On the Structure of Semigroups of Idempotent Matrices

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Abstract

We prove that any pure regular band of matrices admits a simultaneous LU decomposition in the standard form. In case that such a band forms a double-band called a skew lattice, we obtain the standard form without the assumption of purity.

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1 Introduction

A band is a semigroup that consists entirely of idempotents. A commutative band is called a semilattice. On a band $S$ Green’s relations $R$, $L$ and $D$ are defined by

\[
x R y \iff (xy = y \& yx = x),
\]
\[
x L y \iff (xy = x \& yx = y),
\]
\[
x D y \iff (xyx = x \& yxy = y).
\]

The Clifford-McLean Theorem states that $D$ is a congruence relation on any band $S$ and $S/D$ is a semilattice of rectangular bands, i.e. a congruence class $[x]$ is isomorphic to a set $L \times R$ with the product defined by $(a, b)(c, d) = (a, d)$. The congruence classes $[x]$ are called components of $S$.

In [7] a skew lattice $S$ is defined as a set $S$ endowed with operations meet $\land$ and join $\lor$, which are both idempotent, associative, and satisfy the absorption laws $x \land (x \lor y) = x$, $(x \lor y) \land y = y$ and their duals. Skew lattices can be viewed as double bands, since they form a band for either of the two operations. If $S$ is a skew lattice, then Leech’s First Decomposition Theorem states that relations $D$ in respect to $\land$ and $\lor$ coincide, and the factor set $S/D$ forms a lattice, which is called the maximal lattice image of $S$. Components are again skew lattices, since any rectangular band is a skew lattice for the meet operation being multiplication and the join operation defined by $u \lor v = v \land u$.

By a skew lattice in a ring $R$, we mean a set $S \subseteq R$ that forms a skew lattice for the meet operation being ring multiplication and the join defined by $a \nabla b = a + b + ba - aba - bab$. Skew lattices in rings of matrices provide important examples of skew lattices, for instance any normal skew lattice.
(satisfying $ABCD = CDAB$) with a finite lattice image can be embedded into some ring of matrices, see [3].

The structure of bands of matrices was studied by various authors; see for instance [4], [9] and [5]. In [5] the standard form for bands of matrices with the maximal semilattice image forming a chain was introduced. In the present paper we further explore this form and prove that when such a band $S$ is regular, then it admits a simultaneous LU decomposition (a basis exists such that all matrices in $S$ are in the standard form and each matrix is a product of a lower diagonal matrix by an upper diagonal matrix).

In the remainder of the paper we extend the standard form of bands to the standard form of skew lattices as well as explore the structure of skew lattices in rings in the terms defined by Leech in [8]. We shall see that the assumption that $S/D$ is a chain can be omitted in obtaining the standard form for the skew lattice case, and hence the standard form of an arbitrary skew lattice in a matrix ring is obtained; see Theorem 5.

2 The simultaneous LU decomposition of regular bands

A band is called regular if it satisfies the identity $axaya = axya$. Right [left] regular bands are defined by the identity $xyx = yx$ $[yx = xy]$. The Kimura Theorem [6] states that Green’s relations $R$ and $L$ are both congruences on a regular band and any regular bands factors as a fibred product of a left regular band with a right regular band over the common maximal semilattice image, ie. $S/R \times_{S/D} S/L$.

A pure band is a band $S$ such that $S/D$ is a chain (see [4]). In [5] the following standard form of pure bands of matrices was explored. Let $S$ be a pure band in $M_n(F)$, and let $E = \{e_0, e_1, ..., e_r\}$ be a maximal family of commuting idempotents in $S$. Then $E$ contains exactly one element from each component of $S$ [4]. We may assume $e_0 = 0$, $e_r = I$ and $\text{rank}(e_i) < \text{rank}(e_{i+1})$, since the components of $S$ correspond exactly to ranks of matrices in $S$. We denote by $\Delta(s)$ a diagonal matrix with the same diagonal elements as matrix $s$, and $S(m) = \{s \in S; \Delta(s) = \Delta(e_m)\}$. Then there exists such a basis for $F^n$ that each matrix in $S(m)$ can be expressed in block form

\[
\begin{bmatrix}
I_m & x_{21} \\
x_{12} & x_{12}x_{21}
\end{bmatrix},
\]
where $I_m$ is the identity matrix of dimension $n$, and $x_{12}, x_{21}$ are matrices of suitable dimensions. Note that $E = \Delta(S) = \{ \Delta(s) ; s \in S \}$ in this basis.

