Noncritical holomorphic functions on Stein spaces

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The content and references

- A survey of existing results and open problems
- Critical points of holomorphic functions on singular spaces
- The main new result
- Outline of the proof

References:

- F. Forstnerič: *Noncritical holomorphic functions on Stein spaces.* Preprint, 2013. `arxiv.org/abs/1311.1246`
- F. Forstnerič: *Noncritical holomorphic functions on Stein manifolds.* Acta Math. **191** (2003) 143–189.
- R. C. Gunning, R. Narasimhan: *Immersion of open Riemann surfaces.* Math. Ann., **174**, 103–108 (1967)

Stein spaces

1951 **K. Stein** A complex manifold X is said to be a **Stein manifold** if

- **holomorphic functions separate points:**

$$x, x' \in X,\ x \neq x' \Longrightarrow f(x) \neq f(x') \text{ for some } f \in \mathcal{O}(X), \text{ and}$$

- X is **holomorphically convex**: For every compact set $K \subset X$, its $\mathcal{O}(X)$-convex hull $\widehat{K}_{\mathscr{O}(X)}$ is also compact:

$$\widehat{K}_{\mathscr{O}(X)} = \{x \in X \colon |f(x)| \leq \sup_K |f|,\ \forall f \in \mathscr{O}(X)\}.$$

  Equivalently, for every discrete sequence $a_j \in X$ there exists a holomorphic function f on X such that $|f(a_j)| \to +\infty$ as $j \to \infty$.

1955 **H. Cartan; H. Grauert and R. Remmert**
A **Stein space** (or **holomorphically complete space**) is a complex space satisfying these axioms.

Embedding Stein manifolds in Euclidean spaces

1949 **Behnke-Stein** An open Riemann surface is a Stein manifold.

1956-61 **Remmert, Bishop, Narasimhan** A complex manifold X of dimension n is Stein if and only if it is embeddable as a closed complex submanifold of some $\mathbb{C}^N$; one can take $N = 2n + 1$.

Stein manifolds are relatives of affine algebraic manifolds.

1984 **Stout** Every relatively compact domain in a Stein manifold is biholomorphic to a domain in an affine algebraic manifold.

1992 **Eliashberg and Gromov; Schürmann (1997)** A Stein manifold of dimension $n > 1$ is embeddable in $\mathbb{C}^N$ with $N = \left[\frac{3n}{2}\right] + 1$.

1971 **Forster** This N is optimal for every $n > 1$.

Problem Is every open Riemann surface biholomorphic to some closed nonsingular embedded complex curve in $\mathbb{C}^2$?
Recent advances: **Wold & Forstnerič; Ritter**.

Noncritical functions on Stein manifolds

1967 **Gunning and Narasimhan** Every open Riemann surface X admits a holomorphic function $f \in \mathscr{O}(X)$ without critical points: $df_x \neq 0$ for all $x \in X$. The map $f\colon X \to \mathbb{C}$ given by such function is a holomorphic immersion spreading X as a Riemann domain over $\mathbb{C}$.

1986 **Gromov** Does every Stein manifold admit a noncritical holomorphic function? Given a nowhere vanishing holomorphic vector field L on X, does there exist $f \in \mathscr{O}(X)$ such that $L(f)$ has no zeros?

2003 **Forstnerič** Every Stein manifold X admit a noncritical holomorphic function. Furthermore, given a discrete set $P \subset X$, there exists $f \in \mathscr{O}(X)$ with $\mathrm{Crit}(f) = P$.

More generally, if $n = \dim X$ then there exist $q = \left[\frac{n+1}{2}\right] = n - \left[\frac{n}{2}\right]$ holomorphic functions $f_1, \ldots, f_q \in \mathscr{O}(X)$ such that

$$df_1 \wedge df_2 \wedge \cdots \wedge df_q \neq 0 \quad \text{on } X.$$

This number q is maximal for every n by topological reasons.

