Nonlinear holomorphic approximation theory

Franc Forstnerič

Univerza v Ljubljani

Complex Analysis and Related Topics 2018
Euler International Mathematical Institute
Sankt Peterburg, 27 April 2018
Abstract

I will discuss some developments in holomorphic approximation theory of \textbf{Runge, Mergelyan, Carleman and Arakelyan type}, with emphasis on manifold-valued maps.

A more comprehensive discussion of these topics is available in the survey


Runge 1885  If $K$ is a compact set with connected complement in $\mathbb{C}$ then every $f \in \mathcal{O}(K)$ is a uniform limit of holomorphic polynomials. More generally, for every compact $K \subset \mathbb{C}$ we can approximate functions in $\mathcal{O}(K)$ by rational functions with poles in $\mathbb{C} \setminus K$.

Behnke and Stein 1949, Koditz and Timmann 1975  
Let $K$ be a compact set in a Riemann surface $X$. Every $f \in \mathcal{O}(K)$ can be approximated uniformly on $K$ by meromorphic functions on $X$ without poles in $K$, and by functions in $\mathcal{O}(X)$ if $K$ has no holes.

K. Oka 1936, A. Weil 1935  If $X \subset \mathbb{C}^n$ is a domain of holomorphy (or a Stein manifold, K. Stein 1951) and $K \subset X$ is a compact $\mathcal{O}(X)$-convex set, then every $f \in \mathcal{O}(K)$ is a uniform limit of functions in $\mathcal{O}(X)$.
The Oka-Grauert Principle (nonlinear Runge theorem)

**Theorem (Oka 1939, Grauert 1958)**

Let $X$ be a Stein space, $K \subset X$ be a compact $\mathcal{O}(X)$-convex subset, and $X_0 \subset X$ be a closed complex subvariety. Given a complex homogeneous manifold $Y$ and a continuous map $f_0 : X \to Y$ which is holomorphic on $K \cup X_0$, there is a homotopy $f_t : X \to Y$ ($t \in [0, 1]$) such that

- $f_t \in \mathcal{O}(K)$ and $f_t|_K$ approximates $f_0|_K$ for all $t \in [0, 1]$,
- $f_t|_{X_0} = f_0|_{X_0}$ for all $t \in [0, 1]$, and
- $f_1 : X \to Y$ is holomorphic.

For $Y = \mathbb{C}$, this is the **Oka-Weil approximation theorem** combined with the **Oka-Cartan extension theorem**.

The analogous result holds for families of maps depending on a parameter in a compact Hausdorff space. In particular, the natural inclusion

$$\mathcal{O}(X, Y) \hookrightarrow \mathcal{C}(X, Y)$$

is a **weak homotopy equivalence**.
Elliptic manifolds and Oka manifolds

**Gromov 1989** The same holds if the target manifold $Y$ is **elliptic**, i.e., it admits a **dominating holomorphic spray** — a holomorphic map $s : E \rightarrow Y$ from the total space of a holomorphic vector bundle $E \rightarrow Y$ such that

$$s(0_y) = y, \quad ds_{0_y}(E_y) = T_y Y \quad \forall y \in Y.$$  

This holds for example if $TY$ is generated by $\mathbb{C}$-complete holomorphic vector fields (flexible manifolds; [Arzhantsev et al. 2013](#)).

**F., 2005-2010** Let $h : Z \rightarrow X$ be any holomorphic fibre bundle with fibre $Y$ over a Stein base space $X$. Then, the Oka-Grauert principle holds in all forms for sections $X \rightarrow Z$ iff $Y$ satisfies the following condition:

**Every holomorphic map $K \rightarrow Y$ from a compact convex set $K \subset \mathbb{C}^N$ (for any $N \in \mathbb{N}$) is a uniform limit of entire maps $\mathbb{C}^N \rightarrow Y$.**

A complex manifold $Y$ satisfying these equivalent conditions is called an **Oka manifold** ([F. 2009](#)).
What do we know about Oka manifolds?

- A Riemann surface is Oka iff it is not Kobayashi hyperbolic.

- If \( \pi: Z \to X \) is a holomorphic fibre bundle with an Oka fibre \( Y \), then \( X \) is Oka iff \( Z \) is Oka. This holds in particular in a covering space.

- **Kobayashi & Ochiai 1974** A compact complex manifold of general Kodaira type is not dominable, and hence not Oka.

