Divisors defined by noncritical functions

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Before the main feature, a preview...

I have recently proved the following result which answers a question asked by Filippo Bracci (private communication, June 2017).

Theorem (`arxiv.org/abs/1709.01028`)

For every $n > 1$ and $\epsilon > 0$ there exists a proper holomorphic embedding $F\colon \mathbb{D} \hookrightarrow \mathbb{B}^n$ which extends to an injective real analytic immersion $F\colon \overline{\mathbb{D}} \setminus \{1\} \to \overline{\mathbb{B}}^n$ such that $\mathrm{Area}(F(\mathbb{D})) < \epsilon$ and the boundary curve $F(b\mathbb{D} \setminus \{1\})$ is everywhere dense in the sphere $b\mathbb{B}^n$.

Berndtsson 1980 An embedded holomorphic disc of finite area in the ball $\mathbb{B}^2 \subset \mathbb{C}^2$ is the zero set of a bounded holomorphic function on $\mathbb{B}^2$.

Corollary

There is a bounded holomorphic function on $\mathbb{B}^2$ whose zero set is a smooth curve of finite area, biholomorphic to the disc, and its boundary curve is dense in the sphere $b\mathbb{B}^2$.

The main feature: arxiv.org/abs/1709.05147

I will present some new results on the classical subject of **complete intersections in Stein manifolds**. Let us begin with the following result on principal divisors.

Theorem (1)

Let A be a closed complex hypersurface in a Stein manifold X. Assume that there is a continuous function h on X which is holomorphic in a neighborhood of A and whose divisor equals $A = h^{-1}(0)$. Then there exists a holomorphic function f on X whose divisor equals A and whose critical points are precisely the singular points of A:

$$(f) = A, \qquad \mathrm{Crit}(f) = A_{\mathrm{sing}}.$$

Note that the family $\{f^{-1}(c) : c \in \mathbb{C}\}$ is a foliation of X by closed complex hypersurfaces all which, except perhaps the zero fibre $A = f^{-1}(0)$, are smooth. In particular, if the hypersurface A is smooth, it is defined by a holomorphic function without any critical points on X.

Background: Oka's theorem from 1939

What is new in this theorem is that the defining function $f \in \mathcal{O}(X)$ of the hypersurface A can be chosen to have no critical points in $X \setminus A$. Note that if the divisor of $f \in \mathcal{O}(X)$ equals A, then a point $x \in A$ is a critical point of f if and only if x a singular point of A.

Without the condition that f be noncritical on $X \setminus A$, the result is due to **Oka 1939** and follows from the isomorphisms

$$\mathrm{Div}(X)/\sim \ \cong H^1(X; \mathcal{O}_X^*) \cong H^2(X; \mathbb{Z}) \cong H^1(X; \mathcal{C}_X^*).$$

A second Cousin problem on a Stein manifold is solvable by holomorphic functions if it is solvable by continuous functions.

These isomorphisms follow from the cohomology sequences associated to the short exact sequences of sheaf homomorphisms:

$$0 \longrightarrow \mathcal{O}_X^* \hookrightarrow \mathcal{M}_X^* \longrightarrow Div_X \longrightarrow 0$$

$$0 \longrightarrow \mathbb{Z} \hookrightarrow \mathcal{O}_X \xrightarrow{e^{2\pi i \cdot}} \mathcal{O}_X^* \longrightarrow 1.$$

A corollary

Let us consider the special case when $H^2(X;\mathbb{Z}) = 0$. It follows that

$$H^1(X;\mathcal{O}_X^*) = 0$$

which means that every divisor on X is a principal divisor.

Hence, every complex hypersurface in X is defined by a single holomorphic equation. This gives the following corollary:

Corollary

If X is a Stein manifold with $H^2(X;\mathbb{Z}) = 0$, then for every closed complex hypersurface A in X there is a function $f \in \mathcal{O}(X)$ whose divisor equals A and whose critical points are precisely the singular points of A.

