Complete bounded submanifolds in different geometries

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We survey recent constructions of complete bounded submanifolds in several geometries:

- **holomorphic submanifolds** (the problem of Paul Yang, 1977)
- **null holomorphic curves** and **conformal minimal surfaces** in Euclidean spaces (the Calabi-Yau problem, 1965 & 2000)
- **Legendrian curves** in contact complex manifolds.

A noncompact submanifold $M$ (immersed or embedded) of a manifold $X$ is said to be **bounded** if it is relatively compact.

Let $g$ be a Riemannian metric on $X$. A submanifold $M \subset X$ is said to be **complete** if the pull-back of $g$ to $M$ is a complete metric on $M$. Equivalently, every divergent curve in $M$ (i.e., one that leaves every compact subset of $M$) has infinite $g$-length in $X$.

If $M$ is bounded, this notion is independent of the choice of $g$. 

Part I: Complete bounded complex submanifolds of $\mathbb{C}^n$

Paul Yang 1977  Do there exist complete bounded complex submanifolds of complex Euclidean spaces?

Peter Jones 1979 There is a bounded complete holomorphic immersion $\mathbb{D} = \{\zeta \in \mathbb{C} : |\zeta| < 1\} \to \mathbb{C}^2$, embedding $\mathbb{D} \to \mathbb{C}^3$, and proper embedding $\mathbb{D} \to \mathbb{B}^4$. (Based on C. Fefferman: Every $\phi \in BMO_{\mathbb{R}}(\mathbb{T})$ equals $\phi = u + \tilde{v}$ where $u, v \in L^\infty(\mathbb{T})$, $\tilde{v}$ the Hilbert transform of $v$.)

Martin, Umehara and Yamada 2009 There exist complete bounded holomorphic curves in $\mathbb{C}^2$ with arbitrary finite topology.

Theorem

Alarcón and Forstnerič 2013  Every bordered Riemann surface admits a complete proper holomorphic immersion to $\mathbb{B}^2$ and a complete proper holomorphic embedding to $\mathbb{B}^3$. 
A disc on the way of becoming complete

The illustration shows a **minimal disc** solving a **Plateau problem**. By twisting the boundary curve enough to make it everywhere non-rectifiable, the disc becomes complete. Holomorphic disc are minimal, in fact, absolute area minimizers.
Ripples on a disc increase boundary distance
Complete bounded surfaces abound in nature
Let $M$ be a bordered Riemann surface, and let $ds^2$ denote the Euclidean metric on $\mathbb{C}^n$.

- Let $F_0: \overline{M} \to \mathbb{C}^n$ be a holomorphic immersion satisfying $|F_0| \geq r_0 > 0$ on $bM$. We try to increase the boundary distance on $M$ with respect to the induced metric $F_0^* ds^2$ by $\delta > 0$.

- To this end, we approximate $F_0$ uniformly on a compact set in $M$ by an immersion $F_1: \overline{M} \to \mathbb{C}^n$ which at a point $p \in bM$ adds a displacement for approximately $\delta$ in a direction $V \in \mathbb{C}^n$, $|V| = 1$, approximately orthogonal to the point $F_0(p) \in \mathbb{C}^n$. The boundary distance increases by $\approx \delta$, while the outer radius increases by $\delta^2$:

$$|F_1(p)| \approx \sqrt{|F_0(p)|^2 + \delta^2} \approx |F_0(p)| + \frac{\delta^2}{2|F_0(p)|} \leq |F_0(p)| + \frac{\delta^2}{2r_0}.$$  

- Choosing $\delta_j > 0$ such that $\sum_j \delta_j = +\infty$ while $\sum_j \delta_j^2 < \infty$, we obtain by induction a limit immersion $F = \lim_{j \to \infty} F_j: \overline{M} \to \mathbb{C}^n$ with bounded outer radius and with complete metric $F^* ds^2$.
The first main tool – the Riemann-Hilbert problem

This idea can be realized on short arcs \( I \subset bM \), on which \( F_0 \) does not vary too much, by approximately solving a **Riemann-Hilbert problem**.

**Lemma**

Let \( \mathcal{D} = \{ \zeta \in \mathbb{C} : |\zeta| < 1 \} \) and \( \mathcal{T} = b \mathcal{D} = \{ \zeta \in \mathbb{C} : |\zeta| = 1 \} \).

Let \( f \in \mathcal{A}(\mathcal{D}, \mathbb{C}^n) \), and let \( g: \mathcal{T} \times \overline{\mathcal{D}} \to \mathbb{C}^n \) be a continuous map such that for each \( \zeta \in \mathcal{T} \) we have \( g(\zeta, \cdot) \in \mathcal{A}(\mathcal{D}, \mathbb{C}^n) \) and \( g(\zeta, 0) = f(\zeta) \).