- **Lárusson & F. 2014**: About compact complex surfaces:
  1. \( \kappa = -\infty \): Rational surfaces are Oka. A ruled surface is Oka if and only if its base is Oka. Minimal Hopf surfaces and Enoki surfaces are Oka. Inoue surfaces, Inoue-Hirzebruch surfaces, and intermediate surfaces, minimal or blown up, are not Oka. (This covers surfaces of class VII if the global spherical shell conjecture is true.)
  2. \( \kappa = 0 \): Bielliptic surfaces, Kodaira surfaces, and tori are Oka. It is unknown whether any K3 surfaces or Enriques surfaces are Oka.
  3. \( \kappa = 1 \): **Buzzard & Lu 2000** determined which properly elliptic surfaces are dominable. Nothing further is known about the Oka property. No example of an Oka surface with \( \kappa = 1 \) is known.

- Many classes of affine algebraic manifolds are known to be Oka ([Kaliman and Kutzschebauch, Andrist, Leuenberger, ...])
Generalizations and applications

The most general version of this theorem concerns sections of holomorphic submersions \( h : Z \rightarrow X \) over stratified Stein spaces \( X \) with the property that every point in a stratum \( X_0 \subset X \) has a neighborhood \( U \subset X_0 \) such that \( Z|_U \rightarrow U \) admits a dominating fibre-spray.

Some applications:

- **Oka, Grauert**: Classification of principal bundles over Stein spaces.
- **Forster and Ramspott**: the number of equations needed to define a subvariety.
- **Eliashberg and Gromov; Schürmann**: Precise embedding dimension for Stein manifolds and Stein spaces.
- **Ivarsson and Kutzschebauch**: Holomorphic Vaserstein problem.
- **Heizner and Kutzschebauch; Kutzschebauch, Lárusson, Schwarz**: Equivariant h-principle for Stein spaces or bundles with holomorphic group actions.
- **Leiterer**: Similarity of holomorphic matrix-valued functions.
- **Alarcón, López, Drinovec-Drnovšek, F.**: Minimal surfaces in Euclidean spaces.
Mergelyan approximation

**Mergelyan 1951** If $K$ is a compact set in $\mathbb{C}$ without holes, then every function in $\mathcal{A}(K) = \mathcal{C}(K) \cap \mathcal{O}(\hat{K})$ is a uniform limit of entire functions.

In view of Runge’s theorem, Mergelyan’s theorem is equivalent to

\[
\text{The Mergelyan property (MP): } \mathcal{A}(K) = \overline{\mathcal{O}(K)}.
\]

**Vitushkin 1966** Characterization of MP in terms of continuous capacity.

**Bishop 1958 (localization theorem)** Let $K$ be a compact set in a Riemann surface $X$. If every point $p \in K$ has a compact neighborhood $D_p \subset X$ such that $K \cap D_p$ has MP, then $K$ has MP. In particular, a compact set without holes in an open Riemann surface has DP.

**Boivin and Jiang 2004 (the converse to Bishop’s theorem)** If a compact set $K$ in a Riemann surface $X$ has MP, then for every closed coordinate disc $D_p \subset X$ the set $K \cap D_p$ has MP.

**Verdera 1986** Let $K$ be compact set in $\mathbb{C}$. If $f \in \mathcal{C}_0^r(\mathbb{C}) \ (r \in \mathbb{N})$ is such that $\partial f / \partial \bar{z}$ vanishes on $K$ to order $r - 1$, then $f$ can be approximated in $\mathcal{C}^r(\mathbb{C})$ by functions holomorphic in neighborhoods of $K$. 
\(\bar{\partial}\)-proof of Bishop’s localization theorem

**Sakai 1972** Let \( f \in \mathcal{A}(K) \), \( f \) continuous in a neighborhood of \( K \). Cover \( K \) by finitely many compact sets \( D_j \) as in Bishop’s theorem such that \( \hat{D}_j \) is an open cover of \( K \). Let \( \chi_j \) be a subordinate smooth partition of unity. By the assumption, for any \( \epsilon > 0 \) we have functions \( f_j \in \mathcal{C}(D_j) \cap \mathcal{O}(K \cap D_j) \) such that \( \| f_j - f \|_{\mathcal{C}(K \cap D_j)} < \epsilon \). Set

\[
g = \sum_{j=1}^{m} \chi_j f_j.
\]

