HIGHER PRODUCT LEVELS OF DOMAINS

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1. Introduction

The n-th level of a field F was defined by J.-P. Joly in [9] as

$$\mathrm{s}_n(F) = \min\{t| -1 \in \sum_{i=1}^{t} F^n\}$$

where the convention $\min \emptyset = \infty$ is used. He also proved that there exists a function $u\colon \mathbb{N}\times\mathbb{N} \to \mathbb{N}$ such that $\mathrm{s}_{2m}(F) \le u(\mathrm{s}_2(F), 2m)$ for every field F and every $m \in \mathbb{N}$. In particular, if $\mathrm{s}_2(F)$ is finite then $\mathrm{s}_{2m}(F)$ is finite for every $m \in \mathbb{N}$.

In [8], the first author defined the n-th product level of a skew field D as

$$\mathrm{ps}_n(D) = \min\{t| -1 \in \sum_{i=1}^{t} \prod_n (D^\times)\}$$

where $D^\times$ is the multiplicative group of D and $\prod_n(D^\times)$ is the subgroup of $D^\times$ generated by all n-powers and all commutators of elements from $D^\times$. He showed that if $\mathrm{ps}_{2^k}(D)$ is finite for some k then $\mathrm{ps}_{2^k l}(D)$ is finite for every odd l. He also gave an example that this can fail for even l. The aim of this paper is to construct a function $f\colon \mathbb{N}\times\mathbb{N}\times\mathbb{N} \to \mathbb{N}$ such that $\mathrm{ps}_{2^k l}(D) \le f(\mathrm{ps}_{2^k}(D), k, l)$ for every skew field D, every k and every odd l. In the section 5 we extend our result to Ore domains.

2. A bound on higher pythagoras numbers

The construction of Joly's function u is based on the following result of Hilbert from 1909: For every $r, n \in \mathbb{N}$ there exist $\lambda_i \in \mathbb{Q}^+$ $\big(1 \le i \le L(r,n)$ where $L(r,n) := \binom{n+2r-1}{n-1}\big)$ and $a_{ij} \in \mathbb{Z}$ $(1 \le i \le L(r,n)$, $1 \le j \le n)$ such that

$$(x_1^2 + \ldots + x_n^2)^r = \sum_{i=1}^{L(r,n)} \lambda_i (a_{i1}x_1 + \ldots + a_{in}x_n)^{2r}.$$

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Hilbert used this result to prove that the Waring numbers $\mathrm{G}(n)$ are finite. If F is a field of characteristic different from 2 with $t := \mathrm{s}_2(F) < \infty$, then there exist $a_1, \ldots, a_t \in F$ such that $-1 = a_1^2 + \ldots + a_t^2$. Every element $b \in F$ can be written as a sum of $t+1$ squares:

$$b = \left(\frac{1+b}{2}\right)^2 - \left(\frac{1-b}{2}\right)^2 = \left(\frac{1+b}{2}\right)^2 + (a_1^2 + \ldots + a_t^2)\left(\frac{1-b}{2}\right)^2.$$

It follows, that for any odd l we have

$$\begin{aligned}-1 = (-1)^l = (a_1^2 + \cdots + a_t^2)^l = \sum_{j_1=1}^{L(l,t)} \lambda_{j_1} b_{j_1}^{2l} = \\ = \sum_{j_1=1}^{L(l,t)} \lambda_{j_1} (c_{j_1,1}^2 + \cdots + c_{j_1,t+1}^2)^{2l} = \sum_{j_2=1}^{L(2l,t+1)} \sum_{j_1=1}^{L(l,t)} \lambda_{j_1 j_2} b_{j_1 j_2}^{4l} = \\ = \cdots = \sum_{j_k=1}^{L(2^{k-1}l,t+1)} \cdots \sum_{j_2=1}^{L(2l,t+1)} \sum_{j_1=1}^{L(l,t)} \lambda_{j_1 j_2 \ldots j_k} b_{j_1 j_2 \ldots j_k}^{2^k l}.\end{aligned}$$

Therefore $\mathrm{s}_{2^k l}(F) \le G(2^k l) L(l,t) L(2l, t{+}1) \ldots L(2^{k-1}l, t{+}1) =: u(t, 2^k l)$.