Given \( \epsilon > 0 \) and \( 0 < r < 1 \), there are a number \( r' \in [r, 1) \) and a disc \( h \in \mathcal{A}(\mathcal{D}, \mathbb{C}^n) \) with \( h(0) = f(0) \) satisfying the following conditions:

(i) for any \( \zeta \in \mathcal{T} \) we have \( \text{dist}(h(\zeta), g(\zeta, \mathcal{T})) < \epsilon \),

(ii) for any \( \zeta \in \mathcal{T} \) and \( \rho \in [r', 1] \) we have \( \text{dist}(h(\rho \zeta), g(\zeta, \overline{\mathcal{D}})) < \epsilon \),

(iii) for any \( |\zeta| \leq r' \) we have \( |h(\zeta) - f(\zeta)| < \epsilon \), and

(iv) if \( g(\zeta, \cdot) = f(\zeta) \) is the constant disc for all \( \zeta \in \mathcal{T} \setminus J \), where \( J \subset \mathcal{T} \) is an arc, then \( |h - f| < \epsilon \) outside a neighborhood of \( J \) in \( \overline{\mathcal{D}} \).
Proof of the Riemann-Hilbert lemma

Write
\[ g(\zeta, z) = f(\zeta) + \lambda(\zeta, z), \quad \zeta \in \mathbb{T}, \ z \in \overline{\mathbb{D}}, \]
where \( \lambda \) is continuous on \( \mathbb{T} \times \overline{\mathbb{D}} \) and holomorphic in \( z \in \mathbb{D} \), with \( \lambda(\zeta, 0) = 0 \). Approximate \( \lambda \) by Laurent polynomials
\[
\lambda(\zeta, z) = \frac{1}{\zeta^m} \sum_{j=1}^{N} A_j(\zeta) z^j = \frac{z}{\zeta^m} \sum_{j=1}^{N} A_j(\zeta) z^{j-1}
\]
with polynomial coefficients \( A_j(\zeta) \). Choose an integer \( k > m \) and set
\[
h_k(\zeta) = f(\zeta) + \lambda(\zeta, \zeta^k) = f(\zeta) + \zeta^{k-m} \sum_{j=1}^{N} A_j(\zeta) \left(\zeta^k\right)^{j-1}, \quad |\zeta| \leq 1.
\]
This is an analytic disc satisfying \( h_k(0) = f(0) \). For \( \zeta = e^{it} \in \mathbb{T} \) we have
\[
h_k(e^{it}) = f(e^{it}) + \lambda(e^{it}, e^{kjt}) \approx g(e^{it}, e^{ikt}),
\]
and hence (i) holds. It is easy to verify the other conditions for big \( k \).
Exposing boundary points on a Riemann surface

The Riemann-Hilbert method could lead to **sliding curtains** (at least in low dimensions), creating shortcuts in the induced metric on $M$. We **eliminate shortcuts** by the **exposing of points method**.


Set $bM = \bigcup_i C_i$ where $C_i$ is a Jordan curve. Subdivide $C_i = \bigcup_j I_{i,j}$ such that any two adjacent arcs $I_{i,j-1}, I_{i,j}$ meet at a common endpoint $p_{i,j}$.

At the point $x_{i,j} = F_0(p_{i,j}) \in \mathbb{C}^n$ we attach to $F_0(M)$ a smooth real curve $\lambda_{i,j}$ of length $> \delta$ which increases the outer radius by $< \delta^2$. Let $y_{i,j}$ be other endpoint of $\lambda_{i,j}$.

Choose an arc $\gamma_{i,j} \subset \mathbb{R} \setminus M$ attached to $M$ at $p_{i,j}$, with the other endpoint $q_{i,j}$. Extend $F_0$ to a smooth diffeomorphism $\gamma_{i,j} \to \lambda_{i,j}$ mapping $q_{i,j}$ to $y_{i,j}$. Use Mergelyan to approximate $F_0$ by a holomorphic map from a neighborhood of $\overline{M} \cup \gamma_{i,j}$ to $\mathbb{C}^n$. 

Exposing a boundary point

The main point: there is a biholomorphism $\phi: \overline{M} \to \phi(\overline{M}) \subset R$ sending each $p_{i,j} \in bM$ to the other endpoint $q_{i,j}$ of the attached arc $\gamma_{i,j} \subset R$, and close to the identity away from the points $p_{i,j}$. Define $G$ by

$$G = F_0 \circ \phi : M \to \mathbb{C}^n.$$
Increasing the boundary distance

In the metric $G^*(ds^2)$ on $M$, the distance to the yellow neighborhoods of the points $p_{i,j} \in bM$ increased by the length of $\lambda_{i,j}$ which is $\geq \delta$. Apply the Riemann-Hilbert method on the arc $\beta^2_{i,j} \subset bM$ to increase the distance to it by $\geq \delta$. These two deformations are performed in almost orthogonal directions, so they don’t cancel each other. The boundary distance increased by $\geq \delta$ and the outer radius by $< \delta^2$. 
Embedded complete complex submanifolds

This method works well on any bordered Riemann surface $M$ and allows a complete control of the complex structure (i.e., no part of $M$ needs to be cut away in order to keep its image suitably bounded). This was the main novelty with respect to the previous results in the literature.