The h-principle for holomorphic submersions

2003 **F.** Let X be a Stein manifold of dimension $n > 1$ and $q \in \{1, \dots, n-1\}$. Every continuous complex vector bundle surjection $\Theta \colon TX \to X \times \mathbb{C}^q$ is homotopic (through complex vector bundle surjections) to the tangent map Tf of a holomorphic submersion $f = (f_1, \dots, f_q) \colon X \to \mathbb{C}^q$ ($df_1 \wedge df_2 \wedge \cdots \wedge df_q \neq 0$).

2004 **F.** The analogous result hold for submersions $X^n \to Y^q$ to any complex manifold Y^q which satisfies the Runge approximation property for submersions $\mathbb{C}^n \to Y^q$ on compact convex sets $K \subset \mathbb{C}^n$. (The smooth case: Gromov, Philips 1967.)

1986 **Gromov h-principle for folomorphic immersions $X^n \to \mathbb{C}^q$:** If $q > n \geq 1$ then every complex vector bundle injection $TX \to X \times \mathbb{C}^q$ is homotopic (through complex vector bundle injections) to the tangent map of a holomorphic immersion $X \to \mathbb{C}^q$. Such always exists if $q \geq \left[\frac{3n}{2}\right] = n + \left[\frac{n}{2}\right]$.

Problem: Does this h-principle also hold for $q = n > 1$?

Critical points of functions on singular spaces

Let X be a complex space. Notation:

- $\mathcal{O}_{X,x}$ ... the ring of germs of holomorphic function at $x \in X$,
- $\mathfrak{m}_x$ ... the maximal ideal of $\mathcal{O}_{X,x}$, so we have $\mathcal{O}_{X,x}/\mathfrak{m}_x \cong \mathbb{C}$.

Given $f \in \mathcal{O}_{X,x}$ we denote by $f - f(x) \in \mathfrak{m}_x$ the germ obtained by subtracting from f its value $f(x) \in \mathbb{C}$ at x.

Definition

Assume that x is nonisolated point of a complex space X.

(a) A germ $f \in \mathcal{O}_{X,x}$ is said to be critical, and x is a critical point of f, if $f - f(x) \in \mathfrak{m}_x^2$; f is noncritical if $f - f(x) \in \mathfrak{m}_x \setminus \mathfrak{m}_x^2$.

(b) A germ $f \in \mathcal{O}_{X,x}$ is strongly noncritical at x if the restriction $f|_V$ to any local irreducible component V of X is noncritical at x.

(c) Any function is considered noncritical at an isolated point $x \in X$.

Characterization by the differential

One can characterize these notions by the (non) vanishing of the differential df_x on the Zariski tangent space $T_x X = (\mathfrak{m}_x/\mathfrak{m}_x^2)^*$.

The differential $df_x \colon T_x X \to \mathbb{C}$ of $f \in \mathcal{O}_{X,x}$ is determined by the class

$$f - f(x) \in \mathfrak{m}_x/\mathfrak{m}_x^2 = T_x^* X;$$

f is critical at x if and only if $df_x = 0$.

A regular point $x \in X_{\mathrm{reg}}$ is a critical point of f if and only if in some (and hence in any) local holomorphic coordinates $z = (z_1, \ldots, z_n)$ on a neighborhood of x, with $z(x) = 0$, we have

$$\frac{\partial f}{\partial z_j}(0) = 0 \quad \text{for } j = 1, \ldots, n.$$

Hence the set $\mathrm{Crit}(f) \cap X_{\mathrm{reg}}$ is a closed complex subvariety of X_{reg}; on a Stein manifold this set is discrete for a generic choice of $f \in \mathcal{O}(X)$.

The first main result

Theorem (1: Noncritical functions on Stein spaces)

On every reduced Stein space X there exists a holomorphic function which is strongly noncritical at every point.

Furthermore, given a closed discrete set $P = \{p_1, p_2, \ldots\}$ in X, germs $f_k \in \mathcal{O}_{X,p_k}$ and integers $n_k \in \mathbb{N}$, there exists a function $F \in \mathcal{O}(X)$ which is strongly noncritical at every point of $X \setminus P$ and which agrees with the germ f_k to order n_k at each points $p_k \in P$; i.e.,

$$F_{p_k} - f_k \in \mathfrak{m}_{p_k}^{n_k}, \quad \forall k.$$

Corollary

Every 1-convex manifold X admits a holomorphic function which is noncritical outside of the maximal compact complex subvariety of X.