Let $p_n(K)$ denote the n-th Pythagoras number of a commutative field K. The following bound will be used in the sequel.

Lemma 2.1. *For every even m and every field extension $\mathbb{Q}(d)/\mathbb{Q}$*

$$p_m(\mathbb{Q}(d)) \le a_m, \quad a_m := G(m) \max\{25\binom{m+7}{5}, (m+1)u(4,m)\}$$

Proof. We must distinguish three cases:

Case 1: If d is transcendent over $\mathbb{Q}$, then $\mathbb{Q}(d) \cong \mathbb{Q}(x)$. We say that a valuation ring R is real if its residue field is formally real. Let $\sum K^n$ denote the set of all finite sums of n–th powers in a field K. Recall the definition of the real holomorphy ring $\mathrm{H}(K)$ of a formally real field K (see [1]):

$$\mathrm{H}(K) := \bigcap_{\substack{R \text{ is a real} \\ \text{valuation ring} \\ \text{in } K}} R = \left\{ a \in K \,\middle|\, r \pm a \in \sum K^2 \text{ for some } r \in \mathbb{Q}^+ \right\}.$$

Let us denote

$$\begin{aligned}\mu(K) := \inf\{r \in \mathbb{N} \mid \text{ each finitely generated fractional } \mathrm{H}(K)\text{-ideal} \\ \text{can be generated by } r \text{ elements}\}\end{aligned}$$

then by [3, Proposition 2.11]

$$\mu(K) \le p_{2n}(K) \text{ for all } n \in \mathbb{N}. \tag{2.1}$$

Let $L_K^*(r,n,b)$ be the smallest $l \in \mathbb{N}$ such that

$$\left(\sum_{i=1}^{r} x_i^2\right)^n + bx_1^2\left(\sum_{i=1}^{r} x_i^2\right)^{n-1} = \sum_{j=1}^{l} \beta_j q_j(x_1,\dots,x_r)^{2n}.$$

where $b, \beta_j \in \mathbb{Q}^+$, $q_j \in \mathbb{Q}(x_1,\dots,x_r)$, and $L^*(r,n) := \min\{L_K^*(r,n,b) \mid b \in \mathbb{Q}_+^\times\}$ as in [3]. From [11] we have $p_2(\mathbb{Q}(X)) = 5$. From [3, Theorem 2.12] follows $p_m(\mathbb{Q}(X)) \le p_2(\mathbb{Q}(X))\, L^*\left(p_2(\mathbb{Q}(X))+1, \frac{m}{2}\right)\, \mathrm{G}(m)\, \mu(\mathbb{Q}(X))$. Using (2.1) the upper bound transforms into $p_m(\mathbb{Q}(x)) \le 25\, L^*\left(6, \frac{m}{2}\right)\, \mathrm{G}(m)$. By [3, Theorem 2.6] we get $L^*\left(6, \frac{m}{2}\right) \le L^*\left(6, \frac{m}{2}, m\right) = \binom{m+7}{5}$, so the upper bound becomes

$$p_m(\mathbb{Q}(x)) \le 25\binom{m+7}{5}\, \mathrm{G}(m).$$

Case 2: If d is algebraic over $\mathbb{Q}$ and $\mathbb{Q}(d)$ is formally real, then from [3, Theorem 2.12], [3, Theorem 2.6] and (2.1) follows

$$p_m(\mathbb{Q}(d)) \le p_2(\mathbb{Q}(d))^2 \binom{m + p_2(\mathbb{Q}(d)) + 2}{p_2(\mathbb{Q}(d))}\, \mathrm{G}(m).$$

