Rings whose idempotents form a multiplicative set

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Abstract

Let $R$ be a ring whose set of idempotents $E(R)$ is closed under multiplication. When $R$ has an identity 1, $E(R)$ is known to lie in the center of $R$, thus forming a Boolean algebra; moreover each idempotent $e$ induces a decomposition $eR \oplus (1-e)R$ of $R$. In this paper we consider what occurs if $R$ has no identity, in which case $E(R)$ is a possibly noncommutative variant of a generalized Boolean algebra. We explore the effects of $E(R)$ on $R$ with attention given to the decompositions of $R$ induced from decompositions of $E(R)$ as well as to the indecomposable cases.

Keywords: idempotent element, normal band, $D$-class, skew Boolean algebra.

Introduction

A standard exercise in elementary ring theory states that given a ring with identity 1 whose idempotents are closed under multiplication, all the idempotents lie in the center of the ring and thus form a Boolean algebra with operations: $e \wedge f = ef$, $e \vee f = e + f - ef$ (i.e., the circle operation $e \circ f$) and $e' = 1 - e$.

What happens when $E(R)$ is a multiplicative in a ring $R$ without identity? In this case $E(R)$ is immediately seen to be a normal band. That is, $E(R)$ is a band (a semigroup of idempotents) on which $xyzw = xzYW$ holds. For some time it has been known that any band $S$ of idempotents in a ring that is maximal with respect to being normal is likewise closed under a counter-product, $e \nabla f = (e \circ f)^2$, that is also associative and idempotent. In this case $S$ forms a noncommutative variant of a Boolean algebra called a skew Boolean algebra. Its meet is again multiplication, its
join is $\nabla$ and a relative complement is given by $e \setminus f = e - ef$. All of this applies in particular to $E(R)$ when the later is multiplicative. In this paper we explore how this condition can affect the full ring $R$.

Over the past thirty years, skew Boolean algebras have been studied by various authors. In so doing, rings with a plentitude of idempotents have been a fertile source for classes of examples as well as for concepts initially observed in those examples. In particular, the fact that $E(R)$, when multiplicative, forms a skew Boolean algebra first appeared in (Leech [12]). We review the needed details in the first section.

Seeing how $E(R)$ can affect the behavior of all of $R$ requires some assumptions of a reasonably general sort about how $E(R)$ lies in $R$. In much of this paper we assume $R$ to be idempotent-dominated in that every element $x$ is a sum $x_1 + ... + x_n$ of elements $x_i$ that are idempotent-covered, that is, $x_i = e_i x_i = x_i f_i$ for some $e_i, f_i \in E(R)$. This condition and what happens when $E(R)$ is multiplicative are discussed in Section 2.

Like (generalized) Boolean algebras, skew Boolean algebras factor at will. Thus under appropriate conditions, skew Boolean algebras factor into direct products or direct sums (as the case may be) of atomic skew Boolean algebras. The affect of such decompositions of $E(R)$ on all of $R$, and the extent to which $R$ is likewise decomposed, is studied in the third section.

Then in the final, fourth section we look at those rings whose idempotents form primitive (or atomic) skew Boolean algebras, thus providing the pieces from which the larger rings of Section 3 are built.

To our knowledge, skew Boolean algebras were first considered in the Flinders University dissertation of R. J. Bignall [1], although some relevant ideas were anticipated in the 1974 paper of Keimel and Werner [10]. These ideas were developed further in [4] by his advisor, William Cornish. Subsequent papers by Leech [12] and by Bignall and Leech [2] benefited from developments in skew lattices. Since then other papers have appeared on either skew Boolean algebras or their role in closely related topics. (See, e.g., [3], [5] and [14].) For further background on skew lattices and skew Boolean algebras see [2], [11] and [12].

To read this paper, one should have a familiarity with some fundamental concepts and results about bands, including the Green’s relations $L, R$ and $D$, rectangular bands and the Clifford-McLean Theorem. They are given in any standard text on semigroups. (See, e.g., Howie [9].)
1 When $E(R)$ is multiplicative

We begin with the content of the aforementioned standard exercise.

**Proposition 1.1** If $R$ is a ring with identity for which $E(R)$ is multiplicative, then $E(R)$ is in the center of $R$ and thus forms a Boolean algebra $(E; \lor, \land, \neg, 1, 0)$ upon setting $e \land f = ef$, $e \lor f = e + f - ef$ and $e' = 1 - e$. In addition, for all $e \in E(R)$, $eR$ and $(1 - e)R$ are ideals and $R$ is a direct sum of $eR$ and $(1 - e)R$ under the map $x \mapsto (ex, x - ex)$.

**Proof.** To begin, $e(1 - e) = 0$ for all $e \in E(R)$. Hence $ef(1 - e) = [ef(1 - e)]^2 = 0$ and thus $ef = efe$ for all $e, f$ in $E(R)$. Similarly, $fe = efe$ and thus $ef = fe$ for all $e, f$ in $E(R)$. Next, given $e \in E(R)$ and $x \in R$, consider $g = e + ex(1 - e)$. Since $eg = g$ and $ge = e$, we get $g^2 = g(eg) = (ge)g = eg = g$. Hence $e = g$ since $E(R)$ is commutative, so that $ex(1 - e) = 0$. That is, $ex = exe$ and similarly, $xe = exe$ so that $ex = xe$ and the first assertion about $E(R)$ follows. The remaining assertions are easy consequences of this. ■

To move beyond this case, first recall that a band $S$ is normal if all $e, f, g \in S$ satisfy any and hence all of the following equivalent conditions.

i) $efge = egfe$.

ii) $efgh = egfh$.

iii) $efege = egefe$.

**Lemma 1.2** When $E(R)$ is multiplicative, it is normal as a band.

**Proof.** Given $e \in E(R)$, the principal subring $eRe$ has an identity $e$ and thus $eE(R)e = E(eRe)$ is commutative by Proposition 1.1. Hence $E(R)$ satisfies (iii). ■

There is more:

**Lemma 1.3** When $E(R)$ is multiplicative, it is closed under the binary operation, $x \nabla y = x + y + yx - xyx - yxy$, that is also idempotent and associative on $E(R)$.
Proof. Given $e, f, g \in E(R)$, clearly $e \triangledown e = e$. That $(e \triangledown f)^2 = (e + f + fe - efe - ef)^2 = e \triangledown f$ is a routine calculation that uses normality repeatedly. By normality again, both $e \triangledown (f \triangledown g)$ and $(e \triangledown f) \triangledown g$ expand and eventually simplify to a common expression. ■

It seems the operation $\triangledown$ first appeared in Leech [12]. A systematic study of the operation appears in Cvetko-Vah and Leech [7].