Then on some open neighborhood \( U \) of \( K \) we have that

\[
\| g - f \|_{\mathcal{C}(U)} = O(\epsilon), \quad \bar{\partial}g = \sum_{j=1}^{m} f_j \bar{\partial}\chi_j = \sum_{j=1}^{m} (f_j - f) \bar{\partial}\chi_j = O(\epsilon).
\]

Let \( \chi \in \mathcal{C}_0^\infty(U) \) be a cut-off function with \( 0 \leq \phi \leq 1 \) and \( \chi \equiv 1 \) near \( K \). Then \( \| \chi \bar{\partial}g \|_{\mathcal{C}(\bar{U})} = O(\epsilon) \) and so \( T(\chi \cdot \bar{\partial}g) = O(\epsilon) \), where \( T \) is a Cauchy-Green operator (Behnke & Stein 1949). Hence, the function \( \tilde{f} = g - T(\chi \cdot \bar{\partial}g) \in \mathcal{O}(K) \) approximates \( f \) to a precision \( O(\epsilon) \) on \( K \).
Sakai’s proof can be used for a Stein compact $K$ in any complex manifold provided we have solution operators for the $\bar{\partial}$-equation satisfying the same bounds on a suitable basis of Stein neighborhoods of $K$. Here is a brief summary when $K$ is the closure of a pseudoconvex domain $D$ in $\mathbb{C}^n$:

Henkin 1969, Lieb 1969, Kerzman 1971: MP holds for strongly pseudoconvex domains $D$ with smooth boundary: every function in $\mathcal{A}^r(D) = \mathcal{C}^r(\overline{D}) \cap \mathcal{O}(D)$ is a $\mathcal{C}^r(\overline{D})$-limit of functions in $\mathcal{O}(\overline{D})$.

Fornæss 1976: $C^2$ boundary suffices for $C^0$ approximation.

Beatrous & Range 1980 MP holds for $f \in \mathcal{A}(D)$ if $D$ is weakly pseudoconvex and $f$ can be approximated on a neighborhood of the set of weakly pseudoconvex boundary points (the degeneration set).

Diederich & Fornæss 1976: MP fails on a worm domain.

Laurent-Thiébaut & F. 2007: MP holds on a smooth pseudoconvex domain whose degeneration set is a Levi flat hypersurface with Levi foliation defined by a closed nowhere vanishing 1-form.
A submanifold $M$ of a complex manifold $X$ is **totally real** if $T_p M \cap i T_p M = \{0\}$ for all $p \in M$. Models are $\mathbb{R}^m \subset \mathbb{R}^n \subset \mathbb{C}^n$.

**Weierstrass 1885** Every continuous function on a compact interval in $\mathbb{R}$ is a uniform limit of entire functions on $\mathbb{C}$.

**Carleman 1927, Alexander 1979, Gauthier and Zeron 2002**
Approximation in the fine topology on curves in $\mathbb{C}^n$.

**Range & Siu, 1974** $C^k$ approximation by holomorphic functions and $\bar{\partial}$-closed forms on $C^k$ totally real submanifolds of a complex manifold.

**Baouendi & Treves 1981** Local approximation of CR functions on CR submanifolds by entire functions (Gaussian kernel method).

**F., Løw, Øvrelid 2001** Approximation of $\bar{\partial}$-flat functions in tubes around totally real manifolds (Henkin type integral kernel method).

**Manne 1993, Manne, Øvrelid, Wold 2011** Carleman approximation on totally real submanifolds (Gaussian kernel method).
Mergelyan approximation on handlebodies

A compact set $S$ in a complex manifold $X$ is **admissible** if $S = K \cup M$, where $S$ and $K$ are Stein compacts and $M = S \setminus K$ is a totally real submanifold of $X$: $T_pM \cap iT_pM = \{0\}$ for all $p \in M$.

**Theorem (Hörmander & Wermer 1968; F. 2005; Manne, Øvrelid, Wold 2011; Fornæss, F., Wold 2018)**

Let $X$ be a complex manifold.

1. Let $S = K \cup M$ be an admissible set in $X$ with $M$ of class $C^k$. Then for any $f \in C^k(S) \cap \mathcal{O}(K)$ there exists a sequence $f_j \in \mathcal{O}(S)$ such that $\lim_{j \to \infty} \|f_j - f\|_{C^k(S)} = 0$.