The value of $p_2(\mathbb{Q}(d))$ is either 3 or 4 (see [10, Ch. 7, Examples 1.4 (3)]), therefore

$$p_m(\mathbb{Q}(d)) \le 16 \max\left\{\binom{m+x+2}{x} \mid x = 3,4\right\} G(m).$$

Case 3: If d is algebraic over $\mathbb{Q}$ and $\mathbb{Q}(d)$ is not formally real, then [1, Proposition 2.8] implies that $p_m(\mathbb{Q}(d)) \le (m+1)G(m)s_m(\mathbb{Q}(d))$. Hence $p_m(\mathbb{Q}(d)) \le (m+1)\, G(m)\, u(s_2(\mathbb{Q}(d)), m)$. From [10, Ch. 3, Examples 1.2(7)] it follows that $s_2(\mathbb{Q}(d)) \in \{1,2,4\}$, so

$$p_m(\mathbb{Q}(d)) \le (m+1)\, G(m) \max\{u(x,m) \mid x = 1,2,4\}.$$

We define a_m as the maximum of these three upper bounds. □

3. Identities

We need certain identities for the elements of the form $(1+d)^l$ in the field extension $\mathbb{Q}(d)/\mathbb{Q}$.

Proposition 3.1. *For every odd number l, every number $n = 2^k$ and every field extension $\mathbb{Q}(d)/\mathbb{Q}$ there exist $g_0,\dots,g_{n-1} \in \sum(\mathbb{Q}(d))^{nl}$ such that $(1+d)^l$ can be written in the form*

$$(1+d)^l = g_0 + g_1 d^l + \dots + g_{n-1} d^{(n-1)l}. \tag{3.1}$$

Proof. Fix a field extension $\mathbb{Q}(d)/\mathbb{Q}$. For every $j = 0, \dots, k-1$ write

$$P_j = \left\{ \sum_{\alpha=0}^{2^{k-j}-1} f_\alpha \cdot (-(1+d)^{2^j l})^\alpha \,|\, f_\alpha = \sum_{\beta=0}^{2^k-1} p_{\alpha\beta} d^{l\beta};\ p_{ij} \in \sum \mathbb{Q}(d)^{2^k l} \right\}.$$

We claim that $-1 \in P_j$ for every $j = 0, \dots, k-1$. If this is not true, then $-1 \notin P_j$ for some j. Hence, P_j is a preorder of level $2^k l$, so by [1, Lemma 1.3] and [1, Satz 2.17], there exists an ordering P'_j with exponent $2^k l$ containing P_j. By [14, 1.5.2 Theorem], the set $P''_j = \{z \in \mathbb{Q}(d) | z^l \in P'_j\}$ is an ordering with exponent 2^k. Since $d^l \in P_j$ and $-(1+d)^{2^j l} \in P_j$, we have that $d \in P''_j$ and $-(1+d)^{2^j} \in P''_j$, which implies a contradiction $-1 \in P''_j$.

Case 1: Assume that -1 is not of the form

$$-1 = \sum_{\beta=0}^{2^k-1} p_\beta d^{l\beta} \quad \text{for some } p_0, \dots, p_{2^k-1} \in \sum (\mathbb{Q}(d))^{2^k l}. \tag{3.2}$$

Consider the assertions

$$(A_j) \qquad \begin{array}{c} \text{there exist } f_{j,0}, \dots, f_{j,2^k-1} \in \sum (\mathbb{Q}(d))^{2^k l} \text{ such that} \\ (1+d)^{2^j l} = f_{j,0} + f_{j,1} d^l + \dots + f_{j,2^k-1} d^{(2^k-1)l} \end{array}$$

for $j = 0, \dots, k-1$. We will prove that A_{k-1} holds and that $A_{j+1} \Rightarrow A_j$ for every $j = k-2, \dots, 0$. Note that A_0 implies the proposition.

