Skew lattices of matrices in rings

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Abstract. In [6] J. Leech introduced skew lattices in rings. In the present paper we study skew lattices in rings of matrices. We prove that every symmetric, normal skew lattice with a finite, distributive maximal lattice image can be embedded in a skew lattice of upper-triangular matrices.

1. Introduction

We adopt the definition of a skew lattice from [6]. A set $S$ endowed with two operations meet and join, denoted by $\wedge$ and $\vee$, is called a skew lattice if the two operations are both idempotent and associative, and they satisfy the following absorption laws:

\[
a \wedge (a \vee b) = a,
\]

\[
a \vee (a \wedge b) = a,
\]

\[
(a \vee b) \wedge b = b,
\]

\[
(a \wedge b) \vee b = b.
\]

Recall that a band (a semigroup of idempotents) is rectangular if it is isomorphic to a set $X \times Y$ with product $(a, b)(c, d) = (a, d)$. Rectangular bands are characterized by the identity $xyx = x$. On any band $S$ a congruence is defined by $x \equiv y$ if both $xyx = x$ and $yxy = y$. $S/\equiv$ is the maximal lattice image of $S$. Congruence classes of $\equiv$ are called components of $S$. Given $a \in S$, its component $[a]$ is the maximal rectangular subband of $S$ containing $a$. In this sense, every band is a semilattice of rectangular bands. (This is the Clifford–McLean Theorem.) Given a skew lattice $(S, \vee, \wedge)$, $\equiv$ for both operations coincide and thus so do their components with $x \vee y = y \vee x$ holding on each component. Finally, any skew lattice is quasiordered by setting $x \preceq y$ if $x \wedge y \wedge x = x$, or equivalently, $y \vee x \vee y = y$. Clearly, $\equiv$ equals $\preceq \cap \preceq^\vee$.

In [6] skew lattices in rings were introduced. In a ring $R$ there are two natural ways of defining the operation $\vee$ (assuming that $\wedge$ is the multiplication), namely $a \circ b = a + b - ab$, and $a \nabla b = (a \circ b)^2 = a + b + ba - aba - bab$. In particular,
$a \nabla b = a \circ b$ when $a \circ b$ is idempotent. The problem is that the operation $\circ$ need not be idempotent, while the operation $\nabla$ need not be associative. A multiplicative band $S \subseteq R$ which is closed under the operation $\nabla$ is called a $\nabla$-band. In the present paper we focus our attention on skew lattices that are $\nabla$-bands in some matrix ring $M_n(F)$ such that $\nabla$ is associative on the given $\nabla$-band. Throughout this paper, $F$ denotes a field.

It follows from [10] that every band in $M_n(F)$ is simultaneously triangularizable, which means that we may assume for all matrices in $S$ to be in the upper triangular form with 0s and 1s on the diagonal.

Components of an upper-triangular skew lattice $S \subset M_n(F)$ are rectangular bands for the usual multiplication of matrices, and each component consists exactly of all matrices in $S$ which have the same pattern of 1s on the diagonal. It follows from [3] that the elements of each component are simultaneously similar to block matrices with square diagonal blocks of the form

$$\begin{bmatrix} X & XY \\ I_l & Y \end{bmatrix}$$

for some $l$.

Each component is therefore closed under the operation $\nabla$, and $\nabla$ is associative on each component. In fact, we obtain the desired $a\nabla b = ba$ for $a$ and $b$ in the same component. Components however need not be closed under the operation $\circ$. This is the reason why we focus our attention on sets $(S, \cdot, \nabla)$. By a skew lattice of matrices we therefore refer to a subsemigroup of the multiplicative semigroup $M_n(F)$ which is closed under $\nabla$, and $(S, \cdot, \nabla)$ forms a skew lattice.