2. Assume in addition that $K = \overline{D}$ is the closure of a strongly pseudoconvex domain $D \subset X$. Given $f \in C(S) \cap \mathcal{A}(D)$ there is a sequence $f_j \in \mathcal{O}(S)$ such that $\lim_{j \to \infty} \|f_j - f\|_{C(S)} = 0$.

If $f \in C^k(S)$, the convergence takes place in $C^k(S)$.

The analogous results hold for manifold-valued maps.
The use of the Gaussian kernel

We define the bilinear form \( \langle \cdot, \cdot \rangle \) on \( \mathbb{C}^n \) by

\[
\langle z, w \rangle = \sum_{i=1}^{n} z_i w_i, \quad z^2 = \langle z, z \rangle = \sum_{i=1}^{n} z_i^2. \tag{1}
\]

Consider first the standard real subspace \( \mathbb{R}^n \) of \( \mathbb{C}^n \). Recall that

\[
\int_{\mathbb{R}^n} e^{-x^2} \, dx = \int_{\mathbb{R}^n} e^{-\sum_{i=1}^{n} x_i^2} \, dx_1 \cdots dx_n = \left( \int_{\mathbb{R}} e^{-t^2} \, dt \right)^n = \pi^{n/2}.
\]

Hence \( \int_{\mathbb{R}^n} e^{-x^2/\epsilon^2} \, dx = \epsilon^n \pi^{n/2} \), so the family \( \pi^{-n/2} e^{-n} e^{-x^2/\epsilon^2} \) is an approximate identity on \( \mathbb{R}^n \). Given \( f \in \mathcal{C}_0^k(\mathbb{R}^n) \), the entire functions

\[
f_{\epsilon}(z) = \frac{1}{\pi^{n/2} \epsilon^n} \int_{\mathbb{R}^n} f(x) e^{-(x-z)^2/\epsilon^2} \, dx, \quad z \in \mathbb{C}^n, \ \epsilon > 0,
\]

satisfy \( f_{\epsilon} \to f \) in the \( \mathcal{C}^k(\mathbb{R}^n) \) norm as \( \epsilon \to 0 \).

It is remarkable that the same procedure gives local approximation in the \( \mathcal{C}^k \) norm on any totally real submanifold of class \( \mathcal{C}^k \).
Approximation on graphs with small Lip norm

Let $\mathbb{B}_R^n \subset \mathbb{R}^n$ denote the unit ball and $\mathbb{B}_R^n(\epsilon) = \epsilon \mathbb{B}_R^n$.

**Lemma**

Let $\psi : \mathbb{B}_R^n \rightarrow \mathbb{R}^n$ be a $C^k$ map ($k \in \mathbb{N}$) with $\psi(0) = 0$, $(d\psi)_0 = 0$, and set $\phi(x) = x + i\psi(x) \in \mathbb{C}^n$. Then there exists $0 < \delta < 1$ such that the following holds. Let $N \subset \mathbb{B}_R^n$ be a closed set,

$$M = \phi(\mathbb{B}_R^n(\delta) \cap N) \subset \mathbb{C}^n, \quad bM = \phi(b\mathbb{B}_R^n(\delta) \cap N).$$

Given $f \in C_0(M)$, there exist entire functions $f_\epsilon \in O(\mathbb{C}^n)$ ($\epsilon > 0$) satisfying the following conditions as $\epsilon \to 0$:

(a) $f_\epsilon \rightarrow f$ uniformly on $M$, and

(b) $f_\epsilon \rightarrow 0$ on a neighborhood of $bM$.

Moreover, if $N$ is a $C^k$-smooth submanifold of $\mathbb{B}_R^n$ and $f \in C_0^k(M)$, then the approximation in (a) may be achieved in the $C^k$-norm on $M$.

Since functions on $N$ extend to $\mathbb{B}_R^n$ in the appropriate classes, it suffices to prove the lemma in the case $N = \mathbb{B}_R^n$. 
Proof of the lemma

We will need the following fact.

Hörmander 1976 If $A$ is a symmetric $n \times n$ complex matrix with positive definite real part, then

\[
\int_{\mathbb{R}^n} e^{-\langle Au, u \rangle} \, du = \pi^{n/2} (\det A)^{-1/2}.
\]

We shall use this with the matrix

\[
A(x) = \phi'(x)^T \phi'(x), \quad \mathcal{R}A(x) = I - \psi'(x)^T \psi'(x).
\]

Since $\psi'(0) = 0$, there is a number $0 < \delta_0 < 1$ such that $\mathcal{R}A(x) > 0$ is positive definite for all $x \in B^n_{\mathbb{R}}(\delta_0)$, and $\psi$ is Lip-$\alpha$ with $\alpha < 1$ on $B^n_{\mathbb{R}}(\delta_0)$. By using a smooth cut-off function, we extend $\psi$ to $\mathbb{R}^n$ such that $\text{supp}(\psi) \subset B^n_{\mathbb{R}}$, without changing its values on $B^n_{\mathbb{R}}(\delta_0)$.