(A_{k-1}) Because $-1 \in P_{k-1}$, we have $-1 = f_0 - (1+d)^{2^{k-1} l} f_1$ and since -1 is not of the form (3.2), we have $f_1 \neq 0$. Thus,

$$(1+d)^{2^{k-1} l} = \frac{1+f_0}{(f_1)^{2^k l}} (f_1)^{2^k l - 1},$$

which gets the desired form after expanding $(f_1)^{2^k l-1} = \left(\sum_{\beta=0}^{2^k-1} p_{1,\beta} d^{l\beta} \right)^{2^k l - 1}$.

($A_{j+1} \Rightarrow A_j$) Assume that the assertion A_{j+1} holds. Then, $-1 \in P_j$ implies that

$$\begin{aligned} -1 &= - \sum_{\alpha=1, \alpha \text{ odd}}^{2^{k-j}-1} f_\alpha \cdot (1+d)^{2^j l \alpha} + \sum_{\alpha=0, \alpha \text{ even}}^{2^{k-j}-1} f_\alpha \cdot (1+d)^{2^j l \alpha} \\ &= -(1+d)^{2^j l} \sum_{\alpha=0}^{2^{k-j-1}-1} f_{2\alpha+1} \cdot (1+d)^{2^{j+1} l \alpha} + \sum_{\alpha=0}^{2^{k-j-1}-1} f_{2\alpha} \cdot (1+d)^{2^{j+1} l \alpha}. \end{aligned}$$

By A_{j+1} we can replace $(1+d)^{2^{j+1} l}$ by $\sum_{\beta=0}^{2^k-1} f_{j,\beta} d^{l\beta}$, where $f_{j+1,\beta} \in \sum (\mathbb{Q}(d))^{2^k l}$. Expanding, we get $-1 = (\sum_{\beta=0}^{2^k-1} q_\beta d^{l\beta}) - (1+d)^{2^j l} (\sum_{\beta=0}^{2^k-1} r_\beta d^{l\beta})$, where $q_\beta, r_\beta \in \sum(\mathbb{Q}(d))^{2^k l}$. Because -1 is not of the form (3.2), we have $\sum_{\beta=0}^{2^k-1} r_\beta d^{l\beta} \neq 0$, therefore

$$(1+d)^{2^j l} = \frac{1+\sum_{\beta=0}^{2^k-1} q_\beta d^{l\beta}}{\left(\sum_{\beta=0}^{2^k-1} r_\beta d^{l\beta}\right)^{2^k l}} \left(\sum_{\beta=0}^{2^k-1} r_\beta d^{l\beta}\right)^{2^k l-1},$$

which gets the desired form after expanding $\left(\sum_{\beta=0}^{2^k-1} r_\beta d^{l\beta}\right)^{2^k l-1}$. Hence, A_j holds.

Case 2: Now let $-1 = \sum_{\beta=0}^{2^k-1} p_\beta d^{l\beta}$ for some $p_0, \dots, p_{2^k-1} \in \sum(\mathbb{Q}(d))^{2^k l}$. We use the identity

$$m!X = \sum_{h=0}^{m-1} (-1)^{m-1-h} \binom{m-1}{h} ((X+h)^m - h^m) \tag{3.3}$$

from [9]. The identity (3.3) implies that every element of $\mathbb{Q}(d)$ is a difference of two elements from $\sum(\mathbb{Q}(d))^{2^k l}$, in particular $(1+d)^l = q_1 - q_2$, where $q_1, q_2 \in \sum(\mathbb{Q}(d))^{2^k l}$. Now $-1 = \sum_{\beta=0}^{2^k-1} p_\beta d^{l\beta}$ implies that $(1+d)^l = q_1 + \sum_{\beta=0}^{2^k-1} p_\beta q_2 d^{l\beta}$ as desired. □

4. Construction of f

From now on let D denote a skew field and $D^\times$ its group of units. Let $\prod_n(D^\times)$ denote the subgroup of $D^\times$ generated by n-th powers and multiplicative commutators and let $\sum_n(D) := \{\sum_{i=1}^m p_i | \, m \in \mathbb{N}, p_i \in \prod_n(D^\times)\}$. We shall write simply $\sum_n$ when there is no possibility of confusion. Proposition 3.1 and Lemma 2.1 imply the following