To understand the system $(E(R); \triangledown, \cdot)$, recall that a skew lattice $(S; \vee, \wedge)$ is a set $S$ with two idempotent, associative binary operations, the meet $\wedge$ and the join $\vee$, that dualize each other in that $e \wedge f = e$ iff $e \vee f = f$ and $e \wedge f = f$ iff $e \vee f = e$. These dualities are equivalent to satisfying the absorption identities:

$$x \wedge (x \vee y) = x = (y \vee x) \wedge x$$
$$x \vee (x \wedge y) = x = (y \wedge x) \vee x.$$

A number of special conditions can apply to a skew lattice. A skew lattice is symmetric if $e \vee f = f \vee e$ if and only if $e \wedge f = f \wedge e$ for any two elements so that all instances of commutativity are unambiguous. A skew lattice is normal if its $\wedge$-reduct band $(S; \wedge)$ is a normal band. A skew lattice is distributive if $x \wedge (y \vee z) \wedge x = (x \wedge y) \vee (x \wedge z \wedge x)$ and its dual hold. Both identities are equivalent for symmetric skew lattices, but not for arbitrary skew lattices. (See Cvetko-Vah [6] and Spinks [13].) Jointly normal, symmetric and distributive skew lattices are characterized by the identities:

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$$
$$y \vee z \wedge x = (y \wedge x) \vee (z \wedge x).$$

**Proposition 1.4** If $R$ is a ring where $E(R)$ is multiplicative, then $(E(R); \triangledown, \cdot)$ is a normal, symmetric, distributive skew lattice.

**Proof.** Associativity, idempotency and normality have already been noted. Both $e(e \triangledown f) = e = (f \triangledown e)e$ and $e \triangledown (ef) = e = (fe) \triangledown e$ are easily verified. Since $e \triangledown f$ and $f \triangledown e$ differ only in the respective terms $fe$ and $ef$, symmetry is clear. Given these conditions and the ordinary distributivity of rings, both $e(f \triangledown g) = eg \triangledown eh$ and $(f \triangledown g)e = fe \triangledown ge$ hold in $E(R)$ and distributivity follows. ■

We further refine our understanding of $E(R)$, when multiplicative, as follows. A zero in a skew lattice is an element 0 such that $e \wedge 0 = 0 = 0 \wedge e$ or dually $e \vee 0 = e = 0 \vee e$ for all elements $e$. 4
Finally, a skew Boolean algebra $(S; \lor, \land, \setminus, 0)$ is a normal, symmetric skew lattice with 0 such that for each $e \in S$ the principal subalgebra

$$e \land S \land e = \{e \land s \land e \mid s \in S\} = \{s \in S \mid s \leq e\}$$

is a Boolean sublattice of $S$, in which case the relative complement $\setminus$ is defined on $S$ by letting $e \setminus f$ be the complement of $e \land f \land e$ in $e \land S \land e$. The operation $e \setminus f$ is determined also by the identities: $e = (e \land f \land e) \lor (e \setminus f)$ and $(e \land f \land e) \land (e \setminus f) = 0$. Such algebras are always distributive. Since $E(eRe)$ is a Boolean lattice for all principal subrings $eRe$ of any ring $R$ for which $E(R)$ is multiplicative, we have:

**Theorem 1.5** If $E(R)$ is multiplicative, then upon setting $e \setminus f = e - efe$ one obtains a skew Boolean algebra $(E(R); \lor, \cdot, \setminus, 0)$.

We conclude this section by briefly considering the minimal case that is needed in Section 3 and is studied in detail in the last section. Recall that an idempotent $e \succ 0$ in a ring is primitive if no idempotent $f$ exists such that $e \succ f > 0$. $E(R)$ is called primitive if all its non-zero elements are primitive. Recall that a band is rectangular when $efe = e$, or equivalently, $efg = eg$ hold. In the primitive case, we let $M(R)$ denote $E(R) \setminus \{0\}$.

**Theorem 1.6** If $E(R)$ is a primitive band, then $M(R)$ is a rectangular band under multiplication and $e \lor f = fe$ on $M(R)$.

**Proof.** Given the assumptions on $E(R)$, let $e \neq f$ in $M(R)$. Then $e \geq efe \geq 0$ so that $efe$ is either 0 or $e$. If $efe = 0$, then so are $fee, ef$ and $fe$ (by the Clifford-McLean Theorem). It follows that $e + f$ is an idempotent greater than either $e$ or $f$ and hence not primitive. Thus $efe = e$ for all $e, f$ in $M(R)$ and $M(R)$ is a rectangular band. $\blacksquare$

In this paper a primitive band is just a rectangular band $M$ with a distinct element $0 \notin M$ adjoined, so that $x0 = 0 = 0x$. Abstractly viewed, every primitive skew Boolean algebra has operations induced from a primitive band $M^0$: given $x, y \in M$, $0 \land x = 0 = x \land 0$, $0 \lor x = x \lor 0$, $x \land y = xy$, $x \lor y = yx$, $x \setminus 0 = x$ and $0 \setminus x = x \setminus y$. As a whole, primitive bands correspond bijectively to the class of primitive skew Boolean algebras.
2 Idempotent-dominated rings

How does \( E(R) \) being multiplicative affect the structure of \( R \)? To answer this one is pressed to find reasonable assumptions that in some way “bind” \( E(R) \) to all of \( R \) in order to obtain consequences on all of \( R \). To do so we begin with the following lemma.

**Lemma 2.1** Given any ring \( R \), the set

\[
\Gamma(R) = \{ x \mid ex = x = xf, \text{ for some pair } e, f \in E(R) \}
\]

is a multiplicative set containing \( E(R) \) that also has negatives: \( x \in \Gamma(R) \) implies \( -x \in \Gamma(R) \). Moreover, the set \( Q(R) \) of all finite sums of elements in \( \Gamma(R) \) is the subring in \( R \). Finally, when \( E(R) \) is also multiplicative, then \( \Gamma(R) = \bigcup \{ eRe \mid e \in E(R) \} \).

**Proof.** All statements should be clear except perhaps the last. So let \( E(R) \) be multiplicative and let \( x \in \Gamma(R) \) and \( e, f \in E(R) \) be such that \( ex = x = xf \). Set \( e' = e \nabla f \nabla e \) and \( f' = f \nabla e \nabla f \). By absorption, \( ee' = e = e'e \) and \( ff' = f = f'f \). Hence \( e'x = e'ex = x \) and \( xf' = xf' = x \). But \( e' \) and \( f' \) are \( \mathcal{D} \)-related in \( E(R) \) with their product \( e'f' \) in the same \( \mathcal{D} \)-class. In particular \( e'f'e' = e' \) and \( f'e'f' = f' \). Hence both \( (e'f')x = e'f'e'x = e'x = x \) and \( x(e'f') = xf'e'f' = xf' = x \) in the principal subring \( e'f'\mathcal{R}e'f' \).

Elements in \( \Gamma(R) \) are said to be idempotent-covered. The ring \( R \) is idempotent-covered if all its elements are thus. Rings with identity and von Neumann regular rings are trivially idempotent-covered. For such rings we have the following extension of Proposition 1.1. In its statement, \( \text{ann}(e) \) denotes the subring, \( \{ x \in R \mid ex = 0 = xe \} \).

**Theorem 2.2** Given an idempotent-covered ring \( R \), \( E(R) \) is multiplicative if and only if it lies in the center of \( R \). When this occurs, \( E(R) \) forms a generalized Boolean algebra and for each \( e \in E(R) \), both \( eR \) and \( \text{ann}(e) \) are ideals and \( R = eR \oplus \text{ann}(e) \).