For $e_\beta < e_\alpha$, any two elements $a \in [e_\alpha]$, $b \in [e_\beta]$ can therefore be represented by block matrices

$$
a = \begin{bmatrix} I & 0 & a_{13} \\ 0 & I & a_{23} \\ a_{31} & a_{32} & a_{31}a_{13} + a_{32}a_{23} \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} I & b_{12} & b_{13} \\ b_{21} & b_{21}b_{12} & b_{21}b_{13} \\ b_{31} & b_{31}b_{12} & b_{31}b_{13} \end{bmatrix}.
$$

(1)

**Theorem 1** Let $S \subseteq M_n(F)$ be a pure regular band. Then there exist such a basis for $F^n$ that for any pair of components $A > B$, all matrices of the sub-skew lattice $A \cup B$ are in the standard form in this basis and all matrices in $S$ have a simultaneous LU decomposition in this basis.

**Proof.** The first part of the Theorem is exactly the standard form as introduced in [5]. In this basis any component can be written in a block form

$$
a = \begin{bmatrix} I & Y \\ X & XY \end{bmatrix}.
$$

Any two matrices $a, a' \in S$ are $R$-equivalent if and only if they have the same $(2,1)$-entry. Similarly, $a \overline{\mathcal{L}} a'$ if and only if they have the same $(1,2)$-entry. By The Kimura Theorem each component can therefore be viewed as the product of bands

$$
\{S_L = \begin{bmatrix} I & 0 \\ X & 0 \end{bmatrix} ; \exists Y : \begin{bmatrix} I & Y \\ X & XY \end{bmatrix} \in S \} \\
\{S_R = \begin{bmatrix} I & Y \\ 0 & 0 \end{bmatrix} ; \exists X : \begin{bmatrix} I & Y \\ X & XY \end{bmatrix} \in S \}.
$$

One may assume

$$
\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \in S,
$$

which implies that both $S_L$ and $S_R$ are contained in $S$. This yields $Y'X = 0$ for all

$$
\begin{bmatrix} I & Y' \\ X & XY' \end{bmatrix}, \begin{bmatrix} I & Y \\ X & XY \end{bmatrix} \in S.
$$
and \( a = a_La_R \), where

\[
a_L = \begin{bmatrix} I & 0 \\ X & 0 \end{bmatrix} \quad \text{and} \quad a_R = \begin{bmatrix} I & Y \\ 0 & 0 \end{bmatrix},
\]

is the LU decomposition of \( a \).

Let \( S \subseteq M_n(F) \) be a band with components \( A > B \). Assume that the matrices of \( A \cup B \) are in the standard form (1), and denote by \( a_0 \) and \( b_0 \) the diagonal elements in \( A \) and \( B \), respectively.

Furthermore, by notation \( A_{kl} = \{ a_{kl} \mid a \in [a_0] \} \) and \( B_{kl} = \{ b_{kl} \mid b \in [b_0] \} \), the following holds: \( A_{13}A_{3j} = B_{12}B_{21} = B_{13}B_{31} = 0 \), for \( i, j = 1, 2 \), \( A_{13} \subseteq B_{13} \), \( A_{31} \subseteq B_{31} \), \( B_{12}A_{23} \subseteq \text{span}B_{13} \), \( B_{13}A_{32} \subseteq \text{span}B_{12} \), \( A_{23}B_{31} \subseteq \text{span}B_{21} \) and \( A_{32}B_{21} \subseteq \text{span}B_{31} \).

It was proved in [9] that if all \( A_{kl} \) and \( B_{kl} \) are vector spaces over \( F \) that satisfy the above conditions, then the set \( S \) of all matrices of the above forms is a band.

This result was extended in [2], where it was proved that if in addition to the above also \( A_{13} = B_{13} \), \( A_{31} = B_{31} \), \( B_{21}A_{13} \subseteq A_{23} \) and \( B_{31}B_{12} \subseteq A_{32} \), then \( S \) is closed under the operation \( \nabla \).

More can be said about the sets \( A_{kl} \) and \( B_{kl} \) in the skew lattice case.

