Noncritical holomorphic functions on Stein spaces

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A survey of existing results and open problems
Critical points of holomorphic functions on singular spaces
The main new result
Outline of the proof

References:

1951 K. Stein A complex manifold $X$ is said to be a **Stein manifold** if

- **holomorphic functions separate points:**

  $x, x' \in X, \ x \neq x' \implies f(x) \neq f(x')$ for some $f \in \mathcal{O}(X)$, and

- $X$ is **holomorphically convex**: For every compact set $K \subset X$, its $\mathcal{O}(X)$-convex hull $\tilde{K}_{\mathcal{O}(X)}$ is also compact:

$$\tilde{K}_{\mathcal{O}(X)} = \{x \in X : |f(x)| \leq \sup_{K} |f|, \ \forall f \in \mathcal{O}(X)\}.$$ 

Equivalently, for every discrete sequence $a_j \in X$ there exists a holomorphic function $f$ on $X$ such that $|f(a_j)| \to +\infty$ as $j \to \infty$.

1955 H. Cartan; H. Grauert and R. Remmert

A **Stein space** (or **holomorphically complete space**) is a complex space satisfying these axioms.
Embedding Stein manifolds in Euclidean spaces

1949 **Behnke-Stein** An open Riemann surface is a Stein manifold.

1956-61 **Remmert, Bishop, Narasimhan** A complex manifold $X$ of dimension $n$ is Stein if and only if it is embeddable as a closed complex submanifold of some $\mathbb{C}^N$; one can take $N = 2n + 1$.

Stein manifolds are relatives of affine algebraic manifolds.

1984 **Stout** Every relatively compact domain in a Stein manifold is biholomorphic to a domain in an affine algebraic manifold.

1992 **Eliashberg and Gromov; Schürmann (1997)** A Stein manifold of dimension $n > 1$ is embeddable in $\mathbb{C}^N$ with $N = \left\lceil \frac{3n}{2} \right\rceil + 1$.

1971 **Forster** This $N$ is optimal for every $n > 1$.

**Problem** Is every open Riemann surface biholomorphic to some closed nonsingular embedded complex curve in $\mathbb{C}^2$?

Recent advances: **Wold & Forstnerič; Ritter**.
Noncritical functions on Stein manifolds

1967 **Gunning and Narasimhan** Every open Riemann surface $X$ admits a holomorphic function $f \in \mathcal{O}(X)$ without critical points: $df_x \neq 0$ for all $x \in X$. The map $f : X \to \mathbb{C}$ given by such function is a holomorphic immersion spreading $X$ as a Riemann domain over $\mathbb{C}$.

1986 **Gromov** Does every Stein manifold admit a noncritical holomorphic function? Given a nowhere vanishing holomorphic vector field $L$ on $X$, does there exist $f \in \mathcal{O}(X)$ such that $L(f)$ has no zeros?

2003 **Forstnerič** Every Stein manifold $X$ admit a noncritical holomorphic function. Furthermore, given a discrete set $P \subset X$, there exists $f \in \mathcal{O}(X)$ with $\text{Crit}(f) = P$.

More generally, if $n = \dim X$ then there exist $q = \left[ \frac{n+1}{2} \right] = n - \left[ \frac{n}{2} \right]$ holomorphic functions $f_1, \ldots, f_q \in \mathcal{O}(X)$ such that

$$df_1 \wedge df_2 \wedge \cdots \wedge df_q \neq 0 \quad \text{on } X.$$

This number $q$ is maximal for every $n$ by topological reasons.
The h-principle for holomorphic submersions

2003 F. Let $X$ be a Stein manifold of dimension $n > 1$ and $q \in \{1, \ldots, n - 1\}$. Every continuous complex vector bundle surjection $\Theta : TX \to X \times \mathbb{C}^q$ is homotopic (through complex vector bundle surjections) to the tangent map $Tf$ of a holomorphic submersion $f = (f_1, \ldots, f_q) : X \to \mathbb{C}^q$ $(df_1 \wedge df_2 \wedge \cdots \wedge df_q \neq 0)$.