The scheme of proof in the nonsingular case

When X is a Stein manifold, the proof (F., Acta Math. **191** (2003) 143–189) relies on two main ingredients:

- Runge approximation theorem for noncritical holomorphic functions on polynomially convex subset of $\mathbb{C}^n$ by entire noncritical functions.
- A splitting lemma for biholomorphic maps on Cartan pairs. This enables one to extend (by approximation) a noncritical function across a noncritical strongly pseudoconvex Runge pair.
- For the h-principle (when constructing several functions with pointwise independent differentials), we also need a method of passing critical points of a Morse exhaustion function (change of topology). This uses the Gromov-Philips h-principle on totally real handles and a reduction to the noncritical case.

Approximation by noncritical functions

Lemma (The Oka-Weil theorem for noncritical functions)

Let f a noncritical holomorphic function on a neighborhood of a compact polynomially convex set $K \subset \mathbb{C}^n$. Then f can be approximated uniformly on K by noncritical holomorphic functions $F \colon \mathbb{C}^n \to \mathbb{C}$.

Proof.

The proof for $n = 1$ is an elementary application of Runge's and Mergelyan's theorem. Assume now that $n > 1$.

- Approximate f by a generic holomorphic polynomial $h \in \mathbb{C}^{[n]}$ with finite critical locus $\mathrm{Crit}(h) = \{z \in \mathbb{C}^n : dh_z = 0\} \subset \mathbb{C}^n \setminus K$.
- Use Andersén-Lempert theory to find an injective holomorphic map $\phi \colon \mathbb{C}^n \to \mathbb{C}^n$ which is close to the identity map on K and satisfies $\phi(\mathbb{C}^n) \cap \mathrm{Crit}(h) = \emptyset$.
- The composition $F = h \circ \phi : \mathbb{C}^n \to \mathbb{C}$ is then noncritical on $\mathbb{C}^n$ and it approximates f uniformly on K.

□

A splitting lemma for biholomorphic maps

A compact set K in a complex space X is said to be a **Stein compact** if K admits a basis of open Stein neighborhoods in X.

Definition

A pair (A, B) of compact sets in a complex space X is a **Cartan pair** if $D = A \cup B$ and $C = A \cap B$ are Stein compacts and we have

$$\overline{A \setminus B} \cap \overline{B \setminus A} = \emptyset.$$

Lemma (The Splitting Lemma)

Let (A, B) be a Cartan pair in a complex space X, with $B \subset X_{\mathrm{reg}}$. For every biholomorphic map $\gamma \colon U \to \gamma(U) \subset X$ in an open neighborhood U of $C = A \cap B$ which is sufficiently close to Id_U there exist biholomorphic maps α, β, close to Id in small neighborhoods of A and B, respectively, such that α is tangent to the identity along X_{sing} (to any given order) and

$$\gamma = \beta \circ \alpha^{-1} \quad \textit{holds on a neighborhood of } C.$$

The main induction step in the proof of Theorem 1

Corollary

Let (A, B) be a Cartan pair in X such that $C = A \cap B$ is $\mathscr{O}(B)$-convex and B is contained in a coordinate chart of X which is Runge in $\mathbb{C}^n$. Then every noncritical holomorphic function f on a neighbourhood of A can be approximated by noncritical holomorphic functions on a neighbourhood of $D = A \cup B$.

Proof. We may consider $C \subset B$ as subset of $\mathbb{C}^n$, with C polynomially convex. Approximate f uniformly on a neighbourhood of C by a noncritical function g on a neighbourhood of B. Then

$$f = g \circ \gamma,$$

where γ is a biholomorphic map close to the identity near C.
Now $\gamma = \beta \circ \alpha^{-1}$ by the splitting lemma. Hence

$$f \circ \alpha = g \circ \beta \text{ holds near } C,$$

so these functions amalgamate into a noncritical function on D.