Complete intersections of higher codimension

Theorem (2)

Assume that X is a Stein manifold of dimension $n > 1$, $q \in \{1, \ldots, n-1\}$, and $h = (h_1, \ldots, h_q)\colon X \to \mathbb{C}^q$ is a continuous map which is a holomorphic submersion in a neighborhood of its zero fibre $A = h^{-1}(0)$. If there exists a q-tuple of continuous $(1,0)$-forms $\theta = (\theta_1, \ldots, \theta_q)$ on X which are pointwise linearly independent at every point of X and agree with $dh = (dh_1, \ldots, dh_q)$ in a neighborhood of A, then there is a holomorphic submersion $f\colon X \to \mathbb{C}^q$ such that $A = f^{-1}(0)$ and $f - h$ vanishes to a given finite order on A.

Such θ always exists if $q \le \frac{n+1}{2}$; equivalently, if $\dim A \ge \left[\frac{n}{2}\right]$.

Note that the submersion $f\colon X \to \mathbb{C}^q$ defines a nonsingular holomorphic foliation on X by closed embedded complete intersection submanifolds $A_c = \{f = c\}$ $(c \in \mathbb{C}^q)$ of codimension q, with $A_0 = A$.

A remark and a question

One may wonder whether Theorem 2 also holds under the weaker assumption that the map $h = (h_1, \ldots, h_1)\colon X \to \mathbb{C}^q$ is holomorphic in a neighborhood of the zero-fibre $A = h^{-1}(0)$ and generates the ideal of A at every point. We do not know the answer if A has singularities.

When $q = 1$, a generic perturbation of the function h in Theorem 1 which is fixed to the second order on A yields a holomorphic function without critical points in a deleted neighborhood of A in X. This is a starting point of our analysis.

One the other hand, if $q > 1$ then a generic choice of the map $h\colon U \to \mathbb{C}^q$ with $h^{-1}(0) = A$ may have branch locus of dimension $q - 1 > 0$ in a deleted neighborhood of A. If h can be chosen to have no critical points near A, then it can be extended to all of X such that it does not have any critical points on $X \setminus A$.

Analysis of the conditions in Theorem 2

Assume that A is a closed complex submanifold (not necessarily connected) of pure codimension q in a Stein manifold X. Then, the normal bundle $N_{A/X}$ is trivial if and only if there is a neighborhood $U \subset X$ of A and a holomorphic submersion $h \colon U \to \mathbb{C}^q$ such that

$$A = h^{-1}(0), \quad dh_1 \wedge \cdots \wedge dh_q \neq 0.$$

The nontrivial (only if) direction follows from the tubular neighborhood theorem of **Docquier and Grauert 1960**. Assume that this holds.

The second assumption on h in Theorem 2, namely that the differential $dh = (dh_1, \ldots, dh_q)$ extends to a q-tuple of pointwise linearly independent $(1,0)$-forms $(\theta_1, \ldots, \theta_q)$ on X, always holds if

$$q = n - \dim A \leq \frac{n+1}{2} \iff \dim A \geq \left[\frac{n}{2}\right].$$

We use that a Stein manifold X is homotopy equivalent to a CW complex of dimension $\leq \dim_{\mathbb{C}} X$ and a general position argument.

Some results of Forster and Ramspott

Forster & Ramspott 1966 If h extends from a neighborhood of A to a continuous map $h\colon X \to \mathbb{C}^q$ satisfying $h^{-1}(0) = A$, then h can be deformed to a holomorphic map $f = (f_1, \ldots, f_q)\colon X \to \mathbb{C}^q$ that agrees with h to a given finite order on A and satisfies $f^{-1}(0) = A$.

This is an application of the **Oka-Grauert principle** for maps to $\mathbb{C}^q \setminus \{0\}$.

Theorem (Forster and Ramspott 1966)

Let A be a closed complex submanifold of a Stein manifold X.