2004 F. The analogous result hold for submersions $X^n \to Y^q$ to any complex manifold $Y^q$ which satisfies the Runge approximation property for submersions $\mathbb{C}^n \to Y^q$ on compact convex sets $K \subset \mathbb{C}^n$. (The smooth case: Gromov, Philips 1967.)

1986 Gromov h-principle for holomorphic immersions $X^n \to \mathbb{C}^q$: If $q > n \geq 1$ then every complex vector bundle injection $TX \to X \times \mathbb{C}^q$ is homotopic (through complex vector bundle injections) to the tangent map of a holomorphic immersion $X \to \mathbb{C}^q$. Such always exists if $q \geq \left\lceil \frac{3n}{2} \right\rceil = n + \left\lfloor \frac{n}{2} \right\rfloor$.

Problem: Does this h-principle also hold for $q = n > 1$?
Critical points of functions on singular spaces

Let $X$ be a complex space. Notation:

- $\mathcal{O}_{X,x}$ ... the ring of germs of holomorphic function at $x \in X$,
- $m_x$ ... the maximal ideal of $\mathcal{O}_{X,x}$, so we have $\mathcal{O}_{X,x}/m_x \cong \mathbb{C}$.

Given $f \in \mathcal{O}_{X,x}$ we denote by $f - f(x) \in m_x$ the germ obtained by subtracting from $f$ its value $f(x) \in \mathbb{C}$ at $x$.

**Definition**

Assume that $x$ is nonisolated point of a complex space $X$.

(a) A germ $f \in \mathcal{O}_{X,x}$ is said to be **critical**, and $x$ is a **critical point** of $f$, if $f - f(x) \in m_x^2$; $f$ is **noncritical** if $f - f(x) \in m_x \setminus m_x^2$.

(b) A germ $f \in \mathcal{O}_{X,x}$ is **strongly noncritical at** $x$ if the restriction $f|_V$ to any local irreducible component $V$ of $X$ is noncritical at $x$.

(c) Any function is considered noncritical at an isolated point $x \in X$. 
One can characterize these notions by the (non) vanishing of the differential $df_x$ on the Zariski tangent space $T_x X = (m_x/m_x^2)^*$. The differential $df_x : T_x X \rightarrow \mathbb{C}$ of $f \in \mathcal{O}_{X,x}$ is determined by the class

$$f - f(x) \in m_x/m_x^2 = T_x^* X;$$

$f$ is critical at $x$ if and only if $df_x = 0$.

A regular point $x \in X_{\text{reg}}$ is a critical point of $f$ if and only if in some (and hence in any) local holomorphic coordinates $z = (z_1, \ldots, z_n)$ on a neighborhood of $x$, with $z(x) = 0$, we have

$$\frac{\partial f}{\partial z_j}(0) = 0 \quad \text{for } j = 1, \ldots, n.$$

Hence the set $\text{Crit}(f) \cap X_{\text{reg}}$ is a closed complex subvariety of $X_{\text{reg}}$; on a Stein manifold this set is discrete for a generic choice of $f \in \mathcal{O}(X)$. 
The first main result

**Theorem (1: Noncritical functions on Stein spaces)**

On every reduced Stein space $X$ there exists a holomorphic function which is strongly noncritical at every point.

Furthermore, given a closed discrete set $P = \{p_1, p_2, \ldots\}$ in $X$, germs $f_k \in \mathcal{O}_{X,p_k}$ and integers $n_k \in \mathbb{N}$, there exists a function $F \in \mathcal{O}(X)$ which is strongly noncritical at every point of $X \setminus P$ and which agrees with the germ $f_k$ to order $n_k$ at each points $p_k \in P$; i.e.,

$$F_{p_k} - f_k \in m_{p_k}^{n_k}, \quad \forall k.$$  

**Corollary**

Every 1-convex manifold $X$ admits a holomorphic function which is noncritical outside of the maximal compact complex subvariety of $X$. 
The scheme of proof in the nonsingular case

When $X$ is a Stein manifold, the proof (F., Acta Math. 191 (2003) 143–189) relies on two main ingredients:

- Runge approximation theorem for noncritical holomorphic functions on polynomially convex subset of $\mathbb{C}^n$ by entire noncritical functions.