**Proof.** Let \( R \) be idempotent-covered and \( E(R) \) be multiplicative. Then given \( e, f \) in \( E(R) \), Lemma 2.1 implies that \( g \in E(R) \) exists such that \( g(ef - fe) = ef - fe = (ef - fe)g \). By normality,

\[
ef - fe = g(ef - fe)g = gefg - gfg = gefg - gefg = 0.
\]
Thus $E(R)$ is commutative and so forms a generalized Boolean algebra. Given $e \in E(R)$ and $x \in R$, $f \in E(R)$ exists such that $fx = x = xf$. Let $g = e \circ f$. Since $g \geq e$ and $g \geq f$, both $e$ and $x$ lie in the subring $gRg$ which has identity $g$. Hence $e$ and $x$ commute in $gRg$ and thus also in $R$. It follows that $eR$ and $ann(e)$ are indeed ideals. The stated decomposition of $R$ is given by $x \mapsto ex + (x - ex)$. The converse implication is trivial. 

A ring $R$ is idempotent-dominated if it is generated from the set $\Gamma(R)$ of all idempotent-covered elements, or put otherwise, $R = Q(R)$. Clearly idempotent-covered rings are idempotent-dominated, but not conversely. For the remainder of this paper, nearly all rings of interest are assumed to be idempotent-dominated. But this has relevance to all rings in that any ring $R$ has a unique maximal idempotent-dominated subring $Q(R)$.

Being idempotent-dominated can have a side effect. For any idempotent-covered ring $R$, the annihilator ideal vanishes, $ann(R) = \{0\}$. If $R$ is just idempotent-dominated, however, this need not be the case when $E(R)$ is also multiplicative. Indeed, given $e,f$ in a multiplicative $E(R)$, the following small rectangular band arises

\[
\begin{array}{c c c}
 e & e & + \\
 R & ( & R \\
 \mathcal{L} & \mathcal{L} & \mathcal{L} \\
 f & f & + \\
 R & ( & R \\
 fe & ef & \\
 \end{array}
\]

and the combination $\alpha(e,f) = ef + fe - efe - fef$ can lie in $ann(R)$. More precisely:

**Theorem 2.3** When $E(R)$ is multiplicative and $R$ is idempotent-dominated, then $\alpha(e,f) \in ann(R)$ for all $e,f \in E(R)$. In general, given $e \in E(R)$ along with

\[
e_1, \ldots, e_m \in D_e \text{ and } a_1, \ldots, a_m \in eRe \text{ such that } a_1 + \cdots + a_m = 0;
\]

\[
f_1, \ldots, f_n \in D_e \text{ and } b_1, \ldots, b_n \in eRe \text{ such that } b_1 + \cdots + b_n = 0.
\]

Then

\[
\pi = (e_1a_1 + \cdots + e_ma_m)(b_1f_1 + \cdots + b_nf_n) \in ann(R)
\]

as do all sums of such products. When $ann(R) = \{0\}$, all such expressions vanish in $R$. 

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Proof. Observe that $\alpha(e, f) = (efe - fe)(ef - efe)$ is included in the general situation. Given the $e_i$s and $f_j$s and $a_i$s and $b_j$s as described, take $g \in E(R)$. Then by normality

$$g(e_1a_1 + e_2a_2 + \cdots + e_ma_m) = (ge_1e)a_1 + (ge_2e)a_2 + \cdots + (ge_m)e)a_m$$

$$= (gee_1e)a_1 + (gee_2e)a_2 + \cdots + (gee_m)e)a_m$$

$$= (ge)a_1 + (ge)a_2 + \cdots + (ge)a_m$$

$$= (ge)(a_1 + a_2 + \cdots + a_m) = 0.$$ 

It follows that $g\pi = 0$ and likewise $\pi g = 0$. Thus $\pi$ annihilates all idempotents and hence all elements in $\Gamma(R)$. Since $R$ is idempotent-dominated, $\pi$ lies in $\text{ann}(R)$. The assertion about sums of such products is clear, as is the final assertion. ■

While $\text{ann}(R)$ need not vanish, this is not the full story.

Lemma 2.4 If $R$ is idempotent-dominated, then $\text{ann}[R/\text{ann}(R)] = \{0\}$.

Proof. In general, $\text{ann}(R) = \{x \in R \mid xy = 0 = yx \text{ for all } y \in R\}$ and if $\pi : R \to R/\text{ann}(R)$ is the induced homomorphism, then $\text{ann}(R) \subseteq \pi^{-1}\{\text{ann}[R/\text{ann}(R)]\}$ with

$$\pi^{-1}\{\text{ann}[R/\text{ann}(R)]\} = \{x \in R \mid xyz = 0 = yzx = yxz \text{ for all } y, z \in R\}.$$ 

But if $R$ is idempotent-dominated, then $\text{ann}(R) = \{x \in R \mid x_e = 0 = ex \text{ for all } e \in E(R)\}$. Hence $\pi^{-1}\{\text{ann}[R/\text{ann}(R)]\} \subseteq \text{ann}(R)$ and equality follows and $\text{ann}[R/\text{ann}(R)]$ vanishes in $R/\text{ann}(R)$. ■

Lemma 2.5 If $R$ is a ring and $\pi : R \to R/\text{ann}(R)$ is the induced homomorphism $\pi(x) = x + \text{ann}(R)$, then $\pi$ restricts to a bijection $\pi^E : E(R) \to E(R/\text{ann}(R))$. If $E(R)$ is multiplicative, so is $E(R/\text{ann}(R))$ and $\pi^E$ is an isomorphism of skew Boolean algebras.

Proof. Clearly $\pi^E$ is a well-defined map between the stated sets. If $\pi(e) = \pi(f)$ for $e, f \in E(R)$, then $e = f + a$ for some $a \in \text{ann}(R)$ and squaring gives $e = f$. Thus $\pi^E$ is at least injective. Given $x + \text{ann}(R) \in E(R/\text{ann}(R))$, $x^2 = x + a$ for some $a \in \text{ann}(R)$ and hence $x^4 = x^2$ so that $x^2 \in E(R)$ and $\pi^E$ is bijective. Since $\pi$ is a ring homomorphism, the rest of the lemma follows. ■

Since $eRe \cap \text{ann}(R) = \{0\}$ for all idempotents $e$ in any ring $R$, we have:
Theorem 2.6 If \( R \) is idempotent-dominated and \( E(R) \) is multiplicative, then \( R/\text{ann}(R) \) has both properties and \( \text{ann}(R/\text{ann}(R)) = \{0\} \). The natural epimorphism \( \pi : R \to R/\text{ann}(R) \) thus induces a skew Boolean algebra isomorphism \( \pi^E : E(R) \cong E(R/\text{ann}(R)) \) and ring isomorphisms between corresponding principal subrings, \( \pi^e : eRe \cong \pi(e)(R/\text{ann}(R))\pi(e) \).

Thus \( R/\text{ann}(R) \) is a generally cleaner, trimmer version of \( R \), sharing many of its characteristics, but without the presence of a nonvanishing annihilator ideal. Since \( e\nabla f - e \circ f = fe + ef - efe - fef = 0 \) in \( R/\text{ann}(R) \), one has:

Corollary 2.7 If \( E(R) \) is multiplicative and \( R \) is idempotent-dominated, then \( e\nabla f = e \circ f \) in \( E(R/\text{ann}(R)) \). In particular, if \( \text{ann}(R) = \{0\} \) then \( e\nabla f = e \circ f \) on \( E(R) \).

