

HIGHER PRODUCT LEVELS OF NONCOMMUTATIVE RINGS

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The aim of this paper is to introduce the notion of the n -th product level ps_n of an associative unital ring and study its properties. Our main results are that every Noetherian ring A with $\text{ps}_n(A) < \infty$ for some $n \in \mathbb{N}$ has $\text{ps}_{nl}(A) < \infty$ for every odd number l (Theorem 8) and that for every even $n \in \mathbb{N}$ there exists a skew-field D with $\text{ps}_n(D) = 1$ and $\text{ps}_{2n}(D) = \infty$ (Theorem 9). This is in a sharp contrast with the commutative case. Namely, by Proposition 4.6 in [4], for every commutative unital ring R with $\text{ps}_2(R) < \infty$ we have that $\text{ps}_n(R) < \infty$ for every $n \in \mathbb{N}$.

DEFINITIONS AND ELEMENTARY EXAMPLES

Let A be an associative unital ring and $n \in \mathbb{N}$. Write $\Pi_n(A)$ for the set of all elements from A expressible as products of n -th powers of elements from A or permutations of such products. For example, $xyzyxz \in \Pi_2(A)$ for all $x, y, z \in A$. Write $\Sigma_n(A)$ for the set of all finite sums of elements from $\Pi_n(A)$.

If $-1 \in \Sigma_n(A)$, then we look for the shortest expression of -1 as a sum of elements from $\Pi_n(A)$. The number of terms in this expression is denoted by $\text{ps}_n(A)$. If $-1 \notin \Sigma_n(A)$, then we write $\text{ps}_n(A) = \infty$. The invariant $\text{ps}_n(A)$ will be called *the n -th product level of A* . If n is odd, then $\text{ps}_n(A) = 1$ for every A .

The book [20] surveys the results about the level ($= \text{ps}_2$) of commutative rings. The ps_n of commutative rings is considered in [14], [4] and [3]. The ps_n of a commutative ring is the number of terms in the shortest representation of -1 as a sum of n -th powers.

The ps_2 of skew-fields is just the usual product level s_π . This follows from the fact that every multiplicative commutator is a product of three squares. If $\text{ps}_2(D) = \infty$, then D has a linear ordering and it is infinite-dimensional over its center, see [1] and [26] or Section 18 in [15]. In [24], A. Tschimmel and W. Scharlau construct for every positive integer m an infinite-dimensional skew-field D such that $\text{ps}_2(D) = m$. A. Wadsworth [29] shows that every even-dimensional skew-field D has $\text{ps}_2(D) = 1$. In [16], D. B. Leep, J. P. Tignol and N. Vast show that every odd-dimensional non-commutative skew-field whose center is a local or global field has $\text{ps}_2(D) = 2$. They also construct an odd-dimensional skew-field with product level 4.

We start with three elementary examples.

Example 1 Let $U_k(R)$ be the ring of all upper triangular $k \times k$ matrices over an associative unital ring R . For all positive integers n and k we have that $\text{ps}_n(U_k(R)) = \text{ps}_n(R)$.

Proof. If $\text{ps}_n(U_k(R)) \leq m$, then comparing the diagonal entries we see that $\text{ps}_n(R) \leq m$. If $\text{ps}_n(R) \leq m$, then $-I_k$ is a sum of at most m scalar matrices. It follows that $\text{ps}_n(U_k(R)) \leq m$. Q.E.D.

Example 2 For every even $n \in \mathbb{N} \setminus \{0\}$ and every group G we have that $\text{ps}_n(\mathbb{Z}[G]) = \infty$.

Proof. Note that the ring \mathbb{Z} is a homomorphic image of the ring $\mathbb{Z}[G]$. If $\text{ps}_n(\mathbb{Z}[G]) < \infty$, then $\text{ps}_n(\mathbb{Z}) \leq \text{ps}_n(\mathbb{Z}[G]) < \infty$. Hence, n is odd. Q.E.D.