Proof of the theorem for nonsigular X

We exhaust a Stein manifold X by an increasing sequence of Stein compacts

$$A_1 \subset A_2 \subset \cdots \subset \bigcup_{k=1}^{\infty} A_k = X$$

such that for every k we have $A_{k+1} = A_k \cup B_k$ where (A_k, B_k) is a Cartan pair as in the previous corollary.

Two types of Cartan pairs are needed:

- convex bumps, and
- bones (to change the topology).

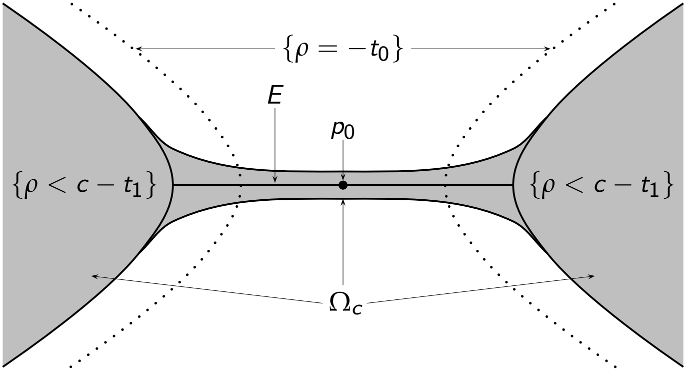
We inductively construct a sequence $f_k \in \mathscr{O}(A_k)$ of noncritical functions (or functions with a given critical locus). If the approximation of f_k by f_{k+1} is close enough at every step then

$$F = \lim_{k\to\infty} f_k \in \mathscr{O}(X)$$

satisfies Theorem 1.

Passing a critical point p_0 of an exhaustion function ρ

To construct several functions with pointwise independent differentials, we also need a method of passing critical points of a Morse exhaustion function. This uses the Gromov-Philips h-principle on totally real handles and a reduction to the noncritical case.



Problems with singular spaces

These tools do not apply directly at singular points of X. In addition, the following two phenomena make the analysis substantially more delicate:

- The critical locus of $f \in \mathcal{O}(X)$ need not be a closed complex subvariety of X near a singularity.

Example

A simple (reducible) example is $X = \{zw = 0\} \subset \mathbb{C}^2$, $f(z,w) = z$, $\mathrm{Crit}(f) = \{(0,w) \colon w \neq 0\}$. Here is an irreducible example:

$$X = \{(z_1, z_2, z_3) \in \mathbb{C}^3 \colon h(z) = z_1^2 + z_2^2 + z_3^2 = 0\},$$

$X_{\mathrm{sing}} = 0 \in \mathbb{C}^3$, $T_0 X = \mathbb{C}^3$.
For any $\lambda = (\lambda_1, \lambda_2, \lambda_3) \in \mathbb{C}^3_*$ the function

$$f_\lambda(z_1, z_2, z_3) = \lambda_1 z_1 + \lambda_2 z_2 + \lambda_3 z_3,$$

restricted to X, is strongly noncritical at $(0,0,0)$. If $\lambda \in X^* = X \setminus \{0\}$ then $\mathrm{Crit}(f_\lambda|_X) = \mathbb{C}_* \lambda$ which is not closed. □

Non stability of noncritical functions

- The class of (strongly) noncritical functions is not stable under small perturbations on compact sets which include singular points of X.

Example

Let $X \subset \mathbb{C}^3$ be as above. Consider the family of functions

$$f_\epsilon(z_1, z_2, z_3) = z_1 + z_1(z_1 - 2\epsilon) + \mathrm{i}z_2, \qquad \epsilon \in \mathbb{C}.$$

Since $(df_\epsilon)_0 = (1 - 2\epsilon)dz_1 + \mathrm{i}dz_2$, $f_\epsilon|_X$ is noncritical at $0 \in \mathbb{C}^3$ for any $\epsilon \in \mathbb{C}$.
For $\epsilon \neq 1/2$ we have

$$C_\epsilon := \{df_\epsilon \wedge dh = 0\} = \{(z_1, z_2, 0) \in \mathbb{C}^3 : z_2 = \mathrm{i}z_1/(2z_1 - 2\epsilon + 1)\};$$

$$X \cap C_\epsilon = \{(0, 0, 0), (\epsilon, \mathrm{i}\epsilon, 0), (\epsilon - 1, -\mathrm{i}(\epsilon - 1), 0)\}.$$