We will show that the lemma holds for any number $\delta$ with $0 < \delta < \delta_0$. 

Proof of condition (b)

Set
\[ f_\epsilon(z) = \frac{1}{\pi^{n/2}\epsilon^n} \int_M f(w)e^{-(w-z)^2/\epsilon^2}dw, \quad z \in \mathbb{C}^n. \]

Writing \( z = x + iy \in \mathbb{C}^n \) and \( w = u + iv = \mathbb{C}^n \), we have that
\[ \left| e^{-(w-z)^2} \right| = e^{-\Re(w-z)^2} = e^{(y-v)^2-(x-u)^2}. \]

For a fixed \( w = u + iv \in \mathbb{C}^n \) let
\[ \Gamma_w = \{z = x + iy \in \mathbb{C}^n : (y-v)^2 < (x-u)^2\}. \]

On \( \Gamma_w \), the function \( e^{-(w-z)^2/\epsilon^2} \) converges to zero as \( \epsilon \to 0 \). Since \( \psi \) is Lip-\( \alpha \) with \( \alpha < 1 \) on \( B^n_{\mathbb{R}}(\delta) \), we have
\[ M \setminus \{w\} \subset \Gamma_w, \quad \forall w \in M = \phi(B^n_{\mathbb{R}}(\delta)). \]

Given \( f \in C_0(M) \) there is an open neighborhood \( U \subset \mathbb{C}^n \) of \( bM \) with \( U \subset \bigcap_{w \in \text{supp}(f)} \Gamma_w \). This establishes condition (b).
Proof of condition (a)

Fix a point \( z_0 = \phi(x_0) \in M \) with \( x_0 \in B^n_{\mathbb{R}}(\delta) \). We have that

\[
f_{\epsilon}(z_0) = \frac{1}{\pi^{n/2} \epsilon^n} \int_M f(z) e^{-(z-z_0)^2/\epsilon^2} \, dz
= \frac{1}{\pi^{n/2} \epsilon^n} \int_{\mathbb{R}^n} f(\phi(x)) e^{-(\phi(x) - \phi(x_0))^2/\epsilon^2} \det \phi'(x) \, dx
= \frac{1}{\pi^{n/2}} \int_{\mathbb{R}^n} f(\phi(x_0 + \epsilon u)) e^{-[u + i(\psi(x_0 + \epsilon u) - \psi(x_0))/\epsilon]^2} \det \phi'(x_0 + \epsilon u) \, du.
\]

The Lipschitz condition on \( \psi \) gives

\[
|e^{-[u + i(\psi(x_0 + \epsilon u) - \psi(x_0))/\epsilon]^2}| \leq e^{-(1-\alpha)|u|^2}
\]

for all \( x_0 \in B^n_{\mathbb{R}}(\delta) \) and \( 0 < \epsilon < \delta_0 - \delta \). By dominated convergence,

\[
\lim_{\epsilon \to 0} f_{\epsilon}(z_0) = \pi^{-n/2} \int_{\mathbb{R}^n} f(\phi(x_0)) e^{-\langle \phi'(x_0) u, \phi'(x_0) u \rangle} \det \phi'(x_0) \, du
= \pi^{-n/2} \int_{\mathbb{R}^n} f(z_0) e^{-\langle \phi'(x_0)^T \phi'(x_0) u, u \rangle} \det \phi'(x_0) \, du = f(z_0).
\]

The last equality follows from (*) applied with the matrix

\[
A = \phi'(x_0)^T \phi'(x_0), \quad \det A = \det \phi'(x_0)^2.
\]
Proof of Mergelyan approximation on handlebodies

Recall that $S = K \cup M$ is an admissible set in $X$, with $M$ a totally real submanifold of class $C^k$, and $f \in C^k(X) \cap \mathcal{O}(K)$.

Special case: $\text{supp} (f) \subset M = S \setminus K$.