Example 3 Let $M_k(R)$ be the ring of all $k \times k$ matrices over an associative unital ring R . Then for every n we have that $\text{ps}_n(M_k(R)) = 1$ if k is even and $\text{ps}_n(M_k(R)) \leq 2$ if k is odd and ≥ 3 .

Proof. Since every associative unital ring contains a copy of the ring \mathbb{Z}_m for some positive integer m and since \mathbb{Z}_m is a homomorphic image of \mathbb{Z} we have that for every n $\text{ps}_n(M_k(R)) \leq \text{ps}_n(M_k(\mathbb{Z}_m)) \leq \text{ps}_n(M_k(\mathbb{Z}))$. Note that every multiplicative commutator belongs to Π_n for every n .

If k is an even number, then the assertion follows from

$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^{-1}.$$

If k is odd and ≥ 3 , then write $k = 2m + 3$, $A_k = (m \times A_2) \oplus A_3$ and $B_k = (m \times B_2) \oplus B_3$, where the matrices A_2, B_2, A_3, B_3 are defined by

$$A_2 = \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}, B_2 = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}, A_3 = \begin{bmatrix} -1 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, B_3 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -1 & -1 \\ 0 & -1 & -1 \end{bmatrix}.$$

We have that $-I_k = A_k + B_k$ and $A_k, B_k \in \mathrm{SL}_k(\mathbb{Z})$. By Lemma 1.3.2 and Exercise 1.4 in [30] we have that $\mathrm{SL}_k(\mathbb{Z}) = E_k(\mathbb{Z}) = [E_k(\mathbb{Z}), E_k(\mathbb{Z})]$, where the second group is defined as the minimal group containing all transvections and the third group is the commutator subgroup of the second one. Q.E.D.

ORDERINGS OF HIGHER LEVEL

Let A be an associative unital ring. A mapping $\sigma : A \rightarrow \mathbb{C}$ is a *signature* if $\sigma(xy) = \sigma(x)\sigma(y)$ for all $x, y \in A$, $\sigma(-1) = -1$, the set $P_\sigma := \sigma^{-1}(\{0, 1\})$ is additive and $\sigma(A)^n = 1$ for some $n \in \mathbb{N} \setminus \{0\}$. Any such number n is an *exponent* of σ . Any set of the form P_σ for some signature σ is called an *ordering*. The set $P_\sigma \cap -P_\sigma = \sigma^{-1}(0)$ is called *the support* of P_σ . The set of all orderings with exponent n on a ring A will be denoted by $\mathrm{Sper}_n(A)$. Orderings on commutative rings are explored in [5] and [6]. Orderings on noncommutative rings are considered in [13, 17] (exponent 2) and in [11, 21, 22, 8, 7, 9] (higher exponents).

Example 4 For every associative unital ring R and every $n, k \in \mathbb{N} \setminus \{0\}$ we have that $\mathrm{Sper}_n(U_k(R)) = \mathrm{Sper}_n(R^k) = \mathrm{Sper}_n(R) \times \{1, \dots, k\}$.

Proof. Note that the set $J_k(R)$ of strictly upper triangular matrices is a two-sided ideal in the ring $U_k(R)$. Since every element of $J_k(R)$ is nilpotent, it is contained in every completely prime ideal of $U_k(R)$. It follows that every signature on $U_k(R)$ factors through $U_k(R)/J_k(R) \cong R^k$. This gives us a bijective correspondence between $\mathrm{Sper}_n(U_k(R))$ and $\mathrm{Sper}_n(R^k)$. The bijective correspondence between $\mathrm{Sper}_n(R^k)$ and $\mathrm{Sper}_n(R) \times \{1, \dots, k\}$ follows from the bijective correspondence between the sets of completely prime ideals $\mathrm{Spec}(R^k)$ and $\mathrm{Spec}(R) \times \{1, \dots, k\}$. Q.E.D.