- A splitting lemma for biholomorphic maps on Cartan pairs. This enables one to extend (by approximation) a noncritical function across a noncritical strongly pseudoconvex Runge pair.

- For the h-principle (when constructing several functions with pointwise independent differentials), we also need a method of passing critical points of a Morse exhaustion function (change of topology). This uses the Gromov-Philips h-principle on totally real handles and a reduction to the noncritical case.
Lemma (The Oka-Weil theorem for noncritical functions)

Let \( f \) a noncritical holomorphic function on a neighborhood of a compact polynomially convex set \( K \subset \mathbb{C}^n \). Then \( f \) can be approximated uniformly on \( K \) by noncritical holomorphic functions \( F : \mathbb{C}^n \to \mathbb{C} \).

Proof.

The proof for \( n = 1 \) is an elementary application of Runge’s and Mergelyan’s theorem. Assume now that \( n > 1 \).

- Approximate \( f \) by a generic holomorphic polynomial \( h \in \mathbb{C}^{[n]} \) with finite critical locus \( \text{Crit}(h) = \{ z \in \mathbb{C}^n : dh_z = 0 \} \subset \mathbb{C}^n \setminus K \).
- Use Andersén-Lempert theory to find an injective holomorphic map \( \phi : \mathbb{C}^n \to \mathbb{C}^n \) which is close to the identity map on \( K \) and satisfies \( \phi(\mathbb{C}^n) \cap \text{Crit}(h) = \emptyset \).
- The composition \( F = h \circ \phi : \mathbb{C}^n \to \mathbb{C} \) is then noncritical on \( \mathbb{C}^n \) and it approximates \( f \) uniformly on \( K \).
A splitting lemma for biholomorphic maps

A compact set $K$ in a complex space $X$ is said to be a **Stein compact** if $K$ admits a basis of open Stein neighborhoods in $X$.

**Definition**

A pair $(A, B)$ of compact sets in a complex space $X$ is a **Cartan pair** if $D = A \cup B$ and $C = A \cap B$ are Stein compacts and we have

$$
\overline{A \setminus B \cap B \setminus A} = \emptyset.
$$

**Lemma (The Splitting Lemma)**

Let $(A, B)$ be a Cartan pair in a complex space $X$, with $B \subset X_{\text{reg}}$. For every biholomorphic map $\gamma: U \to \gamma(U) \subset X$ in an open neighborhood $U$ of $C = A \cap B$ which is sufficiently close to $\text{Id}_U$ there exist biholomorphic maps $\alpha, \beta$, close to $\text{Id}$ in small neighborhoods of $A$ and $B$, respectively, such that $\alpha$ is tangent to the identity along $X_{\text{sing}}$ (to any given order) and

$$
\gamma = \beta \circ \alpha^{-1}
$$

holds on a neighborhood of $C$. 


The main induction step in the proof of Theorem 1

**Corollary**

Let \((A, B)\) be a Cartan pair in \(X\) such that \(C = A \cap B\) is \(\mathcal{O}(B)\)-convex and \(B\) is contained in a coordinate chart of \(X\) which is Runge in \(\mathbb{C}^n\). Then every noncritical holomorphic function \(f\) on a neighbourhood of \(A\) can be approximated by noncritical holomorphic functions on a neighbourhood of \(D = A \cup B\).

**Proof.** We may consider \(C \subset B\) as subset of \(\mathbb{C}^n\), with \(C\) polynomially convex. Approximate \(f\) uniformly on a neighbourhood of \(C\) by a noncritical function \(g\) on a neighbourhood of \(B\). Then

\[ f = g \circ \gamma, \]

where \(\gamma\) is a biholomorphic map close to the identity near \(C\). Now \(\gamma = \beta \circ \alpha^{-1}\) by the splitting lemma. Hence

\[ f \circ \alpha = g \circ \beta \]

holds near \(C\), so these functions amalgamate into a noncritical function on \(D\).
Proof of the theorem for nonsingular $X$

We exhaust a Stein manifold $X$ by an increasing sequence of Stein compacts

$$A_1 \subset A_2 \subset \cdots \subset \bigcup_{k=1}^{\infty} A_k = X$$

such that for every $k$ we have $A_{k+1} = A_k \cup B_k$ where $(A_k, B_k)$ is a Cartan pair as in the previous corollary.

