Long $\mathbb{C}^2$'s without holomorphic functions

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What is a long $\mathbb{C}^n$?

A complex manifold X of dimension n is said to be a **long** $\mathbb{C}^n$ if it is the union of an increasing sequence of domains

$$X_1 \subset X_2 \subset X_3 \subset \cdots \subset \cup_{j=1}^{\infty} X_j = X$$

such that each X_j is biholomorphic to the complex Euclidean space $\mathbb{C}^n$.

Identifying $X_j \cong \mathbb{C}^n$, each inclusion $X_j \hookrightarrow X_{j+1}$ is given by a Fatou-Bieberbach (FB) map $\phi_j \colon \mathbb{C}^n \hookrightarrow \mathbb{C}^n$, and X is the direct limit of the system $(\phi_j)_{j\in\mathbb{N}}$. The elements of X are represented by infinite strings $x = (x_i, x_{i+1}, \ldots)$, where $i \in \mathbb{N}$ and

$$x_{k+1} = \phi_k(x_k), \quad k = i, i+1, \ldots.$$

Another string $y = (y_j, y_{j+1}, \ldots)$ determines the same element of X if and only if one of the following possibilities hold:

- $i = j$ and $x_i = y_i$ (and hence $x_k = y_k$ for all $k > i$);
- $i < j$ and $y_j = \phi_{j-1} \circ \cdots \circ \phi_i(x_i)$;
- $j < i$ and $x_i = \phi_{i-1} \circ \cdots \circ \phi_j(y_j)$.

The first main theorem

Every long $\mathbb{C}$ equals $\mathbb{C}$. The situation is very different for $n > 1$.

Theorem (L. Boc Thaler and F.F., 2015)

For every integer $n > 1$ there exists a long $\mathbb{C}^n$ without any nonconstant holomorphic or plurisubharmonic functions.

This theorem gives a strong counterexample to the classical *union problem*: **is an increasing union of Stein manifolds always Stein?**

Behnke and Thullen 1933, 1939 Yes for domains in $\mathbb{C}^n$.

J.E. Fornæss 1976 NO: There is an increasing union of complex 3-balls that is not holomorphically convex, hence not Stein.

Key ingredient: a biholomorphic map $\Phi\colon \Omega \to \Phi(\Omega) \subset \mathbb{C}^3$ on a bounded neighborhood $\Omega \subset \mathbb{C}^3$ of any compact set $K \subset \mathbb{C}^3$ with $\mathring{K} \neq \emptyset$ such that the polynomial hull of $\Phi(K)$ is not contained in $\Phi(\Omega)$.

Wermer 1959 described this phenomenon with K a polydisk in $\mathbb{C}^3$.

History continued

Fornæss and Stout 1977 A 3-dimensional increasing union of polydiscs without any nonconstant holomorphic functions.

E.F. Wold 2008 Non-Runge Fatou-Bieberbach domains in $\mathbb{C}^2$.

E.F. Wold 2010 Non-Stein (and non-holomorphically convex) long $\mathbb{C}^2$'s, obtained as the direct limit of a sequence of FB maps $\phi_k\colon \mathbb{C}^2 \to \mathbb{C}^2$ $(k \in \mathbb{N})$ with non-Runge images $\phi_k(\mathbb{C}^2) \subset \mathbb{C}^2$.

F.F. 2012 A holomorphic submersion $\pi\colon X^3 \to \mathbb{C}$ such that every fiber $X_z = \pi^{-1}(z)$ is a long $\mathbb{C}^2$, some fibers are $\mathbb{C}^2$'s and other fibers are non-Stein. Both types of fibers can form a dense set in X.

Problems:

- Do there exist several non-equivalent (non-Stein) long $\mathbb{C}^2$'s?
- Does there exist a non-Stein long $\mathbb{C}^2$ with a nontrivial algebra of holomorphic functions?
- Does there exist a Stein long $\mathbb{C}^2$ which is not a $\mathbb{C}^2$?

Uncountably many long $\mathbb{C}^2$'s

Here we give a positive answer to the first question. The other two remain open for the time being.