It is easily seen that the latter case occurs when \( E(R) \) is either left-handed \( (e \land f \land e = e \land f, \) and equivalently, \( e \lor f \lor e = f \lor e \)) or right-handed \( (e \land f \land e = f \land e, \) and equivalently, \( e \lor f \lor e = e \lor f \)).

Idempotent-dominated rings include all idempotent-generated rings. Any ring \( R \) contains a maximal idempotent-generated subring, namely the subring \( Q_0(R) \) generated from \( E(R) \). If \( E(R) \) is a band, then the identity \( xyzw = xzyw \) extends via distribution to all \( Q_0(R) \). This leads us to a class of rings for which \( E(R) \) is always multiplicative.

A ring is weakly commutative if it satisfies the identity \( xyzw = xzyw \). A weakly commutative ring \( R \) has a nil radical \( \mathcal{N}_R \) that consists of all nilpotent elements. \( \mathcal{N}_R \) is indeed an ideal and \( R/\mathcal{N}_R \) is always commutative with vanishing nil radical.

Theorem 2.8 If \( R \) is a weakly commutative ring, then \( E(R) \) is multiplicative and for each \( e \in E(R) \) the subring \( eRe \) is commutative. The converse also holds when \( R \) is idempotent-dominated. Finally, for any ring, \( E(R) \) is multiplicative if and only if \( Q_0(R) \) is weakly commutative.

Proof. Given \( e, f \in E(R) \), \( (ef)^2 = efe = eef = ef \). Also, given \( exe, eye \) in \( eRe \), we have

\[(exe)(eye) = e(exe)(eye)e = e(eye)(exe)e = (eye)(exe).\]

The first implication follows. Conversely, assume that \( R \) is idempotent-dominated with \( E(R) \) being multiplicative and each subring \( eRe \), for \( e \in E(R) \), being commutative. Let \( eae, fbf, geg, hdh \) in \( \Gamma(R) \) be given with \( e, f, g, h \in E(R) \). Since \( E(R) \) is multiplicative, as in the proof of Lemma 2.1, \( e', f', g', h' \) in \( E(R) \) exist such that \( e' \geq e, f' \geq f, g' \geq g, h' \geq h \) with \( e', f', g', h' \) being \( D \)-related.
Thus we may assume at the outset that \( e, f, g \) and \( h \) are \( D \)-related. This plus the assumption that each \( eRe \) be commutative gives

\[
\]

holding in \( \Gamma(R) \). Distribution extends the identity \( xyzw = xzyw \) from \( \Gamma(R) \) to all of \( R \). If we just assume \( E(R) \) is a band, then weak commutativity extends via distribution from \( E(R) \) to the generated subring \( Q_0(R) \). The converse is clear.

We also have: (1) Given a commutative ring \( A \) and a normal band \( S \), the semigroup ring \( A[S] \) is weakly commutative; it is idempotent-dominated when \( A \) is also idempotent-covered. This makes idempotent-dominated rings with multiplicatively closed idempotents easy to find. Indeed all examples in this paper happen to be weakly commutative. (2) Any normal multiplicative band \( S \) inside a ring generates a weakly commutative subring.

In the next section we consider decompositions of \( E(R) \) when the latter is not commutative and study the extent to which such decompositions induce corresponding decompositions of the ring.

3 Decomposing \( E(R) \) and \( R \)

Just as every (generalized) Boolean algebra is a subdirect product of copies of the primitive Boolean algebra \( 2 \), every skew Boolean algebra is a subdirect product of primitive algebras, thanks largely to the next easily verified result. (See [12] Lemma 1.11.)

**Theorem 3.1** Given a \( D \)-class \( A \) of a skew Boolean algebra \( S \), set

\[
S_1 = \{ e \in S \mid e \wedge a \wedge e = e \text{ for some (and hence all) } a \in A \} \quad \text{and} \quad S_2 = \{ f \in S \mid f \wedge a = a \wedge f = 0 \text{ for all } a \in A \}.
\]

Then both \( S_1 \) and \( S_2 \) are subalgebras of \( S \), elements of \( S_1 \) commute with elements of \( S_2 \) and the map \( \mu : S_1 \times S_2 \to S \) defined by \( \mu(e_1, e_2) = e_1 \vee e_2 \) is an isomorphism of skew Boolean algebras. The inverse isomorphism is given by \( \mu^{-1}(e) = (e \wedge a \wedge e, e \wedge a \wedge e) \).

Described otherwise, \( S_1 \) is the union of the \( D \)-class \( A \) and all lower \( D \)-classes in the generalized Boolean lattice \( S/D \), while \( S_2 \) consists of all \( D \)-classes \( B \) that meet \( A \) and its lower \( D \)-classes at
\{0\} in S/D. In fact S₁ and S₂ are ideals of S where by an **ideal** of a skew lattice S we mean any subset I such that \( e \lor f \in I \) for all \( e, f \in I \), and both \( e \land g, g \land e \in I \) for all \( e \in I \) and all \( g \in S \). S₁ corresponds to the principal ideal in S/D determined by the element \( A \) of S/D while S₂ corresponds to the ideal in S/D consisting of all elements of S/D that meet \( A \) at 0.

How does this play out in the multiplicative set E(R) and its context R? Before stating our next theorem, here is a consequence of the Clifford-McLean Theorem.

**Lemma 3.2** If \( E(R) \) is multiplicative and \( ef = 0 \) then \( fe = 0 \) also and \( e \lor f = e + f \).

**Theorem 3.3** Given an idempotent-dominated ring R with E(R) a multiplicative set, let I and J be ideals of \( E(R) \) such that each element \( e \in E(R) \) is uniquely \( f + g \) for some \( f \in I \) and \( g \in J \). Let \( \Gamma_I \) and \( \Gamma_J \) be multiplicative sets defined by:

\[
\Gamma_I = \{ x \in R \mid x = ef \text{ for some } e, f \in I \} \quad \text{and} \quad \Gamma_J = \{ x \in R \mid x = ef \text{ for some } e, f \in J \}.
\]

If \( Q_I \) and \( Q_J \) are the subrings generated from \( \Gamma_I \) and \( \Gamma_J \) respectively under addition, then

i) \( R = Q_I + Q_J \) and both \( Q_I \) and \( Q_J \) are ideals of R.

ii) \( Q_I Q_J = \{ xy \mid x \in Q_I, y \in Q_J \} = \{0\} = Q_I Q_I \).

iii) In general, \( Q_I \cap Q_J \subseteq \text{ann}(R) \) with \( Q_I \cap Q_J \) often exceeding \( \{0\} \).