It is clear that $\text{ps}_n(A) < \infty$ implies that $\text{Sper}_n A = \emptyset$ for every associative unital ring A and every $n \in \mathbb{N} \setminus \{0\}$. It is still an open question whether the converse is true. By Theorem 4 and Proposition 7 in [7] we have that the converse holds true for Noetherian rings. So, we have the following extension of the classical Artin-Schreier Theorem:

Proposition 5 *Let A be an associative unital Noetherian ring. Then $\text{ps}_n(A) = \infty$ if and only if $\text{Sper}_n A \neq \emptyset$.*

The following example shows how to apply Proposition 5 to compute higher product levels.

Example 6 *Let A be an associative unital ring such that either $-1 = x^2 + y^2$ or $-1 = x^2 y^2 = y^2 x^2$ for some $x, y \in A$. Then $\text{ps}_n(A) < \infty$ for every $n \in \mathbb{N}$.*

Proof. Note that in both cases the elements x^2 and y^2 belong to the center of A . It follows that the elements xy and yx commute. Let S be a subring of A generated by the set $\{1, x, y\}$ and let R be a subring of A generated by $\{x^2, xy, yx, y^2\}$. The ring R is Noetherian since it is a factor ring of a commutative polynomial ring in three variables. Note that the set $\{1, x, y\}$ generates S as a left and as a right R -module. It follows by Lemma 1.1.3 in [19] that S is left and right Noetherian.

If $-1 \notin \Sigma_n(S)$ for some $n \in \mathbb{N}$, then by Proposition 5 there exists a signature σ on the subring S with exponent n . Since x^2 and y^2 belong to the center of A , it follows that the element $xy - yx$ anticommutes with x and y . It follows that $\sigma(xy - yx)\sigma(x) = 0$ and $\sigma(xy - yx)\sigma(y) = 0$. But in both cases either $\sigma(x) \neq 0$ or $\sigma(y) \neq 0$. So, $\sigma(xy - yx) = 0$. Consequently, the ring $S/\sigma^{-1}(0)$ is commutative. The signature σ factors through $S/\sigma^{-1}(0)$. By Proposition 4.6 in [4], there exists a signature τ on $S/\sigma^{-1}(0)$ with exponent 2. The extension of the signature τ to the ring A contradicts the assumptions. Q.E.D.

VALUATIONS ON SKEW-FIELDS

Let P be an ordering with exponent n on a skew field D . Write $A(P) = \{a \in A \mid \exists r \in \mathbb{Q}^+ : r \pm a \in P\}$, $\text{Arch}(P) = \{a \in A \mid \forall r \in \mathbb{Q}^+ : r + a \in P\}$ and $I(P) = \text{Arch}(P) \cap -\text{Arch}(P)$.

Proposition 7 *With P and D as above:*

1. *the set $A(P)$ is an invariant valuation ring with maximal ideal $I(P)$,*
2. *we have that $A(P) \setminus I(P) \subseteq P \cup -P$,*
3. *the set $\text{Arch}(P) \cap A(P)$ is an archimedean ordering with exponent 2 on $A(P)$.*

Proof. Assertion 3. is a consequence of 1. and 2. Assertion 1. is an easy consequence of Satz 2.2 from [2]. It is proved in [9]. The commutative case of the assertion 2. is a part of the proof of Satz 2.2 from [2]. The same proof works in the noncommutative case if we use the version of the Kadison-Dubois representation Theorem from [22]. Q.E.D.

Let D be a skew field and Γ a totally ordered group written multiplicatively. A mapping $v : D \rightarrow \Gamma \cup \infty$ is a *valuation* if $v(a) = \infty$ if and only if $a = 0$ and for every $a, b \in D \setminus \{0\}$ we have that $v(ab) = v(a)v(b)$ and $v(a + b) \geq \min(v(a), v(b))$. A valuation v is *invariant* if $v(a) \geq 1$ implies that $v(cac^{-1}) > 1$ for any $a, c \in D \setminus \{0\}$. See [25] or [10] for more details.