Two types of Cartan pairs are needed:
- convex bumps, and
- bones (to change the topology).

We inductively construct a sequence $f_k \in \mathcal{O}(A_k)$ of noncritical functions (or functions with a given critical locus). If the approximation of $f_k$ by $f_{k+1}$ is close enough at every step then

$$F = \lim_{k \to \infty} f_k \in \mathcal{O}(X)$$

satisfies Theorem 1.
Passing a critical point $p_0$ of an exhaustion function $\rho$

To construct several functions with pointwise independent differentials, we also need a method of passing critical points of a Morse exhaustion function. This uses the Gromov-Philips h-principle on totally real handles and a reduction to the noncritical case.
Problems with singular spaces

These tools do not apply directly at singular points of $X$. In addition, the following two phenomena make the analysis substantially more delicate:

- The critical locus of $f \in \mathcal{O}(X)$ need not be a closed complex subvariety of $X$ near a singularity.

Example

A simple (reducible) example is $X = \{zw = 0\} \subset \mathbb{C}^2$, $f(z, w) = z$, $\text{Crit}(f) = \{(0, w): w \neq 0\}$. Here is an irreducible example:

$$X = \{(z_1, z_2, z_3) \in \mathbb{C}^3: h(z) = z_1^2 + z_2^2 + z_3^2 = 0\},$$

$X_{\text{sing}} = 0 \in \mathbb{C}^3$, $T_0X = \mathbb{C}^3$.

For any $\lambda = (\lambda_1, \lambda_2, \lambda_3) \in \mathbb{C}^3_*$ the function

$$f_\lambda(z_1, z_2, z_3) = \lambda_1 z_1 + \lambda_2 z_2 + \lambda_3 z_3,$$

restricted to $X$, is strongly noncritical at $(0, 0, 0)$. If $\lambda \in X^* = X \setminus \{0\}$ then $\text{Crit}(f_\lambda|_X) = \mathbb{C}_* \lambda$ which is not closed.
Non stability of noncritical functions

The class of (strongly) noncritical functions is not stable under small perturbations on compact sets which include singular points of $X$.

Example

Let $X \subset \mathbb{C}^3$ be as above. Consider the family of functions

$$f_\epsilon(z_1, z_2, z_3) = z_1 + z_1(z_1 - 2\epsilon) + iz_2, \quad \epsilon \in \mathbb{C}.$$ 

Since $(df_\epsilon)_0 = (1 - 2\epsilon)dz_1 + idz_2$, $f_\epsilon|_X$ is noncritical at $0 \in \mathbb{C}^3$ for any $\epsilon \in \mathbb{C}$.

For $\epsilon \neq 1/2$ we have

$$C_\epsilon := \{ df_\epsilon \wedge dh = 0 \} = \{(z_1, z_2, 0) \in \mathbb{C}^3 : z_2 = iz_1/(2z_1 - 2\epsilon + 1) \};$$

$$X \cap C_\epsilon = \{(0, 0, 0), (\epsilon, i\epsilon, 0), (\epsilon - 1, -i(\epsilon - 1), 0) \}.$$ 

The second and the third of these points are critical points of $f_\epsilon|_X$ when $\epsilon \notin \{0, 1\}$. For $\epsilon$ close to 0 the point $(\epsilon, i\epsilon, 0)$ lies close to the origin, while the third point is close to $(-1, i, 0)$. Hence $f_0|_X$ is noncritical on $X \cap \{ ||z|| \leq 1/2 \}$, but $f_\epsilon|_X$ for small $\epsilon \neq 0$ is close to $f_0$ and has a critical point $(\epsilon, i\epsilon, 0) \in X$ near the origin.
The main idea

The idea that we use in the construction of noncritical functions on Stein spaces stems from the following elementary observation:

(*) If $S \subset X$ is a local complex submanifold of positive dimension at a point $x \in S$ and if the restriction of a holomorphic function $f \in \mathcal{O}(X)$ to $S$ is noncritical at $x$, then $f$ is noncritical at $x$ (as a function on $X$). If $S$ is contained in every local irreducible component of the germ $X_x$, then $f$ is strongly noncritical at $x$.