Theorem (L. Boc Thaler and F.F., 2015)

Let $n > 1$. To every open set $U \subset \mathbb{C}^n$ one can associate a long $\mathbb{C}^n$, $X(U)$ (containing U in the initial copy of $\mathbb{C}^n \subset X$) such that any biholomorphism $\Phi\colon X(U) \to X(V)$ maps U onto V.

Hence, if $X(U)$ is biholomorphic to $X(V)$ then U is biholomorphic to V.

To distinguish some long $\mathbb{C}^n$'s one from another, we introduce two new invariants of a complex manifold:

stable core **strongly stable core**.

The stable hull property

Definition

Let X be complex manifold. A compact set B in X is said to have the **stable hull property (SHP)** if there exists an exhaustion

$$K_1 \subset K_2 \subset \cdots \subset \cup_{j=1}^{\infty} K_j = X$$

by compact sets such that $B \subset K_1$, $K_j \subset \mathring{K}_{j+1}$ for every $j \in \mathbb{N}$, and the increasing sequence of hulls $\widehat{B}_{\mathscr{O}(K_j)}$ stabilizes, i.e., there is a $j_0 \in \mathbb{N}$ such that

$$\widehat{B}_{\mathscr{O}(K_j)} = \widehat{B}_{\mathscr{O}(K_{j_0})} \quad \text{for all } j \geq j_0.$$

Lemma

Stable hull property is independent of the choice of exhaustion.

Proof.

An exercise. □

The proof (just in case...)

Let $X = \cup_{j=1}^{\infty} K_j = \cup_{\ell=1}^{\infty} L_\ell$ be two exhaustions ($K_j \subset \mathring{K}_{j+1}$ and $L_j \subset \mathring{L}_{j+1}$) and $B \subset X$ a compact set such that for some $j_0 \in \mathbb{N}$:

$$\widehat{B}_{\mathscr{O}(K_j)} = \widehat{B}_{\mathscr{O}(K_{j_0})} \quad \text{for all } j \geq j_0. \tag{1}$$

Set $C := \widehat{B}_{\mathscr{O}(K_{j_0})}$. Pick an integer $\ell_0 \in \mathbb{N}$ such that $C \subset L_{\ell_0}$.

We can find sequences of integers $j_1 < j_2 < j_3 < \cdots$ and $\ell_1 < \ell_2 < \ell_3 < \cdots$ such that $j_0 \leq j_1$, $\ell_0 \leq \ell_1$, and

$$K_{j_0} \subset L_{\ell_1} \subset K_{j_1} \subset L_{\ell_2} \subset K_{j_2} \subset L_{\ell_3} \subset \cdots$$

From this and (1) we obtain

$$C = \widehat{B}_{\mathscr{O}(K_{j_0})} \subset \widehat{B}_{\mathscr{O}(L_{\ell_1})} \subset \widehat{B}_{\mathscr{O}(K_{j_1})} = C \subset \widehat{B}_{\mathscr{O}(L_{\ell_2})} \subset \widehat{B}_{\mathscr{O}(K_{j_2})} = C \subset \cdots.$$

It follows that $\widehat{B}_{\mathscr{O}(L_{\ell_j})} = C$ for all $j \in \mathbb{N}$. Since the sequence of hulls $\widehat{B}_{\mathscr{O}(L_\ell)}$ is increasing with ℓ, we conclude that

$$\widehat{B}_{\mathscr{O}(L_\ell)} = C \quad \text{for all } \ell \geq \ell_1.$$

Hence, B has SHP with respect to the exhaustion $(L_\ell)_{\ell \in \mathbb{N}}$.

The (strongly) stable core

Corollary

If $\Phi\colon X \to Y$ is a biholomorphic map then a compact $B \subset X$ has SHP if and only if its image $\Phi(B) \subset Y$ has SHP.

This enables us to introduce the following **biholomorphic invariants** of a complex manifold.

Definition

Let X be a complex manifold.

(i) The **stable core** of X, $SC(X)$, is the open set consisting of all points $x \in X$ which admit a compact neighborhood $B \subset X$ with the stable hull property.