Direct calculation yields

\[
ab = \begin{bmatrix} I & b_{12} & b_{13} \\ b_{21} & b_{21}b_{12} & b_{21}b_{13} \\ a_{31} + a_{32}b_{21} & a_{31}b_{12} + a_{33}b_{21}b_{12} & a_{31} + a_{32}b_{21}b_{13} \end{bmatrix},
\]

\[
ba = \begin{bmatrix} I & a_{13} + b_{12}a_{23} \\ b_{21} & b_{21}b_{12} + b_{21}b_{13}a_{13} + b_{21}b_{12} \end{bmatrix},
\]

\[
a \nabla b = \begin{bmatrix} I & 0 & a_{13} \\ 0 & I & a_{23} \\ b_{31} - a_{32}b_{21} & a_{32} + b_{31}b_{12} - a_{31}b_{12} - a_{32}b_{21}b_{12} & * \end{bmatrix}
\]

and

\[
b \nabla a = \begin{bmatrix} I & b_{13} - b_{12}a_{23} \\ 0 & I & a_{23} + b_{21}b_{13} - b_{21}a_{13} - b_{21}b_{12}a_{23} \\ a_{31} & a_{32} & a_{31}a_{13} + a_{32}a_{23} \end{bmatrix}.
\]
Since $P = A \cup B$ allows the simultaneous LU decomposition as described in Theorem 1, one obtains additional conditions on $A_{ij}, B_{ij}$, namely $B_{31}B_{12} = B_{21}B_{13} = 0$, $A_{31}B_{12} = B_{21}A_{13} = 0$ and $B_{21}B_{12}A_{23} = A_{32}B_{21}B_{12} = 0$.

**Theorem 2** Let $P = A \cup B \subseteq M_n(F)$ be a set of matrices, where the matrices in $A$ and $B$ are of the forms given above. If the sets $A_{ij}, B_{ij}$ are vector spaces over $F$ and they satisfy the conditions:

1. $A_{i3}A_{3j} = 0$ for $i, j \in \{1, 2\}$,
2. $B_{12}B_{21} = 0$,
3. $A_{31}B_{12} = B_{21}A_{13} = 0$,
4. $A_{13} = B_{13}, A_{31} = B_{31}$,
5. $B_{12}A_{23} \subseteq A_{13}, A_{32}B_{21} \subseteq A_{31}$, and
6. $B_{21}A_{13} \subseteq A_{23}, A_{31}B_{12} \subseteq A_{32}$

then $P$ is a skew lattice.

3 Primitive skew lattices and cosets

A skew lattice consisting of only two components is called a **primitive skew lattice**. The structure of primitive skew lattices was thoroughly studied in [8]. We state the most important concepts and results in order to help the reader follow the rest of the paper.

Let $P$ be a primitive skew lattice with components denoted by $A$ and $B$ and assume $A > B$ in $P/D$.

For $b \in B$ the set $A \wedge b \wedge A = \{a \wedge b \wedge a' | a, a' \in A\}$ is a **coset** of $A$ in $B$; and for $a \in A$ the set $a \wedge B \wedge a = \{a \wedge b \wedge a | b \in B\} = \{b \in B | b \leq a\}$ is the **image** of $a$ in $B$.

Dually, a **coset** of $B$ in $A$ is any subset of the form $B \vee a \vee B$ for any $a \in A$; and for $b \in B$ its **image** in $A$ is $b \vee A \vee b = \{a \in A | b \leq a\}$.

The component $B$ is partitioned by the cosets of $A$ and the image set in $B$ of any element $a \in A$ is a transversal of cosets of $A$ in $B$. Given cosets $A_i$ in $A$ and $B_j$ in $B$ there is a natural bijection of cosets $\varphi_{ji} : A_i \rightarrow B_j$, called **coset bijection**, such that $\varphi_{ji}(x) = y$ iff $x \geq y$, i.e. $x \wedge y = y \wedge x = y$. Moreover, both operations $\wedge$ and $\vee$ are determined by the coset bijections.

6
A skew lattice is **right handed** if it satisfies the identities, \( x \land y \land x = y \land x \) and \( x \lor y \lor x = x \lor y \). Hence \( x \land y = y \) and \( x \lor y = x \) hold on each component. **Left handed** skew lattices are defined by the dual identities. In the right handed case, the description of a coset can be simplified as
\[
A \land b \land A = b \land A \text{ and } B \lor a \lor B = B \lor a.
\]
Indeed, in B one has \( a \land b \land a' = (a \land b) \land (b \land a') = b \land a' \).

Leech’s Second Decomposition Theorem for skew lattices \([7]\) states that skew lattices are regular bands with respect to either of the operations, and every skew lattice is isomorphic to the fibred product of a left handed skew lattice with a right handed skew lattice over a common maximal lattice image, that is to \( S/R \times_{S/D} S/L \), \([7]\). Following Leech’s terminology, we say that a skew lattice is **right primitive** if it is primitive and right handed, and a skew lattice is called **left primitive** if it is primitive and left handed. The skew lattice \( S/R \times_{S/D} S/L \) is called the **left [right] factor** of \( S \).