Corollary 4.1. *For every odd number l, every number $n = 2^k, k \geq 1$ and every $d \in D$ there exist $g_0, \dots, g_{n-1} \in \Sigma_{i=1}^{a_{nl}} \mathbb{Q}(d)^{nl}$ such that $(1+d)^l = g_0 + g_1 d^l + \dots + g_{n-1} d^{(n-1)l}$.*

Form Corollary 4.1, we obtain:

Lemma 4.2. *For every odd number l and every $n = 2^k, k \geq 1$ there exists a number a_{nl} such that for every skew-field D, every $s, t \in D$ and every $i = 0, \dots, n-1$ there exist elements $u_{i,0}, \dots, u_{i,n-1} \in \sum_{\alpha=1}^{a_{nl}} \prod_{nl}(D)$ such that $(s+t)^{il} = u_{i,0}s^{il} + u_{i,1}s^{(i-1)l}t^l + \dots + u_{i,i-1}s^l t^{(i-1)l} + u_{i,i}t^{il} + u_{i,i+1}s^{(n-1)l}t^{(i+1)l} + \dots + u_{i,n-1}s^{(i+1)l}t^{(n-1)l}$.*

Proof. There exists a commutator c such that $(s+t)^{il} = c(1+ts^{-1})^{il}s^{il}$. From Corollary 4.1 it follows that there exist a number $a_{nl} \in \mathbb{N}$ and

$g_1,\dots,g_{n-1}\in\sum_1^{a_{nl}}\mathbb{Q}(ts^{-1})^{nl}$ such that

$$(1+ts^{-1})^l=g_0+g_1(ts^{-1})^l+\dots+g_{n-1}(ts^{-1})^{(n-1)l}.$$

Hence, for all $i=0,\dots,n-1$

$$(1+ts^{-1})^{il}=g_{i,0}+g_{i,1}(ts^{-1})^l+\dots+g_{i,n-1}(ts^{-1})^{(n-1)l},$$

where all $g_{i,j}\in\sum_{i=1}^{a_{nl}}\mathbb{Q}(ts^{-1})^{nl}$. There exist commutators $c,c_0,\dots,c_{2^k-1}$ such that

$$\begin{gathered}(s+t)^{il}=cg_{i,0}s^{il}+cg_{i,1}(ts^{-1})^l s^{il}+\dots+cg_{i,n-1}(ts^{-1})^{(n-1)l}s^{il}=\\ =(cg_{i,0}c_0)s^{il}+(cg_{i,1}c_1)s^{(i-1)l}t^l+\dots+(cg_{i,n-1}c_{n-1}s^{-nl})s^{(i+1)l}t^{(n-1)l}=\\ =u_{i,0}s^{il}+u_{i,1}s^{(i-1)l}t^l+\dots+u_{i,i-1}s^l t^{(i-1)l}+\\ +u_{i,i}t^{il}+u_{i,i+1}s^{(n-1)l}t^{(i+1)l}+\dots+u_{i,n-1}s^{(i+1)l}t^{(n-1)l}.\end{gathered}$$

Note that $u_{i,0},\dots,u_{i,2^k-1}\in\sum_{\alpha=1}^{a_{nl}}\prod_{nl}(D)$. □

Corollary 4.3 can be considered as a noncommutative analogue of Hilbert identities.