**Proof.** From \( e = e + 0 = 0 + e \) one first has \( I \cap J = \{0\} = \text{ann}(R) \), then \( \Gamma_I \Gamma_J = \Gamma_J \Gamma_I = \{0\} \) so that (ii) holds. Given \( x \in \Gamma(R) \), \( e \in I \) and \( f \in J \) exist such that \( x = (e + f)x(e + f) = exe + fxf + ef + fxe \). Clearly, \( exe + fxf \in Q_I + Q_J \). If \( exe + fxe \in Q_I \cap Q_J \), then \( \Gamma(R) \subseteq Q_I + Q_J \) and \( R = Q_I + Q_J \). So observe that \( e + exf \in I \). Indeed, \( (e + exf)^2 = e(e + exf) = e + exf \in E(R) \) and thus lies in the ideal \( I \), so that \( exf = (e + exf) - e \) lies in \( Q_I \). Likewise \( exf \in Q_J \) so that \( exf \in Q_I \cap Q_J \). Similarly \( fxe \in Q_I \cap Q_J \) and (i) follows. The inclusion of (iii) follows from (i) and (ii). That it can be strict is seen in the next example. ■

**Example 1** R is the matrix ring

\[
\begin{bmatrix}
0 & a & b & p \\
0 & u & 0 & c \\
0 & 0 & v & d \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

where \( F \) is any given
In general, an isomorphism \( \sigma \) is as follows:

\[
\begin{bmatrix}
0 & m & n & mp + nq \\
0 & 1 & 0 & p \\
0 & 0 & 1 & q \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

where \( a, b, c, \ldots, q \) vary freely over \( F \). If \( I \) and \( J \) are the primitive skew lattice ideals determined by the middle left and middle right \( D \)-classes, then \( Q_I, Q_J \) and \( \text{ann}(R) = \text{ann}(Q_I \cap J) \) are respectively

\[
\begin{bmatrix}
0 & a & 0 & v \\
0 & u & 0 & c \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix},
\begin{bmatrix}
0 & 0 & b & v \\
0 & 0 & 0 & 0 \\
0 & 0 & u & d \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

and

\[
\begin{bmatrix}
0 & 0 & 0 & x \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

Theorem 3.3 can be applied repeatedly. Doing so leads one to a modification of a direct sum decomposition of a ring. We describe this modification in fuller generality, independent of the special assumptions on \( R \) and \( E(R) \) in this paper.

To begin, given a ring \( R \) and a set of subrings \( \{Q_i | i \in I\} \) define \( \sigma : \bigoplus_{i \in I} Q_i \to R \) by \( \sigma((x_i | i \in I)) = \sum x_i. \) (Here \( \bigoplus_{i \in I} Q_i \) is the direct sum of the \( Q_i \), the subring of the direct product \( \Pi Q_i \) consisting of all \( I \)-tuples for which all but finitely many components are 0.) \( \sigma \) preserves addition. It preserves multiplication if \( (\sum x_i)(\sum y_i) = \sum x_i y_i \) for all \( (x_i | i \in I), (y_i | i \in I) \) in \( \sum_{i \in I} Q_i \). An examination of cases shows this to be equivalent to \( x_i y_j = 0 \) for all \( i \neq j \). If in addition the image \( \sum_{i \in I} Q_i = R \), then each \( Q_i \) must be an ideal of \( R \). When \( R \) is a sum \( \sum_{i \in I} Q_i \) of ideals \( Q_i \), where \( Q_i Q_j = \{0\} \) for all \( i \neq j \), we say that \( R \) is the orthosum of the \( Q_i \). If \( \sigma \) is as above, it is easily seen that \( \sigma((\sum_{i \in I} \text{ann}(Q_i))) = \text{ann}(R) \) and \( \sigma^{-1}[\text{ann}(R)] = \sum_{i \in I} \text{ann}(Q_i). \) From this we get:

**Proposition 3.4** Let \( R \) be an orthosum of ideals \( Q_i \) with \( \sigma : \bigoplus_{i \in I} Q_i \to R \) the sum epimorphism. \( \sigma \) is an isomorphism if and only if it restricts to an isomorphism of \( \sum_{i \in I} \text{ann}(Q_i) \) with \( \text{ann}(R) \).

In general, an isomorphism \( \sigma' : \sum_{i \in I} Q_i/\text{ann}(Q_i) \cong R/\text{ann}(R) \) is defined by \( \sigma'(\langle x_i + \text{ann}(Q_i) | i \in I \rangle) = \sum x_i + \text{ann}(R) \).

We also have:
**Proposition 3.5** If $R$ is the orthosum of ideals $Q_i$ and $\sigma : \bigoplus_{i \in I} Q_i \to R$ is the sum epimorphism, then the restriction $\sigma^E : \bigoplus_{i \in I} E(Q_i) \to E(R)$ is a bijection of sets that is an isomorphism of skew Boolean algebras, whenever $E(R)$ is multiplicative. ($\bigoplus_{i \in I} E(Q_i)$ denotes the subset of $\bigoplus_{i \in I} Q_i$ where all components lie in the various $E(Q_i)$. It equals $E(\bigoplus_{i \in I} Q_i)$ and is multiplicative when each $E(Q_i)$ is thus.)

**Proof.** Since any sum of orthogonal idempotents is also idempotent, $\sigma^E$ is at least well-defined. Let $e \in E(R)$ equal $x_1 + \ldots + x_r$, with each $x_i \in Q_i$. Then $e = e^2 = x_1^2 + \ldots + x_r^2$ and thus for each $i \leq r$, $x_i^2 = x_i + a_i$ where $a_i \in \text{ann}(R)$. Once again one has $x_i^4 = x_i^2$ so that each $x_i^2 \in E(Q_i)$ and $\sigma^E$ is at least surjective. Let $e \in E(R)$ be represented as both $e_1 + \ldots + e_r$ and $f_1 + \ldots + f_r$ where $e_i, f_i \in E(Q_i)$. (By letting some of the values be 0 we may assume a common indexing.) Again each $e_i$ and $f_i$ differ by an element of $\text{ann}(R)$ and hence are equal. Thus $\sigma^E$ is a bijection. Since $\sigma$ is a ring homomorphism, the final assertions are clear. ■

A ring $R$ satisfies the **descending chain condition on idempotents** if any sequence $e_1 \geq e_2 \geq e_3 \geq \ldots$ in $E(R)$ eventually stabilizes: $e_n = e_{n+1} = \ldots$ The **ascending chain condition on idempotents** is defined in dual fashion. The latter implies the former since a descending chain $e_1 \geq e_2 \geq e_3 \geq \ldots$ in $E(R)$ induces a corresponding ascending chain of idempotents $e_1 - e_2 \leq e_1 - e_3 \leq \ldots$ with both stabilizing, if they do, simultaneously. Returning to the specific assumptions of the paper we have:

**Theorem 3.6** Let ring $R$ be idempotent-dominated and let $E(R)$ be multiplicative and satisfy the descending chain condition on idempotents. Then

i) $E(R) = \bigoplus_{i \in I} P_i$ where the $P_i$ are the primitive bands determined by the minimal nonzero $\mathcal{D}$-classes of $E(R)$.

ii) $R$ is an orthosum $\sum_{i \in I} Q_i$ of ideals $Q_i$ where each $Q_i$ is generated from the multiplicative set $\Gamma(P_i) = \{r \in R \mid er = r = re \text{ for some } e \in P_i\}$.

iii) $\sum_{i \in I} Q_i/\text{ann}(Q_i) \cong R/\text{ann}(R)$, with all annihilators vanishing.

iv) As skew Boolean algebras, $E(R) \cong E[R/\text{ann}(R)]$ and each $P_i \cong E[Q_i/\text{ann}(Q_i)]$. 

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Proof. Both (iii) and (iv) follow from (i) and (ii) and Results 2.6, 3.4 and 3.5. The chain condition plus the normality of \( E(R) \) guarantee that each idempotent \( e > 0 \) is a unique sum of primitive idempotents, \( e = p_1 + \cdots + p_n \), where each \( p_i \) covers 0 in \( (E(R), \leq) \) and \( p_1, \ldots, p_n \) are mutually orthogonal, coming from different primitive subalgebras of \( E(R) \). Assertion (i) follows from this.