Theorem 8 *Let A be an associative unital Noetherian ring. If $\text{ps}_n(A) < \infty$ for some $n \in \mathbb{N}$, then $\text{ps}_{nl}(A) < \infty$ for every odd $l \in \mathbb{N}$.*

Proof. Our method is based on an approach of E. Becker, see [2] Korolar 2.3. Let D be a skew-field with $\text{ps}_n(D) = \infty$. By Proposition 5, there exists

an ordering P on D with exponent nl . We will construct an ordering with exponent n on D . By Proposition 5, it follows that $\text{ps}_n(D) = \infty$.

By the first assertion of Proposition 7, the set $A(P)$ is an invariant valuation ring in D . Let v be its (invariant) valuation and Γ its value group. Assume that $s_0 + s_1 + \dots + s_k = 0$ for some $s_0, s_1, \dots, s_k \in \Pi_n(A(P))$. We may assume that $0 \leq v(s_0) = v(s_1) = \dots = v(s_j) < v(s_{j+1}) \leq \dots \leq v(s_k)$ for some $0 \leq j \leq k$. Clearly, we have that $s_i s_0^{-1} \in I(P)$ for all $j < i \leq k$. We claim that $s_i s_0^{-1} \in P$ for every $1 \leq i \leq j$. If $s_i s_0^{-1} \notin P$, then it follows from the fact that $v(s_i s_0^{-1}) = 1$ and the second assertion of Proposition 7 that $s_i s_0^{-1} \in -P$. Since l is odd, we have that $(s_i s_0^{-1})^l \in -P$. On the other side $(s_i s_0^{-1})^l \in \Pi_{nl}(D) \subseteq P$. This implies a contradiction $s_i s_0^{-1} = 0 \in P$ and proves the claim. Now we have a contradiction $-1 \in I(P) \cup (P \cap A(P)) \subseteq \text{Arch}(P) \cap A(P)$ with the third assertion of Proposition 7. It follows that $s_0 = \dots = s_k = 0$. This means that the set $T = \Sigma_n(A(P))$ is a precone on the ring $A(P)$. Note that the ring $A(P)$ is an Ore domain and that D is its skew-field of fractions. By Proposition 3 in [7], it follows that the set $\Pi_n(D)T$ is a precone on D with exponent n . It follows that $\text{ps}_n(D) = \infty$.

Now, take any unital Noetherian ring A with $\text{ps}_{nl}(A) = \infty$. Pick a two-sided ideal J maximal with respect to the property $J \cap (1 + \Sigma_{nl}(A)) = \emptyset$. By Proposition 7 and Corollary 6 in [7] we see that A/J is an Ore domain. Let D be its skew-field of fractions. Since $-\Sigma_{nl}(A/J) \cap \Sigma_{nl}(A/J) = \{0\}$, there exists by Theorem 4 in [7] an ordering with support $\{0\}$ and exponent nl on A/J which extends by Proposition 3 in [7] to an ordering with exponent nl on D . By the first part of the proof, there exists an ordering with exponent n on D . Its restriction to A/J extends to an ordering with exponent n and support J on A . Hence, $\text{ps}_n(A) = \infty$. Q.E.D.

By Proposition 5 and Theorem 8 we see that every Noetherian ring A with $\text{Sper}_{nl}(A) \neq \emptyset$ for some $n, l \in \mathbb{N}$ with l odd has $\text{Sper}_n(A) \neq \emptyset$.

THE MAIN CONSTRUCTION

Let Γ be a totally ordered group, A an associative unital ring and $c : \Gamma \times \Gamma \rightarrow A$ a mapping such that $c(rs, t)c(r, s) = c(r, st)c(s, t)$ for every $r, s, t \in \Gamma$ and $c(u, 1) = c(1, u) = 1$ for every $u \in \Gamma$. For every mapping $\phi : \Gamma \rightarrow A$ define *its support* $\text{supp}(\phi) = \Gamma \setminus \phi^{-1}(0)$. Write $A((\Gamma, c))$ for the set of all mappings $\phi : \Gamma \rightarrow A$ such that $\text{supp}(\phi)$ is a well-ordered subset of Γ . For elements $\phi, \psi \in A((\Gamma, c))$ we define the sum $\phi + \psi$ and the product $\phi * \psi$ by