This naturally leads us to consider complex analytic stratifications.
Stratified noncritical holomorphic functions

A **stratification** $\Sigma = \{S_j\}$ of a complex space $X$ is a subdivision $X = \bigcup_j S_j$ into the union of at most countably many pairwise disjoint connected complex manifolds $S_j$, called the **strata** of $\Sigma$, such that

- every compact set in $X$ intersects at most finitely many strata, and
- for any $S \in \Sigma$, the closure $\overline{S}$ is a closed complex subvariety of $X$ and the boundary $bS = \overline{S} \setminus S$ is a union of lower dimensional strata.

Such a pair $(X, \Sigma)$ is called a **stratified complex space**.

**Definition**

Let $(X, \Sigma)$ be a stratified complex space. A function $f \in \mathcal{O}(X)$ is said to be a **stratified noncritical holomorphic function** on $(X, \Sigma)$, or a **$\Sigma$-noncritical function**, if the restriction $f|_S$ of $f$ to any stratum $S \in \Sigma$ of positive dimension is a noncritical function on $S$.

Clearly the critical locus of a $\Sigma$-noncritical function is contained in the union $X_0$ of all 0-dimensional strata of $\Sigma$ (a discrete subset of $X$).
The second main theorem

**Theorem (2: Stratified noncritical functions)**

*On every stratified Stein space \((X, \Sigma)\) there exists a \(\Sigma\)-noncritical function \(F \in \mathcal{O}(X)\).*

*Furthermore, \(F\) can be chosen to agree to order \(n_x \in \mathbb{N}\) with a given germ \(f_x \in \mathcal{O}_{X,x}\) at any 0-dimensional stratum \(\{x\} \in \Sigma\).*
Choose a stratification $\Sigma$ of $X$ containing a given discrete set $P \subset X$ in the union $X_0 = \{p_1, p_2, \ldots\}$ of zero dimensional strata. For every $i \in \mathbb{N}$ let $X_i$ denote the union of all strata of dimension at most $i$ (the $i$-skeleton of $\Sigma$). Note that $X_i$ is a closed complex subvariety of $X$, the difference $X_i \setminus X_{i-1}$ is either empty or a complex manifold of dimension $i$, and

$$X_0 \subset X_1 \subset X_2 \subset \cdots \subset \bigcup_{i=0}^{\infty} X_i = X.$$ 

Given germs $f_k \in \mathcal{O}_{X, p_k}$ ($p_k \in X_0$) and integers $n_k \in \mathbb{N}$, Theorem 2 furnishes a $\Sigma$-noncritical function $F \in \mathcal{O}(X)$ such that $F_{p_k} - f_{p_k} \in m^n_{p_k}$.

**Claim:** $F$ is strongly noncritical on $X \setminus X_0$. Indeed, given $x \in X \setminus X_0$, pick the smallest $i \in \mathbb{N}$ such that $x \in X_i$, so $x \in X_i \setminus X_{i-1}$. Let $S_i \subset X_i \setminus X_{i-1}$ be the connected component containing $x$. Then the germ of $S_i$ at $x$ is contained in every local irreducible component of $X$ at $x$. It follows by (*) that $F$ is strongly noncritical at $x$ as claimed.

Choosing each germ $f_k$ at $p_k \in X_0$ to be strongly noncritical, we get a function $F \in \mathcal{O}(X)$ that is strongly noncritical on $X$. 

Theorem 2 implies Theorem 1
Analyticity of the critical locus

**Lemma**

Let $f$ be a holomorphic function on a complex space $X$. If $X' \subset X$ is a closed complex subvariety of $X$ containing $X_{\text{sing}}$, then the set

$$C_{X'}(f) := \{x \in X_{\text{reg}} : df_x = 0\} \cup X'$$

is a closed complex subvariety of $X$.

**Proof.**

By Hironaka, there are a complex manifold $M$ and a proper holomorphic surjection $\pi : M \to X$ such that $\pi : M \setminus \pi^{-1}(X_{\text{sing}}) \to X \setminus X_{\text{sing}}$ is a biholomorphism.