Observe that if \( S \subseteq R \) is either a right handed or a left handed skew lattice in a ring, then the \( \nabla \) operation reduces to the circle operation defined by \( a \circ b = a + b - ab \). Hence for an arbitrary skew lattice in a ring the \( \nabla \) operation reduces to the circle operation on the left and right factors of \( S \).

### 4 Coset decomposition of skew lattices in matrix rings

Let \( P \subset M_n(F) \) be a right primitive skew lattice with components \( A > B \) and coset decompositions \( A = \bigcup A_i, \ B = \bigcup B_j \). Using the standard form for bands with two components, one obtains that all matrices in \( A \) have a block decomposition of the form
\[
a = \begin{bmatrix} I & 0 & a_{13} \\ 0 & I & a_{23} \\ 0 & 0 & 0 \end{bmatrix},
\]
while the matrices in \( B \) are of the form
\[
b = \begin{bmatrix} I & b_{12} & b_{13} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.
\]
Computation

\[ ba = \begin{bmatrix} I & b_{12} & a_{13} + b_{12}a_{23} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \]

yields \( a \leq b \) if and only if \( b_{13} = a_{13} + b_{12}a_{23} \).

A coset containing \( a \) is therefore

\[ B \circ a \circ B = \{ \begin{bmatrix} I & b_{13} - b_{12}a_{23} \\ 0 & I & a_{23} \\ 0 & 0 & 0 \end{bmatrix} | b_{12}, b_{13} \text{ s.t.} \begin{bmatrix} I & b_{12} & b_{13} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \in B \} \]

and a coset containing \( b \) is

\[ AbA = \{ \begin{bmatrix} I & b_{12} & a_{13} + b_{12}a_{23} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} | a_{13}, a_{23} \text{ s.t.} \begin{bmatrix} I & 0 & a_{13} \\ 0 & I & a_{23} \\ 0 & 0 & 0 \end{bmatrix} \in A \} \]

Matrices \( a, a' \in A \) lie in the same coset if and only if \( a_{23} = a'_{23} \); similarly \( b, b' \in B \) lie in the same coset if and only if \( b_{12} = b'_{12} \). Denote by \( a_i \) the common \((2, 3)\)-component of all matrices in \( A \) and by \( b_j \) the common \((1, 2)\)-component of all matrices in \( B \). The coset bijection \( A_i \rightarrow B_j \) is obtained by

\[ \begin{bmatrix} I & 0 & a_{13} \\ 0 & I & a_i \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} I & b_j & a_{13} + b_ja_i \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \]

The structure of a left primitive skew lattice \( P \) is obtained in a similar fashion. Again, let \( A > B \) be the components in \( P \) with the coset decompositions as above. In the standard form, any \( b \in B_j \) is of the form

\[ b = \begin{bmatrix} I & 0 & 0 \\ b_j & 0 & 0 \\ b_{31} & 0 & 0 \end{bmatrix} \]

where \( b_j \) denotes the common \((2, 1)\) component of matrices in \( B_j \). Similarly, any \( a \in A_i \) is of the form

\[ a = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ a_{31} & a_i & 0 \end{bmatrix} \]
where \( a_i \) denotes the common \((3,2)\) component of matrices in \( A_i \). The coset bijection \( A_i \to B_j \) is obtained by

\[
\begin{bmatrix}
I & 0 & 0 \\
0 & I & 0 \\
a_{31} & a_i & 0
\end{bmatrix} \to 
\begin{bmatrix}
I & 0 & 0 \\
b_j & 0 & 0 \\
a_{31} + a_i b_j & 0 & 0
\end{bmatrix}.
\]

If \( S \subseteq M_n(F) \) is a pure skew lattice (ie. the multiplicative reduct of \( S \) is a pure band) in the standard form, then each matrix \( a \in S \) is the product of a lower diagonal matrix with an upper diagonal matrix by Theorem 1.

## 5 The coset constants

Let \( S \) be a right handed skew lattice in a ring \( R \).

**Lemma 3** Let cosets \( B_j \in B \) and \( A_i \in A \) and elements \( b_1, b_2 \in B_j \), \( a_1, a_2 \in A_i \) be given. If \( b_1 \leq a_1 \) and \( b_2 \leq a_2 \) then \( b_1 - a_1 = b_2 - a_2 \).