(ii) A compact set $B \subset X$ is called the **strongly stable core** of X, denoted $SSC(X)$, if B has the stable hull property, but any compact set $L \subset X$ with $\mathring{L} \setminus B \neq \emptyset$ fails to have the stable hull property.

The second main theorem

Theorem

Let $n > 1$.

(a) *For every compact polynomially convex set* $B \subset \mathbb{C}^n$ *there exists a long* $\mathbb{C}^n$, $X(B)$, *whose strongly stable core equals* B:

$$SSC(X(B)) = B.$$

(b) *For every open set* $U \subset \mathbb{C}^n$ *there exists a long* $\mathbb{C}^n$, X, *such that*

$$SC(X) \subset U \quad \text{and} \quad \overline{U} = \overline{SC(X)}.$$

Connection to CR geometry

By part (a) of the theorem, every compact polynomially convex domain $B \subset \mathbb{C}^n$ $(n > 1)$ is the strongly stable core of a long $\mathbb{C}^n$.

This shows that the moduli space of long $\mathbb{C}^n$'s contains the moduli space of germs of smooth strongly pseudoconvex real hypersurfaces in $\mathbb{C}^n$.

It has been known since **Poincaré 1907** that most pairs of smoothly bounded strongly pseudoconvex domains in $\mathbb{C}^n$ are not biholomorphic to each other, at least not by maps that extend smoothly to the closed domains. The latter condition always holds (**Fefferman 1974**,...)

For strongly pseudoconvex real-analytic hypersurfaces, a complete set of local biholomorphic invariants is provided by the *Chern-Moser normal form* (**Chern and Moser 1974**). Furthermore, most such domains have no holomorphic automorphisms other than the identity map.

Corollary

There exist uncountably many biholomorphically inequivalent long $\mathbb{C}^2$'s.

The main tool: Andersén-Lempert theory

Varolin 1997 A complex manifold X enjoys the **holomorphic density property** if every holomorphic vector field on X can be approximated, uniformly on compacts, by Lie combinations (sums and commutators) of $\mathbb{C}$-complete holomorphic vector fields on X.

Andersén and Lempert 1992 $\mathbb{C}^n$ for $n > 1$ enjoys the density property.

Varolin 1997, 2000 Any domain $X = (\mathbb{C}^*)^k \times \mathbb{C}^\ell$ with $k + \ell \geq 2$ and $\ell \geq 1$ enjoys the density property.

Theorem (Andersén & Lempert 1992, Rosay & F. 1993, Varolin 2000)

Let X be a Stein manifold with the density property, and let

$$\Phi_t \colon \Omega_0 \longrightarrow \Omega_t = \Phi_t(\Omega_0) \subset X, \quad t \in [0,1]$$

be a smooth isotopy of biholomorphic maps of Ω_0 onto pseudoconvex Runge domains $\Omega_t \subset X$ such that $\Phi_0 = \mathrm{Id}_{\Omega_0}$. Then the map $\Phi_1 \colon \Omega_0 \to \Omega_1$ can be approximated uniformly on compacts in Ω_0 by holomorphic automorphisms of X.

The first main Lemma

Lemma (1)

Let K be a compact set with nonempty interior in $\mathbb{C}^n$ for some $n > 1$.

For every point $a \in \mathbb{C}^n$ there exists an injective holomorphic map $\phi\colon \mathbb{C}^n \hookrightarrow \mathbb{C}^n$ such that

$$\phi(a) \in \widehat{\phi(K)}.$$

More generally, if $L \subset \mathbb{C}^n$ is a compact holomorphically contractible set disjoint from K such that $K \cup L$ is polynomially convex, then there exists an injective holomorphic map $\phi\colon \mathbb{C}^n \hookrightarrow \mathbb{C}^n$ such that

$$\phi(L) \subset \widehat{\phi(K)} \quad \text{and} \quad \widehat{\phi(K)} \setminus \phi(\mathbb{C}^n) \neq \emptyset.$$

We follow **E.F. Wold's construction** (2008, 2010) up to a certain point, adding a new twist at the end. We consider the case $n = 2$.