The second and the third of these points are critical points of $f_\epsilon|_X$ when $\epsilon \notin \{0, 1\}$. For ϵ close to 0 the point $(\epsilon, \mathrm{i}\epsilon, 0)$ lies close to the origin, while the third point is close to $(-1, \mathrm{i}, 0)$. Hence $f_0|_X$ is noncritical on $X \cap \{||z|| \leq 1/2\}$, but $f_\epsilon|_X$ for small $\epsilon \neq 0$ is close to f_0 and has a critical point $(\epsilon, \mathrm{i}\epsilon, 0) \in X$ near the origin.

The main idea

The idea that we use in the construction of noncritical functions on Stein spaces stems from the following elementary observation:

(*) If $S \subset X$ is a local complex submanifold of positive dimension at a point $x \in S$ and if the restriction of a holomorphic function $f \in \mathcal{O}(X)$ to S is noncritical at x, then f is noncritical at x (as a function on X). If S is contained in every local irreducible component of the germ X_x, then f is strongly noncritical at x.

This naturally leads us to consider **complex analytic stratifications**.

Stratified noncritical holomorphic functions

A **stratification** $\Sigma = \{S_j\}$ of a complex space X is a subdivision $X = \bigcup_j S_j$ into the union of at most countably many pairwise disjoint connected complex manifolds S_j, called the **strata** of Σ, such that

- every compact set in X intersects at most finitely many strata, and
- for any $S \in \Sigma$, the closure $\overline{S}$ is a closed complex subvariety of X and the boundary $bS = \overline{S} \setminus S$ is a union of lower dimensional strata.

Such a pair (X, Σ) is called a **stratified complex space**.

Definition

Let (X, Σ) be a stratified complex space. A function $f \in \mathcal{O}(X)$ is said to be a **stratified noncritical holomorphic function** on (X, Σ), or a **Σ-noncritical function**, if the restriction $f|_S$ of f to any stratum $S \in \Sigma$ of positive dimension is a noncritical function on S.

Clearly the critical locus of a Σ-noncritical function is contained in the union X_0 of all 0-dimensional strata of Σ (a discrete subset of X).

The second main theorem

Theorem (2: Stratified noncritical functions)

On every stratified Stein space (X, Σ) *there exists a* Σ*-noncritical function* $F \in \mathcal{O}(X)$.

Furthermore, F *can be chosen to agree to order* $n_x \in \mathbb{N}$ *with a given germ* $f_x \in \mathcal{O}_{X,x}$ *at any* 0*-dimensional stratum* $\{x\} \in \Sigma$.

Theorem 2 implies Theorem 1

Choose a stratification Σ of X containing a given discrete set $P \subset X$ in the union $X_0 = \{p_1, p_2, \ldots\}$ of zero dimensional strata. For every $i \in \mathbb{N}$ let X_i denote the union of all strata of dimension at most i (the i-skeleton of Σ). Note that X_i is a closed complex subvariety of X, the difference $X_i \setminus X_{i-1}$ is either empty or a complex manifold of dimension i, and

$$X_0 \subset X_1 \subset X_2 \subset \cdots \subset \bigcup_{i=0}^{\infty} X_i = X.$$

Given germs $f_k \in \mathcal{O}_{X,p_k}$ $(p_k \in X_0)$ and integers $n_k \in \mathbb{N}$, Theorem 2 furnishes a Σ-noncritical function $F \in \mathcal{O}(X)$ such that $F_{p_k} - f_{p_k} \in \mathfrak{m}_{p_k}^{n_k}$.

Claim: F is strongly noncritical on $X \setminus X_0$. Indeed, given $x \in X \setminus X_0$, pick the smallest $i \in \mathbb{N}$ such that $x \in X_i$, so $x \in X_i \setminus X_{i-1}$. Let $S_i \subset X_i \setminus X_{i-1}$ be the connected component containing x. Then the germ of S_i at x is contained in every local irreducible component of X at x. It follows by (*) that F is strongly noncritical at x **as claimed**.