**Proof.** Consider the coset bijection \( \varphi_{ji} : A_i \to B_j \). Since \( \varphi_{ji}(a_1) = b_1 \) and \( \varphi_{ji}(a_2) = b_2 \), we obtain \( b_2 a_1 = b_1, b_1 a_2 = b_2, b_1 \circ a_2 = a_1 \) and \( b_2 \circ a_1 = a_2 \). Hence \( a_2 = b_2 \circ a_1 = b_2 + a_1 - b_1 \), and \( a_2 - a_1 = b_2 - b_1 \) follows. \( \blacksquare \)

Lemma 3 shows that for any pair of cosets \( B_j \subset B, A_i \subset A \) a constant \( c(B_j, A_i) \in R \) exists such that \( \varphi_{ji}(a) = a + c(B_j, A_i) \) for all \( a \in A_i \). The constants \( c(B_j, A_i) \) were introduced by the author in [1] and we refer to them as coset constants.

Consider next a right handed skew lattice \( S \subset R \) with incomparable components \( A \) and \( B \). Denote the meet class \( A \land B \) by \( M \), and the join class \( A \lor B \) by \( J \). What can be said about the sub skew lattice \( A \cup B \cup M \cup J \), considering all possible maximal primitive sub skew lattices?
Proposition 4 Given the above, all cosets of $A$ in $J$ are of the form $A \circ y$ for $y \in B$, all cosets of $B$ in $M$ are of the form $xB$ for $x \in A$. Moreover given $x \in A$ and $y \in B$,
\[c(xB, M \circ y) = c(xJ, A \circ y).\] (5)

Proof. Take $x \in A, y \in B$. One obtains $xy \leq y$ and $x \leq x \circ y$. By the above,
\[xy = y + c(xB, M \circ y)\]
and
\[x \circ y = x - c(xJ, A \circ y).\] (6)

On the other hand, $x \circ y = x + y - xy = x - c(xB, M \circ y)$ and (5) follows.

6 The standard form of skew lattices

Let $S \subseteq M_n(F)$ be a right handed skew lattice. Consider components $A > B$ in $S$ and assume that all matrices in $A$ and $B$ are of the form (2) and (3), respectively. Consider the coset decomposition that $A$ and $B$ induce on each other. What are the corresponding coset constants?

We have seen that the cosets of $A$ in $B$ correspond exactly to the $b_{12}$ entries, while the cosets of $B$ in $A$ correspond to the $a_{23}$ entries. Consider cosets $A_i$ with the $(2,3)$-entry of all matrices being $a_i$, and $B_j$ with the $(1,2)$-entry of all matrices being $b_j$. The coset bijection $\varphi_{ji}: A_i \rightarrow B_j$ is obtained by $\varphi_{ji}(a) = a + c(B_j, A_i)$, where
\[c(B_j, A_i) = ba - a = \begin{bmatrix} 0 & b_j & b_0a_i \\ 0 & -1 & -a_i \\ 0 & 0 & 0 \end{bmatrix}.\]

If we have a chain of three components $A > B > C$, then each coset constant from $A$ to $C$ is obtained as the sum of any corresponding coset constants from $A$ to $B$ and from $B$ to $C$: $cba - a = (cba - ba) + (ba - a)$. This illustrates the fact that all skew lattices in rings are categorical, which means that cosets and coset bijections form a category, see [8] for details.

Assume next that we have a (maximal) chain of components of some skew lattice $S$ in $M_n(K)$. Can we add a component to obtain a larger skew lattice? The obtained maximal lattice image must still be a distributive lattice [7]. Hence the only way to add a component is to add a component $B$ so that $M < A < J$ is a chain in $S/D$, $A \land B = M$ and $A \lor B = J$ as in (4).
The matrices of $M, A, J$ have the following block forms:

$$m = \begin{bmatrix} 1 & m_{12} & m_{13} & m_{14} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad a = \begin{bmatrix} 1 & 0 & a_{13} & a_{14} \\ 0 & 1 & a_{23} & a_{24} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad j = \begin{bmatrix} 1 & 0 & 0 & j_{14} \\ 0 & 1 & 0 & j_{24} \\ 0 & 0 & 1 & j_{34} \\ 0 & 0 & 0 & 0 \end{bmatrix}. \quad (7)$$