Next let \( x \in \Gamma(R) \) be given. By Lemma 2.1,

\[
x = exe = (p_1 + \cdots + p_n)x(p_1 + \cdots + p_n) = \sum p_i xp_j
\]

for the appropriate primitive idempotents. We show that for \( i \neq j \), \( p_i xp_j = 0 \). Indeed \( pxq = 0 \) for any pair of idempotents \( p, q \) such that \( pq = 0 \) (and hence \( qp = 0 \) by the Clifford-McLean Theorem). First observe that \( p + px - pxp \) is idempotent. Hence \( (p + px - pxp)q \) is also. But since \( q(p + px - pxp) = 0 \), we also have \( (p + px - pxp)q = 0 \). Distributing \( q \) through the parentheses in \( (p + px - pxp)q \) leaves \( pxq = 0 \). Getting back to \( p_1, \ldots, p_n \) we thus have \( x = exe = \sum p_i xp_i \). Thus \( \Gamma(R) = \sum_{i \in I}(P_i) \) with \( \Gamma(P_i) \cap \Gamma(P_j) = \Gamma(P_i) \Gamma(P_j) = \{0\} \) for all \( i \neq j \). That is, \( \Gamma(R) = \sum_{i \in I}^{\oplus} \Gamma(P_i) \) in the sense described in 3.5 above for \( E(R) \). Hence \( \sigma > \sum_{i \in I}^{\oplus} Q_i \rightarrow R \) is a homomorphism and \( R \) is an orthosum of ideals \( Q_i \).

Corollary 3.7 The conclusions of Theorem 3.6 hold if we assume the ascending chain condition on \( E(R) \). In this case the number of summands \( Q_i \) is finite, equaling the number of atoms in \( E(R)/D \). Conversely, when only finitely many summands \( Q_i \) exist, \( E(R) \) satisfies the ascending chain condition.

Proof. The DCC must hold on \( E(R) \) also. The ACC also prevents \( E(R) \) from having an infinite number of 0-minimal \( D \)-classes and thus \( R \) from having an infinite number of ortho-summands \( Q_i \). The converse is clear.

When the descending chain condition holds on \( E(R) \), the question of when \( E(R) \) is multiplicative can be reduced as follows. To begin, let \( M(R) \) denote the set of primitive idempotents of \( E(R) \) and denote the union \( M(R) \cup \{0\} \) by \( M_0(R) \). If \( E(R) \) satisfies this chain condition, then for any \( e > 0 \) in \( E(R) \) an \( m \in M(R) \) exists such that \( e \geq m \); moreover such an \( m \) is unique in its \( D \)-class with respect to \( e \). A result of Dolžan [8] for a case where \( E(R) \) is commutative is easily extened:

Theorem 3.8 If \( E(R) \) satisfies the descending chain condition, then \( E(R) \) is multiplicative if and only if \( M_0(R) \) is multiplicative.
Proof. Let \( M_0(R) \) be multiplicative and let \( S \) consist of all possible finite sums \( \sum e_i \) of elements from distinct \( D \)-classes in \( M_0(R) \). Since all products \( ef \) from distinct \( D \)-classes in \( M_0(R) \) equal 0, \( S \) is also a set of idempotents that is closed under multiplication. Given \( e > 0 \) in \( E(R) \), let \( m_1 \in M(R) \) be such that \( e \geq m_1 \). If \( e = m_1 \), we stop. Otherwise we have \( e > e - m_1 > m_2 \) in \( M(R) \) with \( m_2 \) orthogonal to \( m_1 \) in \( E(R) \). If \( e - m_1 = m_2 \), then \( e = m_1 + m_2 \) with \( m_1 \perp m_2 \) in \( E(R) \). Otherwise, \( e - m_1 - m_2 > 0 \) some \( m_3 \) in \( M(R) \). The descending chain condition insures this process eventually halts to give \( e = m_1 + \cdots + m_n \) and thus \( E(R) = S \). The converse is trivial. \( \blacksquare \)

4 When \( E(R) \) is a primitive band

We begin with an important sub-case that is suggestive of what occurs in general. So let \( A \) be a ring with identity 1 such that \( E(A) = \{0, 1\} \) and let \( S \) be a rectangular band. Consider the semigroup ring \( A[S] \) consisting of all formal sums \( \sigma = a_1 \cdot s_1 + \cdots + a_n \cdot s_n \) for \( a_i \in A \) and \( s_i \in S \) and subject to the rules: \( \sum a_i \cdot s_i + \sum b_i \cdot s_i = \sum (a_i + b_i) \cdot s_i \), \( (a \cdot s)(b \cdot t) = (ab) \cdot (st) \) and \( 0 \cdot s = 0 \). Under addition \( A[S] \) is a free \( A \)-module over set of generators \( S \). If \( s \in S \) is identified with \( 1 \cdot s \in A[S] \) then \( S \) is a multiplicative band sitting in \( A[S] \). But is it a maximal rectangular band in \( A[S] \)? A general answer follows the next lemma.

Lemma 4.1 Given \( A \) and \( S \) as above and \( s \in S \), then in \( A[S] \) we have:
\[
L_S = \{ \sum a_i \cdot s_i \mid \sum a_i = 1 \text{ in } A \& s_i L_S \text{ in } S \} \text{ is the set of idempotents } L\text{-related to } s; \text{ and}
\[
R_S = \{ \sum b_j \cdot t_j \mid \sum b_j = 1 \text{ in } A \& t_j R_S \text{ in } S \} \text{ is the set of idempotents } R\text{-related to } s.
\]
Moreover \( M_S = L_S R_S = \{ \sum (a_i b_j) \cdot (s_i t_j) \mid \sum a_i = 1 = \sum b_j \text{ in } A \text{ with } s_i L_S R t_j \in S \} \) is the maximal rectangular band in \( A[S] \) containing \( s \) and hence all of \( S \).

Proof. Indeed, given \( x = \sum a_i \cdot s_i \) where \( \sum a_i = 1 \) in \( A \) and \( s_i L_S \) in \( S \), one easily sees that \( xs = x \), \( sx = s \) and thus \( x^2 = xsx = x \). Conversely, if \( \sum a_i \cdot s_i \) is an idempotent that is \( L \)-related to \( s \), then \( s = s(\sum a_i \cdot s_i)s = \sum a_i \cdot (ss_i)s) = \sum a_i \cdot s = (\sum a_i) \cdot s \) so that \( \sum a_i = 1 \). Moreover, \( \sum a_i \cdot s_i = (\sum a_i \cdot s_i)s = \sum a_i \cdot (s_i s) \). Since all \( s_i s \) are \( L \)-related to \( s \) in \( S \), by uniqueness of the representation, all \( s_i \)'s in \( \sum a_i \cdot s_i \) are \( L \)-related to \( s \) in \( S \). Thus \( L_S \) and \( R_S \) are indeed as described. Finally, for any rectangular band \( M \), given \( e \in M \) one has \( R_e = eM, L_e = Me \) in \( M \) and thus \( M = MeM = L_e R_e \). Thus we need only show that under multiplication \( M_S \) is a rectangular band. This follows from the easily verified \( (\sum a_i \cdot s_i)(\sum b_j \cdot t_j)(\sum c_i \cdot s_i)(\sum d_j \cdot t_j) = \)
Theorem 4.2 The ring $S$ has an inflated $R$-class with $\geq 3$ distinct elements $a, b, c$, then $a - b + c$ is a new element in the inflated class. If $a \in A \setminus \{0, 1\}$ and $a \neq b$ in an $L$- or $R$-class of $S$, then $aa + (1 - \alpha)b$ is a new element in the inflated class. Returning to $E(A[S])$:

**Theorem 4.2** The ring $A[S]$ is idempotent-dominated and $E(A[S]) = M_S \cup \{0\}$ is a primitive band.

**Proof.** Since $A[S] = \sum_{s \in S} A \cdot s$ and $A \cdot s = sA[S]s$, the first assertion holds. We show that any $f = f^2 \neq 0$ in $A[S]$ lies in $M_S$. Let $f = \sum a_i \cdot s_i$. Given $s$ in $S$ we have $fsf = (\sum a_i \cdot s_i)s(\sum a_j \cdot s_j) = \sum a_i a_j \cdot s_i s_j s_j = \sum a_i a_j \cdot s_i s_j = (\sum a_i \cdot s_i)(\sum a_j \cdot s_j) = ff = f$ and

$$sfs = s(\sum a_i \cdot s_i)s = \sum a_i \cdot (ss_i s) = \sum a_i \cdot s = (\sum a_i) \cdot s.$$ 

Since $sfs$ must be idempotent, $\sum a_i$ is idempotent in $A$. By assumption $E(A) = \{0, 1\}$. If $\sum a_i = 0$, so that $sfs = 0$, then $f = fsfsf = 0$ contradicting $f \neq 0$. This leaves $\sum a_i = 1$ which gives $f = fsf = (\sum a_i \cdot s_i)(\sum a_j \cdot s_j) = (\sum a_i \cdot s_i)(\sum a_j \cdot ss_j) \in L_S R_S = M_S$. ■

In the general primitive case, letting $M(R)$ denote $E(R) \setminus \{0\}$, we have:

**Theorem 4.3** If ring $R$ is idempotent-dominated with $E(R)$ a primitive band, then

i) $\Gamma(R) = \bigcup_{e \in M(R)} eRe$ with $(eRe)(fRf) = eRe$ for all $e, f \in M(R)$.

ii) Given $e, f \in M(R)$, the map $x \mapsto xf$ is a ring isomorphism of $eRe$ with $fRf$. Thus every element $y$ in $fRf$ is uniquely expressed as $xf$ for some $x$ in $eRe$.  

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iii) Given $y = fx f \in fRf$ and $z = gx'g \in gRg$, $yz = (fg)(xx')(fg) \in fgRfg$.

iv) $R = \sum_{e \in M} eRe$ with all summands being isomorphic subrings.

**Proof.** Lemma 2.1 gives $\Gamma(R) = \bigcup_{e \in M(R)} eRe$. The second equality in (i) follows from $(eRe)(fRf) = efeReRfe \subseteq efRef = (efRe)f \subseteq (eRe)(fRf)$. To see (ii), note that $x \mapsto fx f$ is at least an additive homomorphism from $eRe$ to $fRf$. Let $x, y \in eRe$ be given. Then $f(xy)f = f(xey)f = f(xefey)f = (fxf)(fyf)$ so that $x \mapsto fx f$ is a homomorphism of rings. Indeed it is an isomorphism with inverse isomorphism given by $y \mapsto eye$ from $fRf$ back to $eRe$. (iii) follows from $yz = (fx f)(gx'g) = (fxef)(feyg) = (fxf)(fyf)$ so that the final assertion is now clear.

This brings us to the following canonical co-representation:

**Theorem 4.4** Let $R$ be an idempotent-dominated ring with $E(R)$ a primitive band. If $A = e_0Re_0$ for a fixed $e_0 \in M = M(R)$, then $A$ is a ring with identity $e_0$ such that $E(A) = \{0, e_0\}$; moreover the map $\beta : A[M] \rightarrow R$ defined by $\beta(\sum a_i \cdot e_i) = \sum e_0 a_i e_0$ is a surjective ring homomorphism that is bijective between the standard copy of $M$ in $A[M]$ and $M$ in $R$, and also between subrings $A \cdot e$ in $A[M]$ and their images $eAe$ in $R$.

In general, if $\sigma : A[M] \rightarrow R'$ is a surjective ring homomorphism that is bijective between any (and hence all) $A \cdot e$ in $A[M]$ and its image in $R'$, then the image ring $R'$ must also be idempotent-dominated with a primitive band of idempotents.

**Proof.** By Theorem 4.3 (ii) and (iv), $R = \sum_{e \in M} eAe$ with all summands being isomorphic to $A$. Thus $\beta$ is an additive epimorphism that is bijective where stated. That it preserves multiplication follows from Theorem 4.3(iii) and distribution. The last claim follows from a more general assertion in the Theorem 4.6 below.

We turn to the general question of when a homomorphic image of an idempotent-dominated ring $R$ for which $E(R)$ is a primitive band must have the same properties. Our answer is framed in terms of ideals and induced image rings. But first a lemma:

**Lemma 4.5** Given an idempotent $e$ and an ideal $I$ of a ring $R$, $eRe \cap I = eIe$ is an ideal of $eRe$ and the image of $eRe$ in $R/I$ is isomorphic to $eRe/eIe$. Conversely, all ideals of $eRe$ arise in this fashion.
**Theorem 4.6** Given an idempotent-dominated ring $R$ where $E(R)$ is a primitive band and given a proper ideal $I$ of $R$, then $R/I$ is idempotent-dominated also, while $E(R/I)$ is a primitive band if and only if $|E(eRe/eIe)| = 2$ for some and hence all $e \in M(R)$.

**Proof.** In general, homomorphic images of idempotent-dominated rings are idempotent-dominated also. Since all $eRe$ are isomorphic, with each $eRe/eIe$ isomorphic to the image of $eRe$ in $R/I$, the “only if” part follows from Theorem 4.3 (ii). So let $E(eRe/eIe) = 2$ for all $e$ in $M = M(R)$. In particular each $E(eRe/eIe) = \{e + eIe, 0 + eIe\}$. Note that $M \cap I = \emptyset$ since otherwise $M \subseteq I$ so that $I = R$, because $R$ is idempotent-dominated. Note also that $xey = xy$ for all $x, y \in R$ and all $e \in M$, since $R$ is idempotent-dominated and $M$ is a rectangular band.

Let $x \in R$ be such that $x + I \in E(R/I)$. Consider $exe$ for some $e \in M$. $(exe)^2 = ex^2e$ in $R$ so that $exe + I \in E(R/I)$ also. But $exe$ lies in the image of $eRe$ in $R/I$. By our assumption on $eRe/eIe$, either $exe + I = e + I$ or $exe \in I$. If the latter, then $x^3 = xexe \in I$ and $x + I = (x + I)^3 = 0 + I$. Otherwise, $(e + I)(x + I)(e + I) = exe + I = e + I \neq 0 + I$ for all $e \in M$. Given other nonzero idempotents $y + I, z + I$ in $E(R/I)$,

$$[(x + I)(y + I)]^2 = (x + I)(e + I)(y + I)(e + I)(x + I)(e + I)(y + I) = (x + I)(e + I)(y + I) = (x + I)(y + I)$$

and similarly, $(x + I)(y + I)(z + I)$ reduces to $(x + I)(z + I)$. Thus the nonzero elements in $E(R/I)$ form a rectangular band under multiplication. □

This criterion is clearly satisfied by those ideals $I$ for which $eRe \cap I = \{0\}$, which is essentially the case in Theorem 4.4.