Choosing each germ f_k at $p_k \in X_0$ to be strongly noncritical, we get a function $F \in \mathcal{O}(X)$ that is strongly noncritical on X.

Analyticity of the critical locus

Lemma

Let f be a holomorphic function on a complex space X. If $X' \subset X$ is a closed complex subvariety of X containing X_{sing}, then the set

$$C_{X'}(f) := \{x \in X_{\text{reg}} \colon df_x = 0\} \cup X'$$

is a closed complex subvariety of X.

Proof.

By Hironaka, there are a complex manifold M and a proper holomorphic surjection $\pi \colon M \to X$ such that $\pi \colon M \setminus \pi^{-1}(X_{\text{sing}}) \to X \setminus X_{\text{sing}}$ is a biholomorphism.

Given $f \in \mathcal{O}(X)$, consider $F = f \circ \pi \in \mathcal{O}(M)$ and the subvariety $M' = \pi^{-1}(X') \subset M$. Then:

- $C_{M'}(F) := \mathrm{Crit}(F) \cup M'$ is a closed complex subvariety of M.
- As π is proper, $\pi(C_{M'}(F))$ is a closed complex subvariety of X.
- Since π is biholomorphic over X_{reg}, we have $\pi(C_{M'}(F)) = C_{X'}(f)$.

The Stability Lemma

Lemma

Assume that X is a complex space, $X' \subset X$ is a closed complex subvariety containing X_{sing}, and $K \subset L$ are compact subsets of X with $K \subset \mathring{L}$. Let $f \in \mathcal{O}(X)$ be noncritical on $L \setminus X'$. Then there exist $r \in \mathbb{N}$ and $\epsilon > 0$ such that the following holds.
If $g \in \mathcal{O}(L)$ satisfies

- (i) $f - g \in \Gamma(L, \mathcal{J}^r_{X'})$, *where $\mathcal{J}^r_{X'}$ is the r-th power of the ideal sheaf $\mathcal{J}_{X'}$ of the subvariety X', and*
- (ii) $||f - g||_L := \sup_{x \in L} |f(x) - g(x)| < \epsilon$,

then g has no critical points on $K \setminus X'$.

This clearly holds on compact subsets of $X \setminus X' \subset X_{\mathrm{reg}}$, so it suffices to consider the behavior of g near $K \cap X'$.

Proof of the Stability Lemma

Fix $p \in K \cap X'$. Embed a neighborhood $U \subset X$ of p as a complex subvariety of a ball $B \subset \mathbb{C}^N$. Pick a smaller ball $B' \Subset B$ and set $U' := B' \cap U$. There is a linear extension operator T mapping bounded holomorphic functions on U to bounded holomorphic functions on B'.

A point $x \in U' \setminus X' \subset B'$ is a critical point of f if and only if the differential $d\tilde{f}_x \colon T_x\mathbb{C}^N \to \mathbb{C}$ of $\tilde{f} = Tf \in \mathcal{O}(B')$ annihilates the Zariski tangent space $T_x U$.
This is expressed by holomorphic equations on B':

$$F_j(f) = 0 \ (j = 1, \ldots, k); \quad h_1 = 0, \ldots, h_m = 0$$

where $F_j(f)$ involve the first order jets of $\tilde{f} = Tf$ and of some holomorphic defining functions $h_1, \ldots, h_m$ for the subvariety U in B.

By the assumption, this system has no solutions on $U \setminus X'$. If a bounded function $g \in \mathcal{O}(U)$ agrees with f to order r along the subvariety $U \cap X'$, then $F_j(g)|_{U'} - F_j(f)|_{U'}$ vanishes to order $r - 1$ along $U' \cap X'$.

The conclusion now follows from the Łojasiewicz inequality together with the stability of noncritical functions on X_{reg}.

The Genericity Lemma

Lemma

Let X be a Stein space.