The corresponding coset constants are

$$c(mA, M \circ a) = \begin{bmatrix} 0 & m_{12} & m_{12}a_{23} & m_{12}a_{24} \\ 0 & -1 & -a_{23} & -a_{24} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad c(aJ, A \circ j) = \begin{bmatrix} 0 & 0 & a_{13} & a_{13}j_{34} \\ 0 & 0 & a_{23} & a_{23}j_{34} \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \quad (7)$$

Proposition 4 yields that elements of component $B$ must be of the form

$$b = ab - c(aB, M \circ b) = ab - c(aJ, A \circ b)$$

and

$$b = (b \circ a) + c(bJ, B \circ a) = (b \circ a) + c(bM, M \circ a)$$

for $a \in A$. Hence $b$ is of the form

$$b = \begin{bmatrix} 1 & b_{12} & b_{13} & b_{14} \\ 0 & 0 & b_{23} & b_{24} \\ 0 & 0 & 1 & b_{34} \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (8)$$

and $b_{23} = b_{24} = 0, 2b_{13} = 0, b_{13}b_{34} = 0$ follow because $b$ is idempotent. Hence $b_{13} = 0$ if char$F \neq 2$.

**Theorem 5** Let char$F \neq 2$, $S \subseteq M_{n}(F)$ a right handed skew lattice and $E_1 < ... < E_m$ a maximal chain of components of $S$. Then a basis for $F^n$ exists in which:

1. for any two matrices $a \in E_i, b \in E_j, i > j$ a block decomposition exists such that $a$ and $b$ have block forms

$$a = \begin{bmatrix} I & 0 & a_{13} \\ 0 & I & a_{23} \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} I & b_{12} & b_{13} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} ;$$

and
2. for non-comparable components $A$ and $B$ a block decomposition exists such that matrices $a \in A$ and $b \in B$ have block forms

$$
a = \begin{bmatrix} 1 & 0 & a_{13} & a_{14} \\ 0 & 1 & 0 & a_{24} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 1 & b_{12} & 0 & b_{14} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & b_{34} \\ 0 & 0 & 0 & 0 \end{bmatrix}.
$$

**Proof.** It remains to prove $a_{23} = 0$. We obtain $a = ba - c(bA, M \circ a) = ba - c(bJ, B \circ a)$, and

$$
c(bJ, B \vee a) = bj - j = \begin{bmatrix} 0 & b_{12} & 0 & b_{12j24} \\ 0 & -1 & 0 & -j_{24} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.
$$

Hence $a = ba + c(bA, M \circ a) = ba - c(bJ, B \circ a)$ is of the form

$$
a = \begin{bmatrix} 1 & 0 & * & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}
$$

and $a_{23} = 0$ follows. ■

Theorem 5 allows us to extend the standard form to an arbitrary right handed skew lattice in $M_n(F)$. The fibred product of the corresponding left handed and right handed skew lattices yields the standard form of an arbitrary skew lattice in $M_n(F)$.

**Corollary 6** Let $\text{char} F \neq 2$, $S \subseteq M_n(F)$ a skew lattice and $E_1 < \ldots < E_m$ a maximal chain of components of $S$. Then a basis for $F^n$ exists in which:

1. for any two matrices $a \in E_i, b \in E_j, i > j$ a block decomposition exists such that $a$ and $b$ have block forms

$$
a = \begin{bmatrix} I & 0 & a_{13} \\ 0 & I & a_{23} \\ a_{31} & a_{32} & a_{31}a_{13} + a_{32}a_{23} \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} I & b_{12} & b_{13} \\ b_{21} & b_{21b12} & b_{21b13} \\ b_{31} & b_{31b12} & b_{31b13} \end{bmatrix};
$$

and
2. for non-comparable components $A$ and $B$ a block decomposition exists such that matrices $a \in A$ and $b \in B$ have block forms

\[
a = \begin{bmatrix}
1 & 0 & a_{13} & a_{14} \\
0 & 1 & 0 & a_{24} \\
a_{31} & 0 & a_{31}a_{13} & a_{31}a_{14} \\
a_{41} & a_{42} & a_{41}a_{13} & a_{41}a_{14} + a_{42}a_{24}
\end{bmatrix}
\quad \text{and} \quad
b = \begin{bmatrix}
1 & b_{12} & 0 & b_{14} \\
b_{21} & b_{21}b_{12} & 0 & b_{21}b_{14} \\
0 & 0 & 1 & b_{34} \\
b_{41} & b_{41}b_{12} & b_{43} & b_{41}b_{14} + b_{43}b_{34}
\end{bmatrix}.
\]

The form of matrices in the bases as described in Corollary 6 is called the **standard form** for $S$.

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**References**