(a) *If $\dim A < \frac{1}{2} \dim X$ and the normal bundle $N_{A/X}$ is trivial, then A is a complete intersection in X.*

(b) *For $X = \mathbb{C}^n$, the same conclusion holds if $\dim A \le \frac{2}{3}(n-1)$.*

(c) *If $X = \mathbb{C}^n$ with $n \le 6$ then A is a complete intersection if and only if $c_1(A) = 0$ (i.e., the first Chern class of A vanishes).*

A Corollary to Theorem 2

Combining these results, we obtain items (1)–(3) in the following corollary. Item (4) relies on a result of **Forster and Ohsawa 1985**.

Corollary

1. *If X is a Stein manifold of dimension $n = 2k + 1$ and $A \subset X$ is a closed complex submanifold of dimension k with trivial normal bundle, then there exists a holomorphic submersion $f\colon X \to \mathbb{C}^{k+1}$ such that $A = f^{-1}(0)$.*
2. *If A is a closed complex submanifold of $\mathbb{C}^n$ with trivial normal bundle and $\left[\frac{n}{2}\right] \le \dim A \le \frac{2}{3}(n-1)$, then A is the zero fibre of a holomorphic submersion $f\colon \mathbb{C}^n \to \mathbb{C}^{n-\dim A}$.*
3. *If A is a closed complex submanifold of $\mathbb{C}^n$ with $\left[\frac{n}{2}\right] \le \dim A < n \le 6$, then A satisfies (2) if any only if $c_1(A) = 0$.*
4. *If A is an algebraic submanifold of $\mathbb{C}^n$ of pure codimension 2 with topologically trivial canonical bundle (for example, a smooth algebraic curve in $\mathbb{C}^3$), then A is the zero fibre of a holomorphic submersion $f = (f_1, f_2)\colon \mathbb{C}^n \to \mathbb{C}^2$.*

Low dimensions

Example

If $A \subset \mathbb{C}^n$ is a closed complex submanifold of pure dimension k with trivial normal bundle, then A is the zero fibre of a holomorphic submersion $\mathbb{C}^n \to \mathbb{C}^{n-k}$ in each of the following cases:

$$k = 1,\ n \in \{2, 3\}; \quad k = 2,\ n \in \{4, 5\}; \quad k = 3,\ n \in \{6, 7\}.$$

Smooth hypersurfaces containing a given submanifold

Clearly, a complete intersection submanifold $A \subset X$ in Theorem 2 is contained in a smooth hypersurface $H \subset X$. Indeed, if $A = f^{-1}(0)$ for a holomorphic submersion $f = (f_1, \ldots, f_q) \colon X \to \mathbb{C}^q$ then the preimage $H = f^{-1}(H')$ of any smooth complex hypersurface $H' \subset \mathbb{C}^q$ with $0 \in H'$ is such. The proof of Theorem 1 also gives the following result which holds in a bigger range of dimensions than Theorem 2.

Corollary

Let A be a closed complex submanifold in a Stein manifold X. If

$$3 \dim A + 1 \leq 2 \dim X$$

then there exists a holomorphic foliation of X by closed complex hypersurfaces such that A is contained in one of the leaves.

In particular, A is contained in a smooth complex hypersurface.

The last statement is due to **Jelonek and Kucharz 2016**.

Smooth hypersurfaces containing a given submanifold

The proof of the last Corollary goes as follows. The assumption on $k = \dim A$ is equivalent to

$$q := \dim X - k > \left[\frac{k}{2}\right].$$

It follows that the conormal bundle $N^*_{A/X} \to A$ admits a nonvanishing holomorphic section ξ over A.

By the Docquier-Grauert theorem there is a holomorphic function h in an open neighborhood of A that vanishes on A and satisfies $dh_x = \xi_x \neq 0$ for all $x \in A$. Our proof then furnishes a function $f \in \mathcal{O}(X)$ without critical points that agrees with h to the second order on A.

Then, $\{f^{-1}(c) : c \in \mathbb{C}\}$ is a foliation of X by closed complex hypersurfaces such that $A \subset f^{-1}(0)$.

