dots = q_{m-1} = 1$  and  $q_m \neq 1$ . Write  $Q_m = (1^{m-1}, q_m)$  and  $\Gamma_m$  for the upper left  $m \times m$  submatrix of  $\Gamma$ . Let  $S_m$  be a subring of  $A_m^{Q_m, \Gamma_m}$  generated by  $\bar{x}_1, \dots, \bar{x}_m, \bar{y}_1, \dots, \bar{y}_{m-1}$ . The commutation relations of  $S_m$  contain only parameters  $\gamma_{ij}$ ,  $i, j = 1, \dots, m$  which are positive by the assumption. Hence,  $S_m$  has a support zero ordering by the same argument as in the first paragraph. We also have a unital homomorphism  $\phi: R \rightarrow (S_m)_{\bar{x}_m}$  defined by

$$\begin{aligned} \phi(x_i) &= \bar{x}_i, & \phi(y_i) &= \bar{y}_i, & i &= 1, \dots, m-1, \\ \phi(x_m) &= \bar{x}_m, & \phi(y_m) &= \frac{1}{1-q_m} \bar{x}_m^{-1}, \\ \phi(x_j) &= 0, & \phi(y_j) &= 0, & j &= m+1, \dots, n. \end{aligned}$$

Hence,  $R$  is semireal. □

If  $q_1 \neq 1, \dots, q_n \neq 1$  then  $\text{Fract}(A_n^{Q, \Gamma})$  is a quantum Weyl field and the parameters  $q_i$  and  $\gamma_{ij}$ ,  $i, j = 1, \dots, n$  appear as entries of its matrix, [12, Section 3.1]. The claim about formal reality then follows from Proposition 1.

**Remark.** The structure of the prime spectrum of algebras  $A_n^{Q, \Gamma}$  is discussed in [1]. If  $q_1 \neq 1, \dots, q_n \neq 1$ , then for every prime ideal  $J$  of  $A_n^{Q, \Gamma}$ , the skew field  $\text{Fract}(A_n^{Q, \Gamma}/J)$  is a quantum Weyl field over a finitely generated extension of  $k$ , see [3].

## 6. FINAL COMMENTS AND OPEN PROBLEMS

- (1) The description of orderings in the quantum case is much easier than in the classical case. The description of the real spectrum of  $A_1(\mathbb{R}) = \mathbb{R}\langle x, y \rangle / (yx - xy - 1)$  and  $U(\mathfrak{sl}_2(\mathbb{R}))$  is still an open problem, see [22].
- (2) Quantum groups usually have nontrivial involutions. Can the results of this paper be extended to  $*$ -orderings? See [24, 23, 7].
- (3) As noted by Ringel, [26],  $U^+$  is an iterated skew polynomial ring so further analysis of  $\text{Sper}(U^+)$  is possible. What are the

- best results for minimal generation of basic semialgebraic sets? See [22] for usual orderings and [23] for  $*$ -orderings.
- (4) The results of this paper can probably be extended to orderings of higher level. See [6] for the classification of orderings of higher level on quantum polynomials. See also [25].
  - (5) Quantized enveloping algebras are graded by their root lattice. Positivstellensätze for noncommutative graded rings have been developed by Igor Klep, see [15].
  - (6) Is there a reasonable stratification theory for real spectra of quantum groups? See [10].
  - (7) Let  $A$  and  $B$  be unital  $k$ -algebras with with support zero orderings which induce the same ordering on  $k$ . Is it always true that  $A \otimes_k B$  has a support zero ordering extending the orderings on  $A$  and  $B$ ?

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