(i) *For a generic $f \in \mathcal{O}(X)$ the set $\mathrm{Crit}(f|_{X_{\mathrm{reg}}})$ is discrete in X.*

(ii) *If $X' \subset X$ is a closed complex subvariety containing X_{sing} and $g \in \mathcal{O}(X')$, then a generic holomorphic extension $f \in \mathscr{O}(X)$ of g is noncritical on a deleted neighborhood of X' in X.*

(iii) *If g is holomorphic on an open neighborhood of X' in X, then for any $r \in \mathbb{N}$, the conclusion of part (ii) holds for a generic extension $f \in \mathcal{O}(X)$ of $g|_{X'}$ which agrees with g to order r along X'.*

Proof.

This is an application of Cartan's Theorem B, the jet transversality theorem for holomorphic functions $X \to \mathbb{C}$, and the fact that irreducible components of the subvariety $C_{X'}(f) = \mathrm{Crit}(f) \cup X'$ do not cluster on a compact subset of X. □

Construction of stratified noncritical functions

We proceed by induction on skeleta in the given stratification.

Let (X, Σ) be a stratified Stein space. For every integer $i \in \mathbb{Z}_+$ we let Σ_i denote the collection of all strata of dimension at most i in Σ, and let X_i denote the union of all strata in the family Σ_i (the i-skeleton of Σ).

Note that the 0-skeleton $X_0 = \{p_1, p_2, \ldots\}$ is a discrete subset of X.

Since the boundary of any stratum is a union of lower dimensional strata, X_i is a closed complex subvariety of X of dimension $\leq i$ for every $i \in \mathbb{Z}_+$. Clearly $\dim X_i = i$ precisely when Σ contains at least one i-dimensional stratum; otherwise $X_i = X_{i-1}$.
We have

$$X_0 \subset X_1 \subset \cdots \subset \bigcup_{i=0}^{\infty} X_i = X,$$

the sequence X_i is stationary on any compact subset of X, and (X_i, Σ_i) is a stratified Stein subspace of (X, Σ) for every i.

Construction of stratified noncritical functions – page 2

Choose any germs $f_j \in \mathscr{O}_{X,p_j}$ at the points of $X_0 = \{p_1, p_2, \ldots\}$.

We construct a sequence $F_i \in \mathscr{O}(X_i)$ of (X_i, Σ_i)-noncritical functions whose germ at any point $p_j \in X_0$ agrees with $f_j|_{X_i}$, and such that F_{i+1} agrees with F_i along the subvariety X_i for every $i \in \mathbb{Z}_+$.

How to get F_{i+1} from F_i:

- Apply the Genericity Lemma to find $G_i \in \mathscr{O}(X_{i+1})$ which agrees with F_i along the subvariety X_i, its germ at any point $p_j \in X_0$ agrees with $f_j|_{X_{i+1}}$, and G_i is noncritical in a deleted neighbourhood of X_i in X_{i+1}.
- Apply the approximation and gluing procedure (using the Splitting Lemma) within $X_{i+1} \setminus X_i$ to find $F_{i+1} \in \mathscr{O}(X_{i+1})$ which agrees with G_i to a high order along X_i and is noncritical on $X_{i+1} \setminus X_i$.

The function $F \in \mathscr{O}(X)$ with $F|_{X_i} = F_i$ $(\forall i \in \mathbb{N})$ satisfies Theorem 2.

Open problems

Let X be a Stein space of pure dimension $n > 1$.

- What is the maximal number $q \in \{1, \ldots, n\}$ of holomorphic functions $f_1, \ldots, f_q \in \mathscr{O}(X)$ such that

$$df_1 \wedge df_2 \wedge \cdots \wedge df_q \neq 0 \quad \text{at each point of } X?$$

- What is the answer locally at an isolated singularity?

- **F., 2003** If X is nonsingular then $q_{\max} = \left[\frac{n+1}{2}\right] = n - \left[\frac{n}{2}\right]$.

- Let X be a Stein manifold of dimension $n > 1$ with trivial tangent bundle. Does X admit a holomorphic immersion (= submersion) $X \to \mathbb{C}^n$?
  Equivalently: do we have Runge approximation property for locally biholomorphic maps $D \to \mathbb{C}^n$ on **convex domains** $D \subset \mathbb{C}^n$?

DEAR ORGANIZERS:

THANK YOU

for having proved beyond any doubt that

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WITH STYLE