- By using a partition of unity, we may assume that $\text{supp} (f) \subset M_0 \subset M$ where $M_0$ is small enough such that the lemma holds on $M_0$.

- Hence, we get a neighborhood $V \subset X$ of $M_0$ and $f_j \in \mathcal{O}(V)$ with $f_j \to f$ on $M_0$ and $f_j \to 0$ in a neighborhood $U$ of $bM_0$ as $j \to \infty$.

- Since $S$ is a Stein compact, we can solve the Cousin-I problem on a Cartan pair $(A, B)$, where $S \setminus M_0 \subset A$ and $M_0 \subset B \subset V$ are open sets such that $A \cap B \subset U$ and $A \cup B$ is a Stein domain. That is, we glue $f_j \in \mathcal{O}(B)$ with the zero function on $A$.

- This gives $C^k$ approximation of $f$ on $M_0$ by functions $F_j \in \mathcal{O}(S)$ converging to 0 on $S \setminus \text{supp} (f) \supset S \setminus M_0$. 
General case: supp $(f)$ intersects $K$.
Let $U \supset K$ be an open neighborhood such that $f|_U \in \mathcal{O}(U)$.

- Since $S$ is a Stein compact and we can (by the special case) approximate smooth functions supported on $M = S \setminus K$ by functions in $\mathcal{O}(S)$ which are small on $K$, there is a Stein neighborhood $\Omega$ of $S$ such that $K_0 := \hat{K}_{\mathcal{O}(\Omega)} \subset U$.

- Pick an $\mathcal{O}(\Omega)$-convex compact set $K_1 \subset U$ with $K_0 \subset \hat{K}_1$ and a smooth cut-off function $\chi$ with supp $(\chi) \subset K_1$ and $\chi = 1$ near $K_0$.

- By Oka-Weil, there exist functions $g_j \in \mathcal{O}(\Omega)$ such that $g_j \to f$ uniformly on $K_1$ as $j \to \infty$.

- It follows that

$$\tilde{f}_j := \chi g_j + (1 - \chi)f = g_j + (1 - \chi)(f - g_j) \to f \quad \text{as } j \to \infty.$$ 

As $g_j \in \mathcal{O}(S)$, it remains to approximate the functions $(1 - \chi)(f - g_j) \in \mathcal{C}^k(S)$ (whose support does not intersect $K_0 \supset K$) by functions in $\mathcal{O}(S)$ that are small outside their support.
Mergelyan approximation of manifold-valued maps

The following simple lemma is useful in reducing the Mergelyan approximation problem for manifold-valued maps to the case of functions.

**Lemma**

Assume that $K \subset X$ is a compact set with MP, i.e., $\mathcal{A}(K) = \overline{\mathcal{O}}(K)$. Let $Y$ be a complex manifold and $f \in \mathcal{A}(K, Y)$. If the graph $G_f = \{(x, f(x)) : x \in K\}$ has a Stein neighborhood, then $f \in \overline{\mathcal{O}}(K, Y)$.

**Proof.**

Let $W \subset X \times Y$ be a Stein neighborhood of the graph $G_f$.

**Docquier-Grauert 1960** There are a proper holomorphic embedding $\iota : W \hookrightarrow \mathbb{C}^N$ and a holomorphic retraction $\rho : \Omega \to W$ from an open Stein neighborhood $\Omega \subset \mathbb{C}^N$.

Assuming that $\mathcal{A}(K) = \overline{\mathcal{O}}(K)$, we can approximate the map $K \ni x \mapsto \iota(x, f(x)) \in \mathbb{C}^N$ by holomorphic maps $F : U \to \Omega$ on open neighborhoods $U \subset X$ of $K$.

The map $pr_Y \circ \iota^{-1} \circ \rho \circ F : U \to Y$ then approximates $f$ on $K$. □
Theorem (E. Poletsky 2013)

Let $K$ be a Stein compact in a complex manifold $X$ and let $f \in \mathcal{A}(K, Y)$, where $Y$ is an arbitrary complex manifold. Assume that every point $p \in K$ has a neighborhood $V_p \subset X$ such that

$$f|_{K \cap \overline{V}_p} \in \mathcal{O}(K \cap \overline{V}_p).$$

Then the graph $G_f$ is a Stein compact in $X \times Y$.

Poletsky’s proof is similar in spirit to the proof of Siu’s theorem (1976) that every Stein subvariety in a complex space admits a basis of open Stein neighborhoods.