We consider the question, when can a given skew lattice be embedded in a skew lattice of matrices. We shall see that every skew lattice of matrices is symmetric with a finite, distributive maximal lattice image. And, on the other hand, every normal, symmetric skew lattice with a finite, distributive maximal lattice image can be embedded in a skew lattice of matrices. For a definition of a symmetric skew lattice see Definition 3.1.

Skew lattices arise from pseudolattices and near lattices, see [11] and [12]. For further reading on skew lattices the reader should refer to [6] and [9]. For basic definitions and concepts on semigroups and lattices see [5] and [4], respectively.

The following results were proved in [1], and might help the reader follow the rest of the paper.

**Lemma 1.1.** Any $\nabla$-band $S$ in a ring $R$ is a regular band. That is, the identity $xyxzx = yzx$ holds in $S$.

**Lemma 1.2.** Let $S$ be a multiplicative band in a ring $R$. If $x \preceq y$ in $S$, then $x \nabla y$ reduces to $y + yx - yxy$, $y \nabla x$ reduces to $y + xy - yxy$ with both being idempotent.
in $R$. Moreover, $xy, yx \equiv x$, while $x \vee y, y \vee x \equiv y$, provided $x \vee y$ and $y \vee x$ also lie in $S$.

2. Normal skew lattices of matrices

Let $S \subset M_n(F)$ be a band which is closed under $\vee$. It is easy to prove that $(S, \cdot, \vee)$ is a skew lattice, provided that $\vee$ is associative. The following example shows that not every $\vee$-band yields a skew lattice.

Example 2.1. Let $S \subset M_4(F)$ consist of all matrices of the form

$$a = \begin{bmatrix}
0 & x & xz_1 & xz_2 \\
0 & 1 & z_1 & z_2 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix} \quad \text{or} \quad b = \begin{bmatrix}
0 & y_1 & y_2 & y_1w_1 + y_2w_2 \\
0 & 1 & 0 & w_1 \\
0 & 0 & 1 & w_2 \\
0 & 0 & 0 & 0
\end{bmatrix}.$$  

Matrices of either form are idempotent. Moreover, we obtain

$$ab = \begin{bmatrix}
0 & x & xz_1 & xw_1 + xz_1w_2 \\
0 & 1 & z_1 & w_1 + z_1w_2 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}, \quad ba = \begin{bmatrix}
0 & y_1 & y_2 & y_1z_1 + y_1z_2 \\
0 & 1 & 0 & z_1 \\
0 & 0 & 1 & z_2 \\
0 & 0 & 0 & 0
\end{bmatrix},$$

$$a \vee b = b + ba - bab = \begin{bmatrix}
0 & y_1 & y_2 & y_1z_2 - y_1z_1w_2 + y_2w_2 \\
0 & 1 & 0 & z_2 - z_1w_2 \\
0 & 0 & 1 & w_2 \\
0 & 0 & 0 & 0
\end{bmatrix}.$$ 

and

$$b \vee a = b + ab - bab = \begin{bmatrix}
0 & x & y_2 + xz_1 - y_1z_1 & xw_1 + y_2w_2 + xz_1w_2 - y_1z_1w_2 \\
0 & 1 & 0 & w_1 \\
0 & 0 & 1 & w_2 \\
0 & 0 & 0 & 0
\end{bmatrix}.$$ 

Therefore, $S$ is closed under $\vee$. Let

$$a = \begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}, \quad b = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}, \quad c = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}.$$ 

Then

$$c \vee (a \vee b) = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}.$$
Next, we state a sufficient condition for $\nabla$ to be associative. Since all $\nabla$-bands are regular by Lemma 1.1, the next step is to explore normality of such bands. Recall that a band $S$ is called a normal band if it satisfies the identity $abcd = acbd$. A skew lattice $(S, \land, \lor)$ is called a normal skew lattice if $(S, \land)$ is a normal band. Normal skew lattices have been studied in [8]. The following proposition is a special case of a result from [7]. Here we give a direct, algebraic proof.