Approximation of noncritical nonvanishing maps

In the proof of the main results we need the following approximation theorem for holomorphic submersions between Euclidean spaces whose range avoids the origin. Without the condition on the range, these results are proved in my paper in Acta Math. 2003.

Theorem

Let K be a compact convex set in $\mathbb{C}^n$, $q \in \{1, \ldots, n-1\}$, and let $f \colon U \to \mathbb{C}^q \setminus \{0\}$ be a holomorphic submersion on a neighborhood $U \subset \mathbb{C}^n$ of K. Given $\epsilon > 0$ there exists a holomorphic submersion $g \colon \mathbb{C}^n \to \mathbb{C}^q \setminus \{0\}$ such that $\sup_K |f - g| < \epsilon$.

We may assume that the set U is convex. Consider first the case $q = 1$. Let $h \colon U \to \mathbb{C}$ be a holomorphic logarithm of f, so $f = e^h$. Then, $df = e^h dh$ and hence h has no critical points on U. By F. (2003) we can approximate h as closely as desired uniformly on K by a holomorphic function $\tilde{h} \colon \mathbb{C}^n \to \mathbb{C}$ without critical points on $\mathbb{C}^n$. The function $g = e^{\tilde{h}} \colon \mathbb{C}^n \to \mathbb{C} \setminus \{0\}$ then satisfies the conclusion of the theorem.

The case $q > 1$

In this case, the individual components of f may have zeros on U. Instead, we proceed as follows.

Pick a compact convex set $L \subset U$ with $K \subset \mathring{L}$ and approximate f uniformly on L by a polynomial map $P\colon \mathbb{C}^n \to \mathbb{C}^q$. Assuming that the approximation is sufficiently close and P is chosen generic, the set

$$\Sigma = \{z \in \mathbb{C}^n : P(z) = 0 \ \text{ or } \ \mathrm{rank}\, dP_z < q\}$$

is an algebraic subvariety of $\mathbb{C}^n$ of codimension $\geq \min\{q, n - q + 1\} \geq 2$ which does not intersect K. Hence, there is a biholomorphic map

$$\phi\colon \mathbb{C}^n \to \phi(\mathbb{C}^n) \subset \mathbb{C}^n \setminus \Sigma$$

which approximates the identity as closely as desired uniformly on K (this follows from Andersén-Lempert theory). The map

$$g = P \circ \phi\colon \mathbb{C}^n \to \mathbb{C}^q \setminus \{0\}$$

is then a holomorphic submersion approximating f on K.

Proof of Theorem 1

We begin by recalling some basic facts from analytic geometry.

A complex hypersurface A in a complex manifold X is locally at each point $x_0 \in A$ the zero set of a single holomorphic function h which generates the ideal $\mathscr{J}_{A,x}$ of A at every point $x \in A$ in an open neighborhood of x_0.

Every other holomorphic function g on X near x_0 that vanishes on A is divisible by h, i.e. $g = uh$ for some holomorphic function u in a neighborhood of x_0. The function g also generates $\mathscr{J}_{A,x_0}$ if and only if $u(x_0) \neq 0$. In particular, if the difference $g - h$ belongs to the square $\mathscr{J}^2_{A,x_0}$ of the ideal $\mathscr{J}_{A,x_0}$ then g is another local generator of $\mathscr{J}_{A,x_0}$.

We say that g agrees with h to order $r \in \mathbb{Z}_+ = \{0, 1, 2, \ldots\}$ on A if their difference $g - h$ is a section of the sheaf $\mathscr{J}^{r+1}_A$ on their common domain of definition.

Proof of Theorem 1

We make the following assumptions:

- X is a Stein manifold,
- A is a closed complex hypersurface in X,
- $K \subset L$ are compact $\mathcal{O}(X)$-convex subsets of X with $K \subset \mathring{L}$, and
- h is a holomorphic function in a neighborhood $U \subset X$ of A which generates the ideal sheaf $\mathscr{J}_A$ at every point.