Corollary 4.3. *Let $s_1,\dots,s_t\in\prod_{2^k}(D^\times)$ and let l be an arbitrary odd number. Then $(s_1+\dots+s_t)^l\in\sum_{2^k l}(D)$.*

Theorem 4.4. *For every number $n=2^k, k\geq 1$ and every odd number l, $\operatorname{ps}_{nl}(D)\leq(na_{nl})^{2\operatorname{ps}_n(D)}$.*

Proof. Let $t=\operatorname{ps}_n(D)$. There exist $p_1,\dots,p_t\in\prod_n(D)$ such that $-1=p_1+\ldots+p_t$. Pick a number r such that $2^{r-1}<t\leq 2^r$ and write $p_{t+1}=\ldots=p_{2^r}=0$. For every $i=0,\dots,n-1$ write $f_{i,r}$ for the smallest number such that $(s_1+\ldots+s_{2^r})^{il}\in\sum_{j=1}^{f_{i,r}}\prod_{nl}(D)$ for every $s_1,\dots,s_{2^r}\in\prod_n(D)$. Note that $f_{0,r}=1$ for every r. From Lemma 4.2, it follows that $f_{i,r}\leq a_{n,l}(f_{i,r-1}f_{0,r-1}+f_{i-1,r-1}f_{1,r-1}+\ldots+f_{0,r-1}f_{i,r-1}+f_{n-1,r-1}f_{i+1,r-1}+\ldots+f_{i+1,r-1}f_{n-1,r-1})$. Writing $F_r:=\max_i f_{i,r}$ we get $F_0=1$ and $F_r\leq na_{nl}F_{r-1}^2$. It follows that $\operatorname{ps}_{nl}(D)\leq F_r\leq(na_{n,l})^{2^r-1}\leq(na_{nl})^{2\operatorname{ps}_n(D)}$. □

Therefore $f(t,k,l):=(2^k a_{2^k l})^{2t}$ is the desired function. If there exists a $d\in D$, such that $\mathbb{Q}(d)$ is not formally real, then $ps_{nl}(D)\leq s_{nl}(\mathbb{Q}(d))\leq u(s_2(\mathbb{Q}(d)),nl)\leq u(4,nl)$ where u is the Joly's function.

5. Ore domains

Let R be a domain and $\prod_n(R)$ the set of all permuted products of n-th powers of elements from R. If $\sum(\prod_n(R))\cap-\sum(\prod_n(R))=\{0\}$, then write $\operatorname{ps}_n(R)=\infty$. Otherwise write

$$ps_n(R)=\min\{t|\ \exists p_0,\dots,p_t\in\prod_n(R)\setminus\{0\}:0=p_0+\dots+p_t\}.$$

Let $\overline{\prod_n(R)} := \{x \in R | x \prod_n(R) \cap \prod_n(R) \neq 0\}$ be the division closure of $\prod_n(R)$. From [7], we know that $\overline{\prod_n(R)} \cdot \overline{\prod_n(R)} \subseteq \overline{\prod_n(R)}$ and $\prod_n(D) = \overline{\prod_n(R)}(\overline{\prod_n(R)})^{-1} = (\overline{\prod_n(R)})^{-1}\overline{\prod_n(R)}$.

Theorem 5.1. *If R is an Ore domain and D is its skew field of fractions, then* $\mathrm{ps}_n(R) = \mathrm{ps}_n(D)$ *for every integer n.*

Proof. If $\mathrm{ps}_n(R) < \infty$, then clearly $\mathrm{ps}_n(D) \leq \mathrm{ps}_n(R) < \infty$.

If $\mathrm{ps}_n(D) < \infty$, then there exist nonzero $p_0, \ldots, p_k \in \prod_n(D)$ such that $0 = p_0 + \cdots + p_k$. Replacing $\prod_n(D)$ with $\overline{\prod_n(R)}(\overline{\prod_n(R)})^{-1}$ and clearing denominators, we may assume that $p_0, \ldots, p_k \in \prod_n(R)$. Hence $\mathrm{ps}_n(R) \leq \mathrm{ps}_n(D) < \infty$. □

Theorem 5.1 implies that Theorem 4.4 holds if D is an Ore domain.

6. Open problems

(1) Is Theorem 4.4 true for non-Ore domains?
(2) Can $f(t, k, l)$ be polynomial in t?
(3) Is Corollary 4.3 true for even l?

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