Proof

Stolzenberg 1966 There is a compact set $M \subset \mathbb{C}^* \times \mathbb{C}$ such that:

- M is a disjoint union of two smooth, embedded, totally real discs;
- M is holomorphically convex in $\mathbb{C}^* \times \mathbb{C}$;
- the polynomial hull $\widehat{M}$ of contains the origin $(0,0) \in \mathbb{C}^2$.

We find ϕ as a composition

$$\phi = \psi_2 \circ \psi_1 \circ \theta$$

where

- $\theta \colon \mathbb{C}^2 \hookrightarrow \mathbb{C}^* \times \mathbb{C}$ is an FB map with Runge image;
- $\psi_1 \in \mathrm{Aut}(\mathbb{C}^* \times \mathbb{C})$ approximates the identity on $\theta(L)$;
- M is contained in the interior of $K' := \psi_1(\theta(K)) \subset \mathbb{C}^* \times \mathbb{C}$. Hence $\widehat{K'} \setminus K'$ contains a ball $\overline{B}$ around $(0,0) \in \mathbb{C}^2$;
- $\psi_2 \in \mathrm{Aut}(\mathbb{C}^* \times \mathbb{C})$ approximates the identity map on K' and $\psi_2(\psi_1(\theta(L))) \subset B$. Hence $\phi(L) \subset \widehat{\phi(K)} \not\subset \phi(\mathbb{C}^2)$.

Proof of the first main theorem

Pick a compact set $K \subset \mathbb{C}^n$ with nonempty interior and a countable dense sequence $\{a_j\}_{j \in \mathbb{N}}$ in $\mathbb{C}^n$. Set $K_1 = \widehat{K}$.

Lemma 1 furnishes an FB map $\phi_1 \colon \mathbb{C}^n \to \mathbb{C}^n$ such that

$$\phi_1(a_1) \in \widehat{\phi_1(K_1)} =: K_2.$$

Applying Lemma 1 to the set K_2 and the point $\phi_1(a_2) \in \mathbb{C}^n$ gives an injective holomorphic map $\phi_2 \colon \mathbb{C}^n \hookrightarrow \mathbb{C}^n$ such that

$$\phi_2(\phi_1(a_2)) \in \widehat{\phi_2(K_2)} =: K_3.$$

From the first step, we also have $\phi_1(a_1) \in K_2$, and hence

$$\phi_2(\phi_1(a_1)) \in K_3.$$

Proof, conclusion

Continuing inductively, we get a sequence of injective holomorphic maps $\phi_j \colon \mathbb{C}^n \to \mathbb{C}^n$ for $j = 1, 2, \ldots$ such that, setting

$$\Phi_k = \phi_k \circ \cdots \circ \phi_1 \colon \mathbb{C}^n \hookrightarrow \mathbb{C}^n,$$

we have for all $k \in \mathbb{N}$ that

$$\Phi_k(a_j) \in \widehat{\Phi_k(K)} \quad \text{for } j = 1, \ldots, k.$$

In the limit manifold $X = \cup_{k=1}^{\infty} X_k$ determined by $(\phi_k)_{k=1}^{\infty}$ the $\mathscr{O}(X)$-hull of the initial set $K \subset \mathbb{C}^n = X_1 \subset X$ clearly contains the set $\Phi_k(K) \subset X_{k+1}$ for each $k = 1, 2, \ldots$.
(We have identified the k-th copy of $\mathbb{C}^n$ in the sequence with its image $\psi_k(\mathbb{C}^n) = X_k \subset X$.)
It follows that

$$\{a_j\}_{j \in \mathbb{N}} \subset \widehat{K}_{\mathscr{O}(X)}.$$

Since the set $\{a_j\}_{j \in \mathbb{N}}$ is everywhere dense in $\mathbb{C}^n = X_1$, it follows that every holomorphic function on X is bounded on $X_1 = \mathbb{C}^n$ and hence constant. By the identity principle it is constant on all of X.