Theorem 1 follows from the following Lemmas by an obvious induction with respect to an exhaustion

$$K_1 \subset K_2 \subset \cdots \subset \bigcup_{j=1}^{\infty} K_j = X$$

of X by compact $\mathcal{O}(X)$-convex sets such that $K_1 \subset U$ and $K_j \subset \mathring{K}_{j+1}$ for all $j \in \mathbb{N}$. At the j-step of the induction, apply the lemmas with $K = K_j$ and $L = K_{j+1}$.

The main lemmas

Lemma (1)

Let $r \in \mathbb{N}$. Assume that f is a continuous function on X with $f^{-1}(0) = A$ that is holomorphic on a neighborhood V of K, has no critical points on $V \setminus A$, and agrees with h to order r on $A \cap V$. Then there exists a continuous function g on X with $g^{-1}(0) = A$ that is holomorphic in a neighborhood $W \subset X$ of $K \cup (A \cap L)$, has no critical points on $W \setminus A$, approximates f as closely as desired uniformly on K, and agrees with h to order r along A.

Lemma (2)

Assume that g is a continuous function on X with $g^{-1}(0) = A$ that is holomorphic in a neighborhood $W \subset X$ of $K \cup (A \cap L)$, has no critical points on $W \setminus A$, and agrees with h to order r along A. Given $r \in \mathbb{N}$ there exists a continuous function $\tilde{f} \colon X \to \mathbb{C}$ with $\tilde{f}^{-1}(0) = A$ that is holomorphic on an open neighborhood $\widetilde{V}$ of L, agrees with g to order r along $A \cap \widetilde{V}$, approximates g as closely as desired uniformly on K, and has no critical points in $\widetilde{V} \setminus A$.

Proof of Lemma 1

Since g vanishes on A and agrees with h to order r along $A \cap V$, an application of Cartan's division theorem and the Oka-Weil approximation theorem gives a function $\tilde{f}$ that is holomorphic on a neighborhood W of $K \cup (A \cap L)$, approximates g uniformly on a neighborhood of K as closely as desired, and agrees with h to order r along A.

Assuming as we may that $\tilde{f}$ is close enough to g on a neighborhood of K, it follows that $\tilde{f}$ has no critical points there, except perhaps at the points of A. Note also that $\tilde{f}$ has no zeros in $W \setminus A$ provided that the neighborhood W of $K \cup (A \cap L)$ is chosen small enough.

As shown in [F., JEMS 2016], a generic choice of $\tilde{f}$ has no critical points on $W \setminus A$.

Using a smooth cutoff function we can extend $\tilde{f}$ to a continuous function on X such that $(\tilde{f})^{-1}(0) = A$. This proves Lemma 1.

Proof of Lemma 2

The proof amounts to a finite induction where at every step we enlarge the domain on which the function is holomorphic and noncritical off A.

We begin with the initial function $f_0 = g$ on an initial compact strongly pseudoconvex domain W_0 with $K \cup (A \cap L) \subset \mathring{W}_0 \subset W_0 \subset W$, and the process terminates in finitely many steps by reaching a function $\tilde{f}$ which is holomorphic on a neighborhood of L and satisfies the stated conditions. Every step amounts to one of the following two types of operations:

(a) **The noncritical case:** we attach a small convex bump to a given strongly pseudoconvex domain in X.

(b) **The critical case:** we attach a handle of index $\leq n = \dim X$ to a given strongly pseudoconvex domain in X.

We explain the induction step in each of these two cases.

Proof of Lemma 2

In case (a) we are given a pair of compact sets $D_0, D_1 \subset X$ with the following properties:

- $D = D_0 \cup D_1$ is a Stein compact (i.e., it admits a basis of Stein neighborhoods),
- $\overline{D_0 \setminus D_1} \cap \overline{D_1 \setminus D_0} = \varnothing$, and
- the set $D_1 \subset X \setminus A$ is contained in a holomorphic coordinate chart $V \subset X$ such that $C = D_0 \cap D_1$ is convex in that chart.