It relies on the technique of fusing plurisubharmonic functions.
Mergelyan’s theorem for manifold-valued maps

**Corollary**

- If a compact set $K$ in a Riemann surface $X$ has the MP for functions, it has the MP for maps to an arbitrary complex manifold.
- The same holds if $K$ is a Stein compact with $C^1$ boundary in an arbitrary complex manifold $X$.
- In particular, a compact strongly pseudoconvex Stein domain satisfies MP for maps to any complex manifold.

**Proof.**

Let $f \in \mathcal{A}(K, Y)$. We cover $K$ by the interiors of closed coordinate discs $D_1, \ldots, D_k$ such that each $f(D_j)$ is contained in a coordinate chart of $Y$. Since $K$ has MP for functions, the theorem of Boivin and Jiang (2004) shows that $f|_{K \cap D_j} \in \mathcal{O}(K \cap D_j, Y)$ for every $j = 1, \ldots, k$.

By Poletsky’s theorem the graph $G_f$ is a Stein compact. The assumption that $K$ has MP then implies $f \in \overline{\mathcal{O}}(K, Y)$ in view of the lemma.

If $K$ has $C^1$ boundary then it obviously satisfies the local Mergelyan property for functions, so the above proof applies.
Carleman approximation on totally real manifolds

Let $X$ be a Stein manifold and $M \subset X$ be a closed subset. Set

$$\widehat{M}_{\mathcal{O}(X)} = \widehat{M} = \bigcup_{j=1}^{\infty} \widehat{M}_j$$

where $M_j$ is a normal exhaustion of $M$ by compacts.

The set $M$ has **bounded exhaustion hulls** if for any compact $K \subset X$ there is a bigger compact $K' \subset X$ such that

$$\widehat{K \cup M} \subset (K \cup M) \cup K'.$$

For closed sets $M \subset \mathbb{C}$ this is the classical **Arakelyan condition**.

**Manne 1993** If $X$ is Stein and $M \subset X$ is a closed $C^k$ totally real submanifold that is $\mathcal{O}(X)$-convex and has bounded exhaustion hulls, then $M$ admits $C^k$-Carleman approximation by entire functions.

**Magnusson & Wold 2016** If closed holomorphically convex set $M$ in a Stein manifold $X$ admits $C^0$ Carleman approximation, then $M$ has bounded exhaustion hulls.
Theorem (Chenoweth 2018)

Let $X$ be a Stein manifold and $Y$ be an Oka manifold. Assume that $K \subset X$ is a compact $\mathcal{O}(X)$-convex subset and $M \subset X$ is a closed totally real submanifold of class $\mathcal{C}^k$ ($k \in \mathbb{N}$) which is $\mathcal{O}(X)$-convex, has bounded exhaustion hulls, and such that $S = K \cup M$ is $\mathcal{O}(X)$-convex.

Then, every map $f \in \mathcal{C}^k(X, Y)$ which is $\bar{\partial}$-flat to order $k$ on $S$ and holomorphic on a neighbourhood of $K$ can be approximated in the fine $\mathcal{C}^k$ topology on $S$ by holomorphic maps $F: X \to Y$.

The proof combines most of the methods presented above:

- Mergelyan approximation of functions on handlebodies,
- existence of Stein neighborhoods of graphs over handlebodies (this gives Mergelyan approximation of manifold-valued maps),
- the techniques of Oka theory.
Arakelyan theorem for maps from plane domains to compact homogeneous manifolds

**Arakelyan 1964–1971** The following conditions are equivalent for a closed set $E$ in a domain $X \subset \mathbb{C}$:

(a) Every function in $\mathcal{A}(E)$ is a uniform limit of functions in $\mathcal{O}(X)$.

(b) The complement $X^* \setminus E$ of $E$ in the one point compactification $X^* = X \cup \{\ast\}$ of $X$ is connected and locally connected. (Equivalently, $E$ has bounded exhaustion hulls.)

**Scheinberg 1978** Generalization to $X$ an open Riemann surface.

---

**Theorem (F. 2018)**

Assume that $Y$ is a compact complex homogeneous manifold. If $E$ is an Arakelyan set in a domain $X \subset \mathbb{C}$, then every continuous map $X \rightarrow Y$ which is holomorphic in $\tilde{E}$ can be approximated uniformly on $E$ by holomorphic maps $X \rightarrow Y$. 
THANK YOU

FOR YOUR ATTENTION