$$(\phi + \psi)(t) = \phi(t) + \psi(t),$$

$$(\phi * \psi)(t) = \sum_{rs=t} c(r, s)\phi(r)\psi(s).$$

It is shown in Section 4 of [18] that both operations are well-defined and that $(A((\Gamma, c)), +, *)$ is an associative unital ring. Define a mapping $\pi : A \times \Gamma \rightarrow A((\Gamma, c))$ by $\pi(x, r)(s) = x$ if $r = s$ and $\pi(x, r)(s) = 0$ if $r \neq s$. We will write xr instead of $\pi(x, r)$ and r instead of $1r = \pi(1, r)$. Note that $xr * ys = (c(r, s)xy)rs$ for every $x, y \in A$ and $r, s \in \Gamma$. If A is a skew field and for every $y, z \in A \setminus \{0\}$ and $s, t \in \Gamma$ there exists an element $x \in A \setminus \{0\}$ such that $xts^{-1} * ys = zt$, then the ring $A((\Gamma, c))$ is a skew-field by Satz 2.4 in [27] (see also Theorem 5.7 in [18]). We will apply this construction with $A = \mathbf{Q}$, $\Gamma = \Gamma_m$ and $c = c_m$ where Γ_m and c_m are defined below.

For every $m \in \mathbf{N}$ define a groupoid $\Gamma_m = (\mathbf{Z} \times \mathbf{Z}[\frac{1}{m}], \circ)$ where $(a, u) \circ (b, v) = (a + b, u + m^a v)$. Note that Γ_m is a group with identity $(0, 0)$ and inverse $(a, u)^{-1} = (-a, \frac{-u}{m^a})$. Set $(a, b) < (a', b')$ if and only if $a < a'$ or $a = a'$ and $b < b'$ and note that $(\Gamma_m, <)$ is a totally ordered group. Clearly $\Gamma_1 = \mathbf{Z} \times \mathbf{Z}$. For every odd m define a mapping $c_m : \Gamma_m \times \Gamma_m \rightarrow \mathbf{Q}$ by $c_m((a, bm^j), (c, dm^k)) = (-1)^{ad}$ for all $a, b, c, d, j, k \in \mathbf{Z}$. If $dm^k = d'm^{k'}$, then $(-1)^{ad} = (-1)^{ad'}$ since m is odd. Hence, the mapping c_m is well-defined. If $r = (a, bm^j)$, $s = (c, dm^k)$ and $t = (e, fm^l)$, then write $p = |k| + |l| + |c|$.

We have that $c_m(r \circ s, t)c_m(r, s) = (-1)^{(a+c)f}(-1)^{ad} = (-1)^{a(d+f)}(-1)^{cf} = (-1)^{a(dm^k+fm^{l+c})m^p}(-1)^{cf} = c_m(r, s \circ t)c_m(s, t)$ and $c_m(1, r) = 1 = c_m(r, 1)$. For any $y, z \in \mathbf{Q} \setminus \{0\}$ write $x = c_m(t \circ s^{-1}, s)zy^{-1}$ to get $x(t \circ s^{-1}) * ys = zt$. By the previous paragraph, the ring $D_m := \mathbf{Q}((\Gamma_m, c_m))$ is a skew-field. Define a mapping $o : D_m \rightarrow \Gamma_m \cup \{\infty\}$ by setting $o(d) = \min(\text{supp } d \cup \{\infty\})$ and note that o is a valuation.

Theorem 9 *For every odd number $m \geq 3$ we have that $\text{ps}_{m-1}(D_m) = 1$ and $\text{ps}_{2m-2}(D_m) = \infty$.*

Proof. Write $d_1 = (1, 0) \in D_m$ and $d_2 = (0, 1) \in D_m$ to get $-1 = -(0, 0) = -(1, m) * (-(-1, -1)) = ((1, 0) * (0, 1)) * ((-1, 0) * (0, -m)) = (d_1 * d_2) * (d_1^{-1} * d_2^{-m}) \in \Pi_{m-1}(D_m)$. If $m \geq 3$, then $\text{ps}_{m-1}(D_m) = 1$.