Furthermore, we are given a holomorphic function f on a neighborhood of D_0 without zeros or critical points on $D_0 \setminus A$. (This is one of the functions in the inductive construction.)

By the approximation theorem above, we can approximate f uniformly on a neighborhood of C by a holomorphic function ξ on a neighborhood of D_1 which has no zeros or critical points.

Proof of Lemma 2

Assuming that the approximation is close enough, there are a neighborhood U of C and a biholomorphic map $U \to \gamma(U) \subset X$ close to the identity such that

$$f = \xi \circ \gamma \quad \text{holds on} \quad U.$$

By a splitting lemma proved in [F., Acta Math 2003] (a simpler proof is given in my Ergebnisse book, 2nd edn., 2017), there are biholomorphic maps α and β on neighborhoods of D_0 and D_1, close to the identity on their respective domains, such that α is tangent to the identity to order r on $A \cap D_0$ and

$$\gamma \circ \alpha = \beta \ \text{ holds on a neighborhood of } C.$$

Hence $$f \circ \alpha = \xi \circ \beta \quad \text{holds on a neighborhood of } C.$$

This defines a holomorphic function $\tilde{f}$ on a neighborhood of $D = D_0 \cup D_1$ which is close to f on D_0 and is tangent to f to order r along $A \cap D_0$. Furthermore, the construction ensures that $\tilde{f}$ has no zeros or critical points on $D \setminus A$. This completes the induction step in case (a).

Proof of Lemma 2

In case (b) we are given a compact strongly pseudoconvex domain $D_0 \subset X$ with an attached totally real embedded disc $M \subset X \setminus (A \cup \mathring{D}_0)$ whose boundary sphere $bM \subset bD_0 \setminus A$ is complex tangential to bD_0.

Furthermore, $D_0 \cup M$ is a Stein compact admitting small strongly pseudoconvex neighborhoods D (handlebodies with core $D_0 \cup M$).

We are also given a continuous function $f \colon X \to \mathbb{C}$ with $f^{-1}(0) = A$ which is holomorphic near D_0 and has no critical points off A.

By a $\mathscr{C}^0$-small deformation of f, keeping it fixed in a neighborhood of D_0, we may assume that f is smooth and nonvanishing in a neighborhood of M and df_x is $\mathbb{C}$-linear and nonvanishing at every point of M.

Applying the Mergelyan theorem we obtain a holomorphic function $\tilde{f}$ on a neighborhood of $D_0 \cup M$ with the desired properties. We extend it to X by using a cutoff function. This completes the induction step in case (b).

Proof of Theorem 2

The scheme of proof is as in Theorem 1. Step (a) goes through as before. In step (b) we must extend $f = (f_1, \ldots, f_q)$ smoothly across the disc M such that it has no zeros there and its differential is $\mathbb{C}$-linear and of maximal rank q at every point of M. The nonvanishing condition holds if we keep the deformation uniformly sufficiently small.

The second condition concerning the differential holds generically when $\dim M < 2(n - q + 1)$; the number on the right is the real codimension of the variety of $q \times n$ matrices of less than maximal rank. Since $\dim M \leq n$, this holds if $q \leq \frac{n+1}{2}$, so in this case there always exists a q-coframe $(\theta_1, \ldots, \theta_q)$ on X extending $dh = (dh_1, \ldots, dh_q)$.

If $q > \frac{n+1}{2}$ and we already have a q-coframe as in the theorem, we can achieve the required condition on the differential df_x for points $x \in M$ by applying Gromov's h-principle; the partial differential relation controlling this problem is ample in the coordinate directions. This requires a deformation that is big in the $\mathscr{C}^1$ norm but arbitrarily small in the $\mathscr{C}^0$ norm; hence we do not introduce any zeros on M. We complete the proof as before by applying the Mergelyan theorem.