Second main lemma

Lemma (2)

Let $n > 1$. Assume that

- *B is a compact, strongly pseudoconvex and polynomially convex domain in $\mathbb{C}^n$, and*
- *$K_1, \ldots, K_m$ are pairwise disjoint compact sets with nonempty interiors in $\mathbb{C}^n \setminus B$ such that $B \cup (\cup_{j=1}^m K_j)$ is polynomially convex.*

Then there exists a Fatou–Bieberbach map $\phi \colon \mathbb{C}^n \hookrightarrow \mathbb{C}^n$ such that

(i) $\widehat{\phi(B)} = \phi(B)$ *and* $\phi|_B$ *is close to* $\mathrm{Id}|_B$*;*

(ii) $\widehat{\phi(K_j)} \not\subset \phi(\mathbb{C}^n)$ *for all* $j = 1, \ldots, m$. *(The* **Wold process***.)*

Proof: Let $n = 2$. Choose a closed ball $\mathcal{B}$ containing B in its interior. By a suitable choice of coordinates we may assume that $\mathcal{B} \subset \mathbb{C}^* \times \mathbb{C}$.

Proof of Lemma 2

Choose an FB map $\theta_1\colon \mathbb{C}^2 \hookrightarrow \mathbb{C}^* \times \mathbb{C}$ whose image is Runge in $\mathbb{C}^2$.
Hence, the set $\theta_1(\mathcal{B})$ is polynomially convex.
Since $\mathcal{B}$ is contractible, we can connect the identity map on $\mathcal{B}$ to $\theta_1|_{\mathcal{B}}$ by an isotopy of biholomorphic maps $h_t\colon \mathcal{B} \to \mathcal{B}_t$ $(t \in [0,1])$ with Runge images in $\mathbb{C}^* \times \mathbb{C}$.

The AL Theorem furnishes $\theta_2 \in \mathrm{Aut}(\mathbb{C}^* \times \mathbb{C})$ which approximates θ_1^{-1} on the set $\theta_1(\mathcal{B})$.
The composition $\theta = \theta_2 \circ \theta_1\colon \mathbb{C}^2 \hookrightarrow \mathbb{C}^* \times \mathbb{C}$ is then a FB map which is close to the identity on $\mathcal{B}$.
The set $B' := \theta(B)$ is a small smooth perturbation of B, hence polynomially convex. Set

$$K = \cup_{j=1}^m K_j, \quad K_j' = \theta(K_j), \quad K' = \theta(K) = \cup_{j=1}^m K_j'.$$

Note that the set $B' \cup K' = \theta(B \cup K) \subset \mathbb{C}^* \times \mathbb{C}$ is $\mathscr{O}(\mathbb{C}^* \times \mathbb{C})$-convex.

Choose m pairwise disjoint copies $M_1, \ldots, M_m \subset (\mathbb{C}^* \times \mathbb{C}) \setminus B'$ of Stolzenberg's compact set M.
By placing the sets M_j far apart and away from B', we may assume that

$$B' \cup (\cup_{j=1}^m M_j) \text{ is } \mathscr{O}(\mathbb{C}^* \times \mathbb{C})\text{-convex.}$$

For every $\epsilon > 0$ the AL theorem gives $\psi \in \mathrm{Aut}(\mathbb{C}^* \times \mathbb{C})$ such that

(i) $|\psi(z) - z| < \epsilon$ for all z in a neighborhood of B';

(ii) $M_j \subset \psi(K'_j)$ for all $j = 1, \ldots, m$.

Hence, the injective holomorphic map

$$\phi := \psi \circ \theta \colon \mathbb{C}^2 \hookrightarrow \mathbb{C}^* \times \mathbb{C}$$

approximates the identity on B and satisfies

$$M_j \subset \psi(K'_j) = \phi(K_j) \quad \text{for } j = 1, \ldots, m.$$

It follows that

$$\widehat{\phi(K_j)} \cap (\{0\} \times \mathbb{C}) \neq \emptyset \;\; \text{for all } j = 1, \ldots, m.$$

If the approximation is close enough, the set $\phi(B) = \psi(B')$ is still polynomially convex. Clearly, ϕ satisfies Lemma 2.