**Proposition 2.2.** Every normal $\nabla$-band in a ring forms a skew lattice.

**Proof.** It suffices to prove that $\nabla$ is associative. We observe this by direct calculation:

\[
(a \nabla b) \nabla c = (a + b + ba - aba - bab) \nabla c
\]
\[
= a + b + ba - aba - bab + c + ca + cb + cba - caba - cbab
\]
\[
- (a + b + ba - aba - bab) (ca + cb + cba - caba - cbab)
\]
\[
- cac - cbc - cbac + cabac + cbabc
\]
\[
= a + b + ba - aba - bab + c + ca + cb + cba - cba - cab
\]
\[
- aca - acb - abca + abca + acb - bca - bcb - bca + bca + bacb
\]
\[
- bca - bacb - bca + bca + bacb + abc + abc + abca - abca - acb
\]
\[
+ bca + bacb + bca - bca - bacb - cac - cbc - cabc + cabc + cabc
\]
\[
= a + b + c + ba + ca + cb - aba - bab - cab - acb - bca - bab
\]
\[
- cac - cbc + bacb + abca + cabc,
\]
and
\[
a \nabla (b \nabla c) = a \nabla (b + c + cb - bcb - cbc) \\
= a + b + c + cb - bcb - cbc + ba + ca + cba - bcb - cbca \\
- aba - aca - abca + abca + abca \\
- (ba + ca - bca) (b + c + cb - bcb - cbc) \\
= a + b + c + cb - bcb - cbc + ba + ca - bca \\
- aba - aca + abca - bab - bac + bacb + bacb + bac \\
- cab - cac - cab + cab + cabc + bacb + bac + bacb - bacb - bac \\
= a + b + c + ba + ca + cb - bcb - cbc - bca - aba - aca - bab \\
- cab + cac + abca + bacb + cabc
\]
\[
= (a \nabla b) \nabla c. \quad \square
\]

However, the condition for \( S \) to be a normal band is not necessary in order for \( S \) to be a skew lattice. The following example gives a band of matrices that yields a skew lattice which is not normal.

**Example 2.3.** Let \( S \) consists of \( \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \) and all \( 2 \times 2 \) matrices of the form \( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \). Let \( a = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \) and \( b = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \). Then \( ab = b \), and \( a \nabla b = a \circ b \). Hence \( \nabla \) is associative. Moreover, since \( S \) contains the identity matrix, normality would imply commutativity, which is a contradiction. Thus skew lattices in matrix rings need not be normal.

The following is an example of a normal skew lattice.

**Example 2.4.** Let \( S \) consist of all matrices of the form
\[
\begin{bmatrix}
0 & a_1 & a_2 & \cdots & a_k & c \\
0 & e_1 & 0 & \cdots & 0 & b_1 \\
0 & 0 & e_2 & \cdots & 0 & b_2 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & e_k & b_k \\
0 & 0 & 0 & \cdots & 0 & 0
\end{bmatrix}
\]
where each \( e_i = 0 \) or 1, each \( a_i = a_i e_i \), each \( b_j = e_j b_j \) and \( c = \sum a_i b_i \). Thus \( a_i = 0 \) if \( e_i = 0 \) and likewise \( b_j = 0 \) when \( e_j = 0 \). Denoting such a matrix by the triple \( \langle \{a_i\}, \{e_i\}, \{b_i\} \rangle \) one has
\[
\langle \{a_i\}, \{e_i\}, \{b_i\} \rangle \langle \{a'_i\}, \{e'_i\}, \{b'_i\} \rangle = \langle \{a_i e'_i\}, \{e_i e'_i\}, \{e_i b'_i\} \rangle.
\]
It follows that $S$ is a set of idempotents that is closed under multiplication and forms a normal band. Moreover,

$$\langle \{a_i\}, \{e_i\}, \{b_i\} \rangle \triangledown \langle \{a'_i\}, \{e'_i\}, \{b'_i\} \rangle = \langle \{a_i + a'_i - a_i e'_i\}, \{e_i + e'_i - e_i e'_i\}, \{b_i + b'_i - e_i b'_i\} \rangle.$$ 

Thus $S$ is a normal $\triangledown$-band, that is, a normal skew lattice.