In the remainder of this paper we will return to the description of $\Gamma(R)$ given in Theorem 4.3. Viewing $A = e_0Re_0$ for $e_0$ in $M(R)$ and $M = M(R)$ as multiplicative semigroups, we denote by $A \times_0 M$ the image of the direct product $A \times M$ obtained by collapsing the semigroup ideal \{(0, e) | e \in M\} to a single point, 0. Theorem 4.3 implies that $\zeta : A \times_0 M \rightarrow \Gamma(R)$ defined by $\zeta(a, e) = eae$ and $\zeta(0) = 0$ is at least a semigroup epimorphism. It is an isomorphism when $eAe \cap fAf = \{0\}$ for all $e \neq f$ in $M$. In this case $\Gamma(R)$ is said to be **partitioned** (although it is $\Gamma(R) \setminus \{0\}$ that is partitioned in the usual sense of the term). In this case $\Gamma(R)$ is a “rectangular band of rings” in the strongest possible sense. We present some results about when $\Gamma(R)$ is partitioned and hence $\zeta$ is a multiplicative isomorphism.
Proposition 4.7 Given an idempotent-dominated ring $R$ for which $E(R)$ is a primitive band. If $A$ denotes $e_0Re_0$ for some $e_0 \in M = M(R)$, then the following hold:

i) For all $e, f \in M$, $eAe \cap fAf = \{eae | eae = faf, \text{ for some } a \in A\}$. When $A$ is commutative, then $eAe \cap fAf$ is a common ideal of both subrings.

ii) In particular, if $A$ is a field, then $\Gamma(R)$ is partitioned.

iii) If $\Gamma(R)$ is partitioned with $M$ nontrivial, then $A$ has no divisors of zero.

Proof. (i) Consider $eae = fbf$ in $eAe \cap fAf$ where $a, b \in A$. But here $a = e_0ee_0ee_0e_0 = e_0fe_0be_0fe_0 = b$. In general $eAe \cap fAf$ is a subring of $R$. Given $eae = faf$ in $eAe \cap fAf$ along with any $ebe \in eAe$, then $eaebe = ebeeae$ in $eAe$ when $eAe$ is also commutative. But also

$$f(eaebe) = f(fafebe) = fafebe = eaebe$$

and

$$(eaebe)f = (ebeeae)f = (ebefaf)f = ebeffae = ebeeae = eaebe$$

so that $eaebe \in fRf = fAf$ also. That is, $eaebe = ebeeae \in eAe \cap fAf$. Thus $eAe \cap fAf$ is an ideal in $eAe$. Similarly, it is an ideal in $fAf$. (ii) follows immediately from (i).

To see (iii), suppose instead that $A$ has proper divisors of zero, say $a, b \in A \setminus \{0\}$ such that $ab = 0$ (but $ba$ need not equal 0). Setting $e_0 = e$, suppose $fLe$ in $M$ exists with $e \neq f$. Then $e$ and $f$ are $L$-equivalent to $g = e - a + faf$ in $M$. If $g = e$, then $eae = a = faf \neq 0$. If $g = f$, then $e - a + f(a - e)f = 0$ or $e(e - a)e = f(e - a)f \neq 0$ since zero divisor $a$ cannot equal $e$. If $g$ is neither $e$ nor $f$, then by $L$-equivalence,

$$(e - a + faf)ebe(e - a + faf) = (e - a + faf)ebe = (e - a)be + fabe = ebe.$$  

Thus $\Gamma(R)$ is not partitioned in all subcases. The case for $eRf$ in $M$ with $e \neq f$ is similar. Since one of these cases holds if $M$ is nontrivial, the contrapositive of (iii) is shown. □

When $A$ is commutative and thus $A[S]$ is weakly commutative we have:

Theorem 4.8 Given a commutative ring $A$ with identity for which $E(A) = \{1, 0\}$ and a rectangular band $S$, $\Gamma(A[S])$ is partitioned and $\zeta : A \times_0 M_s \to \Gamma(A[S])$ is thus a multiplicative isomorphism precisely when $A$ is an integral domain.
**Proof.** If \( \Gamma(A[S]) \) is partitioned, then \( A \) is an integral domain by Proposition 4.7(iv). Conversely, let \( e = (\sum a_i \cdot s_i)(\sum b_j \cdot t_j), f = (\sum c_i \cdot s_i)(\sum d_j \cdot t_j) \in \mathcal{M} \) be given with all \( s_i \in L \) and all \( t_j \in R \) and the conditions of Lemma 4.1 holding on the coefficients. [By using 0-coefficients as needed, we may assume a common indexing in both expressions for the \( s_i \)'s and for the \( t_j \)'s.] Suppose that for some nonzero \( p \in A \) we have

\[
(\sum a_i \cdot s_i)(\sum b_j \cdot t_j)p \cdot s(\sum a_i \cdot s_i)(\sum b_j \cdot t_j) = (\sum c_i \cdot s_i)(\sum d_j \cdot t_j)p \cdot s(\sum c_i \cdot s_i)(\sum d_j \cdot t_j)
\]

which simplifies to \( \sum(a_ipb_j) \cdot (s_it_j) = \sum(c_ipd_j) \cdot (s_it_j) \). Since we are working in a free \( A \)-module, this gives \( a_ipb_j = c_ipd_j \) for all \( i, j \). If \( A \) is an integral domain, this reduces to \( a_ib_j = c_id_j \) for all \( i, j \) so that

\[
e = (\sum a_i \cdot s_i)(\sum b_j \cdot t_j) = \sum(a_ib_j) \cdot (s_it_j) = \sum(c_id_j) \cdot (s_it_j) = (\sum c_i \cdot s_i)(\sum d_j \cdot t_j) = f.
\]

More generally, but also in the case where \( A \) is commutative, we may add:

**Theorem 4.9** Given a commutative ring \( A \) with identity such that \( E(A) = \{1, 0\} \) and a rectangular band \( S \), \( A[S]/\mathcal{N} \cong A/N_A \) where \( N_A \) is the nil radical of \( A \). If \( A \) has no nilpotent elements, and especially if \( A \) is an integral domain, then \( A[S]/\mathcal{N} \cong A \).

**Proof.** Note that \( e - f \in \mathcal{N} \) for all \( e, f \) in \( S \). Indeed \( (e - f)^4 = (e + f - ef - fe)^2 \) expands as

\[
(e + ef - ef - e) + (fe + f - f - fe) - e(fe + f - f - fe) - f(e + ef - ef - e) = 0.
\]

Let \( \mathcal{K} \) be the ideal generated from all differences of idempotents in \( S \). In general, \( \mathcal{K} \) is the kernel of the canonical epimorphism \( \eta : A[S] \to A \) defined by \( \eta(\sum a.s) = \sum a_s \). But \( \mathcal{K} \subseteq \mathcal{N} \) with \( \mathcal{N}/\mathcal{K} \) being isomorphic to the nil radical of \( A \).

**References**