Proof of the second main theorem

Recall:

Theorem (2)

(a) *For every compact polynomially convex set $B \subset \mathbb{C}^n$ there exists a long $\mathbb{C}^n$, $X(B)$, whose strongly stable core equals B:*

$$SSC(X(B)) = B.$$

(b) *For every open set $U \subset \mathbb{C}^n$ there exists a long $\mathbb{C}^n$, X, such that*

$$SC(X) \subset U \quad \text{and} \quad \overline{U} = \overline{SC(X)}.$$

The proof amount to a recursive application of Lemma 2. Assume that $n = 2$. We consider part (a); part (b) is similar.

Choose a decreasing sequence of compact strongly pseudoconvex polynomially convex domains in $\mathbb{C}^2$ whose intersection equals B:

$$B_1 \supset B_2 \supset B_3 \supset \cdots \supset \bigcap_{k=1}^{\infty} B_k = B.$$

Step 1: Pick a countable dense set

$$A_1 = \{a_{1,j} : j \in \mathbb{N}\} \subset \mathbb{C}^2 \setminus B_1.$$

Let $r_1 > 0$ be small enough such that $\overline{\mathbb{B}}(a_{1,1}, r_1) \cap B_1 = \emptyset$ and $\overline{\mathbb{B}}(a_{1,1}, r_1) \cup B_1$ is polynomially convex.

Choose the first FB map $\phi_1 : \mathbb{C}^2 \hookrightarrow \mathbb{C}^2$ such that

$$\phi_1 \text{ is close to the identity on } B_1$$

and the set

$$C^1_{1,1} = \phi_1(\overline{\mathbb{B}}(a_{1,1}, r_1))$$

satisfies

$$\widehat{C^1_{1,1}} \setminus \phi_1(\mathbb{C}^2) \neq \emptyset.$$

Step 2: Choose a countable dense set

$$A_2 = \phi_1(A_1) \cup A_2' \subset \mathbb{C}^2 \setminus \phi_1(B_2).$$

Choose an FB map $\phi_2 \colon \mathbb{C}^2 \hookrightarrow \mathbb{C}^2$ such that

$$\phi_2 \approx \mathrm{Id} \text{ on the set } \phi_1(B_2)$$

and ϕ_2 performs the Wold process on small balls centered at the first two points of the set $\phi_1(A_1)$ and the first point of A_2'.
(This means that the polynomial hull of the ϕ_2-image of each of these three balls is not contained in $\phi_2(\mathbb{C}^2)$.) Set

$$\Phi_2 = \phi_2 \circ \phi_1.$$

The k-th step: We have found an FB map

$$\Phi_k = \phi_k \circ \cdots \circ \phi_1 \colon \mathbb{C}^2 \hookrightarrow \mathbb{C}^2$$

which is close to Id on B_k and ϕ_k performs the Wold process on the first $k(k+1)/2$ points of a countable dense set $A_k \subset \mathbb{C}^2 \setminus \Phi_{k-1}(B_k)$.

We add countably many points to $\phi_k(A_k)$ to get a countable dense set

$$A_{k+1} = \phi_k(A_k) \cup A'_{k+1} \subset \mathbb{C}^2 \setminus \Phi_k(B_{k+1}).$$

Lemma 2 furnishes an FB map $\phi_{k+1} \colon \mathbb{C}^2 \hookrightarrow \mathbb{C}^2$ which is close to Id on $\Phi_k(B_{k+1})$ and which performs the Wold process on the first $(k+1)(k+1)/2$ points of the set A_{k+1}.

Set

$$\Phi_{k+1} = \phi_{k+1} \circ \Phi_k = \phi_{k+1} \circ \phi_k \circ \cdots \phi_1.$$

The process may continue.

In the limit manifold X, the set $B = \bigcup_{k=1}^{\infty} B_k$ is the strongly stable core. This completes the proof.

***** THANK YOU FOR YOUR ATTENTION *****