3. Representations of skew lattices

We adopt the following definition from [6].

**Definition 3.1.** A $\triangledown$-band $S$ is symmetric if for all $x, y \in S$ the equivalence $xy = yx \iff x \triangledown y = y \triangledown x$ holds.

The following lemma is an easy observation.

**Lemma 3.2.** Every $\triangledown$-band in a ring is symmetric.

A homomorphism $\Phi$ from a skew lattice $S$ to a ring $R$ is any homomorphism $\Phi: (S, \wedge, \lor) \to (R, \cdot, \triangledown)$. In the case when $R = M_n(F)$ for some field $F$, $\Phi$ is also called a matrix representation. When $\Phi$ is injective, then $\Phi$ is called an embedding. As mentioned, each component of a skew lattice $S \subset M_n(F)$ consists exactly of matrices corresponding to a certain pattern of 1s on the diagonal. The maximal lattice image of such a skew lattice can therefore be identified with a sublattice of the power set $2^{\{1, 2, \ldots, n\}}$. In order to embed a skew lattice $S$ into $M_n(F)$, $S$ must therefore have a maximal lattice image $T$ of such a form as well, which implies that $T$ is finite and distributive (see [4], Corollary II.1.11). Note that not every normal skew lattice has a distributive maximal lattice image. See [8] or [9] for a characterization of normal skew lattices with a distributive maximal lattice image.

We have proved the following result.

**Proposition 3.3.** If a skew lattice $S$ can be embedded in $M_n(F)$ for some $n$ and $F$, then $S$ is symmetric and its maximal lattice image is finite and distributive.

In the remainder of this section we shall prove the following.

**Theorem 3.4.** Let $S$ be a symmetric, normal skew lattice with a finite, distributive maximal lattice image $T$. Then for some appropriately large field $F$, $S$ can be embedded in $M_{n+2}(F)$, where $n$ is the number of join irreducible elements of $T$. Furthermore, there exists an embedding of $S$ in $M_{n+2}(F)$ such that all images of elements of $S$ are upper triangular matrices.
Recall that for a lattice $L$ a set $P \subseteq L$ is called a *filter* if $P$ is a sublattice of $L$ such that if $x \in P$ and $y \in L$, then $x \lor y \in P$. A filter $P \subseteq L$ is called a *prime filter* if $x \lor y \in P$ implies $x \in P$ or $y \in P$.

By [8] Theorem 3.6, a symmetric, normal skew lattice $S$ for which $T = S/\equiv$ is finite and distributive is *decomposable* in that $S$ is isomorphic to a fibre product over $T$, $S' = \prod_{T'} \{ T [X_P, P] : P \in \mathcal{F}(T) \}$, where $\mathcal{F}(T)$ is the set of prime filters of $T$. For $P \in \mathcal{F}(T)$, $X_P$ is a rectangular skew lattice assigned to $P$ and $T [X_P, P] = P \times X_P \cup (T - P)$ is a skew lattice where $\lor$ and $\land$ on $P \times X_P$ or on $T - P$ are straightforward, but mixed compositions are given by

$$(p, x) \lor t = t \lor (p, x) = (p \lor t, x) \quad \text{and} \quad (p, x) \land t = t \land (p, x) = p \land t$$

for $p \in P$, $x \in X_P$ and $t \in T - P$. If $\pi(T)$ is the set of join irreducible elements of $T$, including 0, then $\mathcal{F}(T) = \{ p \lor t; p \in \pi(T) \}$ where $p \lor T$ denotes the filter $\{ p \lor t ; t \in T \}$. Deleting repeated $T$-coordinates, the components of $S'$ when parameterized by $T$ are

$$K(t) = \prod \{ X_P ; P \in \mathcal{F}(T) \quad \text{and} \quad t \in P \} = \prod \{ X_{p \lor T} ; p \in \pi(T) \quad \text{and} \quad t \geq p \}$$

for $t \in T$. In what follows, we view $S'$ as the disjoint union of all such direct products, allowing for singleton $X_P$ as well as for singleton direct products. In particular, the various $X_P$ are assumed to be disjoint.