Define a mapping $q_m : \mathbf{Z}[\frac{1}{m}] \rightarrow \mathbf{C}$ by setting $q_m(bm^j) = (-1)^{bj}e^{\frac{b\pi i}{m-1}}$ for every $b, j \in \mathbf{Z}$. One has to verify that $bm^j = b'm^{j'}$ implies that $q_m(bm^j) = q_m(b'm^{j'})$. If $b' = mb$ and $j' = j - 1$ then $q_m(b'm^{j'}) = (-1)^{mb(j-1)}e^{\frac{mb\pi i}{m-1}} = (-1)^{mb(j-1)}e^{b\pi i + \frac{b\pi i}{m-1}} = (-1)^{bj}e^{\frac{b\pi i}{m-1}} = q_m(bm^j)$. The general case follows by induction on $|j - j'|$. Every nonzero element $d \in D_m$ can be uniquely decomposed as $d = xg + d'$ with $o(d) < o(d')$. If $g = (a, u)$ then write $\sigma_m(d) = \text{sign}(x)q_m(u)$. Write $\sigma_m(0) = 0$. We claim that σ_m is a signature on D_m with exponent $2m-2$. This proves the second claim of the theorem.

Clearly, $\sigma_m(-1) = \text{sign}(-1)q_m(0) = -1$ and $\sigma_m(D_m)^{2m-2} = 1$. If $\sigma_m(d) = \sigma_m(e) = 1$ for some $d, e \in D_m$, then write $d = xg + d'$ and $e = yh + e'$ where $o(d) < o(d')$ and $o(e) < o(e')$. If $g > h$, then $\sigma_m(d + e) = \sigma_m(e) = 1$. If $g < h$, then $\sigma_m(d + e) = \sigma_m(d) = 1$. If $g = h = (a, u)$, then $\text{sign}(x)q_m(u) = \sigma_m(d) = 1 = \sigma_m(e) = \text{sign}(y)q_m(u)$. It follows that $\text{sign}(x) = \text{sign}(y) = \text{sign}(x + y)$. Hence, $\sigma_m(d + e) = \text{sign}(x + y)q_m(u) = \text{sign}(x)q_m(u) = \sigma_m(d) = 1$.

Pick any $x, y \in \mathbf{Q}$ and $g, h \in \Gamma_m$. Write $g = (a, bm^j)$, $h = (c, dm^k)$ and $l = |j| + |k| + |a|$ for some $a, b, c, d, j, k \in \mathbf{Z}$. We have that $\sigma_m(xg * yh) =$

$\sigma_m((-1)^{ad}xy(a+c, bm^j+dm^{a+k})) = (-1)^{ad} \text{sign}(xy)q_m((bm^{j+l}+dm^{a+k+l})m^{-l}) =$
 $(-1)^{ad} \text{sign}(xy) (-1)^{(bm^{j+l}+dm^{a+k+l})(-l)} e^{\frac{bm^{j+l}+dm^{a+k+l}}{m-1}\pi i} = (-1)^{ad} \text{sign}(xy)$
 $(-1)^{(b+d)l} e^{\frac{m^{j+l}-1}{m-1}b\pi i} e^{\frac{m^{a+k+l}-1}{m-1}d\pi i} e^{\frac{b\pi i}{m-1}} e^{\frac{d\pi i}{m-1}} = (-1)^{ad} \text{sign}(xy) (-1)^{(b+d)l}$
 $(-1)^{(j+l)b} (-1)^{(a+k+l)d} e^{\frac{b\pi i}{m-1}} e^{\frac{d\pi i}{m-1}} = \text{sign}(x) \text{sign}(y) (-1)^{bj+dk} e^{\frac{b\pi i}{m-1}} e^{\frac{d\pi i}{m-1}} =$
 $\sigma_m(xg)\sigma_m(yh)$. It follows that σ_m is multiplicative. Q.E.D.

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