Given $f \in \mathcal{O}(X)$, consider $F = f \circ \pi \in \mathcal{O}(M)$ and the subvariety $M' = \pi^{-1}(X') \subset M$. Then:

- $C_{M'}(F) := \text{Crit}(F) \cup M'$ is a closed complex subvariety of $M$.
- As $\pi$ is proper, $\pi(C_{M'}(F))$ is a closed complex subvariety of $X$.
- Since $\pi$ is biholomorphic over $X_{\text{reg}}$, we have $\pi(C_{M'}(F)) = C_{X'}(f)$. 
The Stability Lemma

Lemma

Assume that $X$ is a complex space, $X' \subset X$ is a closed complex subvariety containing $X_{\text{sing}}$, and $K \subset L$ are compact subsets of $X$ with $K \subset \hat{L}$. Let $f \in \mathcal{O}(X)$ be noncritical on $L \setminus X'$. Then there exist $r \in \mathbb{N}$ and $\epsilon > 0$ such that the following holds.

If $g \in \mathcal{O}(L)$ satisfies

(i) $f - g \in \Gamma(L, \mathcal{J}^r_{X'})$, where $\mathcal{J}^r_{X'}$ is the $r$-th power of the ideal sheaf $\mathcal{J}_{X'}$ of the subvariety $X'$, and

(ii) $\|f - g\|_L := \sup_{x \in L} |f(x) - g(x)| < \epsilon$,

then $g$ has no critical points on $K \setminus X'$.

This clearly holds on compact subsets of $X \setminus X' \subset X_{\text{reg}}$, so it suffices to consider the behavior of $g$ near $K \cap X'$. 
Proof of the Stability Lemma

Fix \( p \in K \cap X' \). Embed a neighborhood \( U \subset X \) of \( p \) as a complex subvariety of a ball \( B \subset \mathbb{C}^N \). Pick a smaller ball \( B' \subset B \) and set \( U' := B' \cap U \). There is a linear extension operator \( T \) mapping bounded holomorphic functions on \( U \) to bounded holomorphic functions on \( B' \).

A point \( x \in U' \setminus X' \subset B' \) is a critical point of \( f \) if and only if the differential \( d\tilde{f}_x : T_x \mathbb{C}^N \to \mathbb{C} \) of \( \tilde{f} = Tf \in \mathcal{O}(B') \) annihilates the Zariski tangent space \( T_x U \).

This is expressed by holomorphic equations on \( B' \):

\[
F_j(f) = 0 \quad (j = 1, \ldots, k); \quad h_1 = 0, \ldots, h_m = 0
\]

where \( F_j(f) \) involve the first order jets of \( \tilde{f} = Tf \) and of some holomorphic defining functions \( h_1, \ldots, h_m \) for the subvariety \( U \) in \( B \).

By the assumption, this system has no solutions on \( U \setminus X' \). If a bounded function \( g \in \mathcal{O}(U) \) agrees with \( f \) to order \( r \) along the subvariety \( U \cap X' \), then \( F_j(g)|_{U'} - F_j(f)|_{U'} \) vanishes to order \( r - 1 \) along \( U' \cap X' \).

The conclusion now follows from the Łojasiewicz inequality together with the stability of noncritical functions on \( X_{\text{reg}} \).
The Genericity Lemma

**Lemma**

Let $X$ be a Stein space.

(i) For a generic $f \in \mathcal{O}(X)$ the set $\text{Crit}(f|_{X_{\text{reg}}})$ is discrete in $X$.

(ii) If $X' \subset X$ is a closed complex subvariety containing $X_{\text{sing}}$ and $g \in \mathcal{O}(X')$, then a generic holomorphic extension $f \in \mathcal{O}(X)$ of $g$ is noncritical on a deleted neighborhood of $X'$ in $X$.

(iii) If $g$ is holomorphic on an open neighborhood of $X'$ in $X$, then for any $r \in \mathbb{N}$, the conclusion of part (ii) holds for a generic extension $f \in \mathcal{O}(X)$ of $g|_{X'}$ which agrees with $g$ to order $r$ along $X'$.