**Proof of Theorem 3.4.** We first refine the remarks of the previous paragraph. Given the finite distributive lattice $T = S/\equiv$, for all $x \in T$ set $\pi(x) = \{ p \in \pi(T) ; x \geq p \}$. (The resulting map $\pi : T \rightarrow 2^{\pi(T)}$ is a lattice embedding since $T$ is distributive and $x = \sup \{ \pi(x) \}$ for all $x \in T$.) Also, each filter $P = p \lor T$ of $T$ is assigned a rectangular band $X_P = R_P \times L_P$ where we may assume that $R_P$ and $L_P$ are collectively pairwise disjoint. Under the isomorphism of $S$ with $S'$, each $s \in S$ corresponds to a $\pi(x)$-tuple $(\langle \alpha_p, \delta_p \rangle)_{p \in \pi(x)}$ with $(\alpha_p, \beta_p) \in R_p \times L_p$ and $x = [s]$.

Given $s' \rightarrow \langle (\alpha_p, \delta_p) \rangle_{p \in \pi(y)}$ where $y = [s']$, we obtain

$$s \land s' \rightarrow \langle (\alpha_p, \delta_p) \rangle_{p \in \pi(x) \cap \pi(y)}$$

and

$$s \lor s' \rightarrow \langle (\gamma_p, \beta_p) \rangle_{p \in \pi(x) \cap \pi(y)} \cup \langle (\alpha_p, \beta_p) \rangle_{p \in \pi(x) - \pi(y)} \cup \langle (\gamma_p, \delta_p) \rangle_{p \in \pi(y) - \pi(x)}.$$

Let $F$ be a field large enough so that $|F| \geq \sum |R_P \times L_P|$. Indeed, upon replacing elements if need be, we may assume that $\bigcup_p (R_P \cup L_P) \subseteq F \setminus \{0\}$. If $|\pi(T)| = n$, then a 1–1 correspondence $\varphi$ from $\{1, 2, \ldots, n\}$ to $\pi(T)$ induces an injective map.
\[ \Phi: S \rightarrow M_{n+2}(F) \] defined by the function sequence

\[ s \rightarrow \langle (\alpha_p, \beta_p) \rangle_{p \in \pi(x)} \rightarrow \langle a_i, e_i, b_i \rangle = \begin{bmatrix}
0 & a_1 & a_2 & \cdots & a_n & c \\
0 & \theta_1 & 0 & \cdots & 0 & b_1 \\
0 & 0 & \theta_2 & \cdots & 0 & b_2 \\
0 & 0 & 0 & \cdots & e_n & b_n \\
0 & 0 & 0 & \cdots & 0 & 0
\end{bmatrix} \]

where the first function is the isomorphism of \( S \) with \( S' \). The second function is given by \( a_i = \alpha_{\varphi(i)} \), \( b_i = \beta_{\varphi(i)} \) and \( e_i = 1 \) if \( \varphi(i) \in \pi(x) \) for \( x = [s] \), and \( a_i = b_i = e_i = 0 \) otherwise, with \( c = \sum a_i b_i \). From the behavior of \( \land \) and \( \lor \) on \( S' \) and the behavior of multiplication and \( \nabla \) in Example 2.4, \( \Phi: S \rightarrow M_{n+2}(F) \) is a skew lattice embedding.

Theorem 3.4 gives a family of skew lattices of matrices. Another family of skew lattices of matrices that is of a natural interest is formed from skew lattices whose maximal lattice images are chains. Such skew lattices arise from pure bands, which were introduced in [2], and were studied in [1].

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