**Proof.**

This is an application of Cartan’s Theorem B, the jet transversality theorem for holomorphic functions $X \to \mathbb{C}$, and the fact that irreducible components of the subvariety $C_{X'}(f) = \text{Crit}(f) \cup X'$ do not cluster on a compact subset of $X$. □
Construction of stratified noncritical functions

We proceed by induction on skeleta in the given stratification.

Let $(X, \Sigma)$ be a stratified Stein space. For every integer $i \in \mathbb{Z}_+$ we let $\Sigma_i$ denote the collection of all strata of dimension at most $i$ in $\Sigma$, and let $X_i$ denote the union of all strata in the family $\Sigma_i$ (the $i$-skeleton of $\Sigma$).

Note that the 0-skeleton $X_0 = \{p_1, p_2, \ldots\}$ is a discrete subset of $X$.

Since the boundary of any stratum is a union of lower dimensional strata, $X_i$ is a closed complex subvariety of $X$ of dimension $\leq i$ for every $i \in \mathbb{Z}_+$. Clearly $\dim X_i = i$ precisely when $\Sigma$ contains at least one $i$-dimensional stratum; otherwise $X_i = X_{i-1}$.

We have

$$X_0 \subset X_1 \subset \cdots \subset \bigcup_{i=0}^{\infty} X_i = X,$$

the sequence $X_i$ is stationary on any compact subset of $X$, and $(X_i, \Sigma_i)$ is a stratified Stein subspace of $(X, \Sigma)$ for every $i$. 

Choose any germs \( f_j \in \mathcal{O}_{X, p_j} \) at the points of \( X_0 = \{ p_1, p_2, \ldots \} \).

We construct a sequence \( F_i \in \mathcal{O}(X_i) \) of \((X_i, \Sigma_i)\)-noncritical functions whose germ at any point \( p_j \in X_0 \) agrees with \( f_j|_{X_i} \), and such that \( F_{i+1} \) agrees with \( F_i \) along the subvariety \( X_i \) for every \( i \in \mathbb{Z}_+ \).

**How to get \( F_{i+1} \) from \( F_i \):**

- Apply the Genericity Lemma to find \( G_i \in \mathcal{O}(X_{i+1}) \) which agrees with \( F_i \) along the subvariety \( X_i \), its germ at any point \( p_j \in X_0 \) agrees with \( f_j|_{X_{i+1}} \), and \( G_i \) is noncritical in a deleted neighbourhood of \( X_i \) in \( X_{i+1} \).

- Apply the approximation and gluing procedure (using the Splitting Lemma) within \( X_{i+1} \setminus X_i \) to find \( F_{i+1} \in \mathcal{O}(X_{i+1}) \) which agrees with \( G_i \) to a high order along \( X_i \) and is noncritical on \( X_{i+1} \setminus X_i \).

The function \( F \in \mathcal{O}(X) \) with \( F|_{X_i} = F_i \) (\( \forall i \in \mathbb{N} \)) satisfies Theorem 2.
Open problems

Let $X$ be a Stein space of pure dimension $n > 1$.

- What is the maximal number $q \in \{1, \ldots, n\}$ of holomorphic functions $f_1, \ldots, f_q \in \mathcal{O}(X)$ such that
  
  $$df_1 \wedge df_2 \wedge \cdots \wedge df_q \neq 0$$

  at each point of $X$?

- What is the answer locally at an isolated singularity?

- **F., 2003** If $X$ is nonsingular then $q_{\text{max}} = \left\lceil \frac{n+1}{2} \right\rceil = n - \left\lfloor \frac{n}{2} \right\rfloor$.

- Let $X$ be a Stein manifold of dimension $n > 1$ with trivial tangent bundle. Does $X$ admit a holomorphic immersion (= submersion) $X \to \mathbb{C}^n$?
  
  Equivalently: do we have Runge approximation property for locally biholomorphic maps $D \to \mathbb{C}^n$ on **convex domains** $D \subset \mathbb{C}^n$?
DEAR ORGANIZERS:

THANK YOU

for having proved beyond any doubt that

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WITH STYLE