**Disadvantages:**
- It does not give complete bounded embeddings into $\mathbb{C}^2$, and
- it does not work on higher dimensional manifolds.

**Another idea:** start with a closed complex submanifold $X \subset \mathbb{C}^n$.

In the ball $\mathbb{B}^n \subset \mathbb{C}^n$, choose a suitable labyrinth $\mathcal{F} = \bigcup_j K_j$, where each $K_j$ is a closed ball (or polytope) in an affine real hyperplane $\Lambda_j \subset \mathbb{C}^n$, such that any path in $\mathbb{B}^n \setminus \mathcal{F}$ terminating on $b\mathbb{B}^n$ has infinite length.

Then, use holomorphic automorphisms of $\mathbb{C}^n$ to push $X$ away from $\mathcal{F}$. 
A complex subvariety avoiding a labyrinth

A labyrinth consisting of tangent balls. Any divergent curve in $\mathbb{B}^n$ avoiding all except finitely many of these balls has infinite length.

The subvariety $X \subset \mathbb{C}^n$ is twisted by holomorphic automorphisms so that it avoids the labyrinth $\mathcal{F}$. The image is ambiently complete.
A theorem of Globevnik, Alarcón and López

**Theorem**

For every closed complex submanifold $X \subset \mathbb{C}^n$ and compact set $L \subset X \cap \mathbb{B}^n$ there exists a Runge domain $\Omega \subset X \cap \mathbb{B}^n$ with $L \subset \Omega$ which admits a complete proper holomorphic embedding into $\mathbb{B}^n$.

In particular, every open orientable surface $S$ admits a complex structure $J$ such that the Riemann surface $R = (S, J)$ admits a complete proper holomorphic embedding to $\mathbb{B}^2$.

This gives an affirmative answer to Yang’s original question in all dimensions and codimensions. The shortcoming is that one cannot control the complex structure of the examples.

Also, this uses that $\mathbb{C}^n$ has a lot of holomorphic automorphisms (the Andersén-Lempert theory), which fails in most other interesting geometries that we wish to consider.
A **holomorphic directed system** on a complex manifold $X$ is given by a conical complex subvariety $\mathcal{G} \subset TX$ of the tangent bundle. Holomorphic **integral curves** are complex curves tangent to $\mathcal{G}$.

**Example (Pfaffian and contact systems)**

Let $\xi \subset TX$ be a holomorphic vector subbundle. A complex curve $F : M \to X$ is **horizontal**, or **isotropic**, or an **integral curve** if

$$dF_x(T_x M) \subset \xi_{F(x)} \quad \text{for all } x \in M.$$  

The case of interest is when $\xi$ is **completely nonintegrable**, in the sense that repeated commutators of vector fields tangent to $\xi$ span $TX$. When $\dim X = 2k + 1$, $\operatorname{rank} \xi = 2k$ and first order commutators span, we have $\xi = \ker \alpha$ where $\alpha$ is a holomorphic 1-form satisfying

$$\alpha \wedge \alpha^k \neq 0 \quad \ldots \quad \text{a contact form}.$$  

**Darboux 1882**: Locally near each point we have $\xi = \ker \alpha_0$ with

$$\alpha_0 = dz + \sum_{j=1}^k x_j dy_j.$$
Consider the standard contact space $(\mathbb{C}^{2k+1}, \alpha_0)$. Holomorphic integral curves are called **Legendrian curves**. They are plentiful:

**Theorem (Alarcón, F., López 2016)**

1. Every immersed Legendrian curve $\hat{M} \rightarrow \mathbb{C}^{2k+1}$ can be approximated uniformly on compacts by properly embedded Legendrian curves.

2. Let $M$ be a compact bordered Riemann surface. Every Legendrian curve $\hat{M} \rightarrow \mathbb{B}^{2k+1}$ can be approximated uniformly on compacts in $\hat{M}$ by complete proper Legendrian embeddings $\hat{M} \rightarrow \mathbb{B}^{2k+1}$.

3. Let $M$ be a compact bordered Riemann surface. Every Legendrian curve $\hat{M} \rightarrow \mathbb{C}^{2k+1}$ of class $\mathcal{A}^1(M)$ can be uniformly approximated by topological embeddings $F : M \rightarrow \mathbb{C}^{2k+1}$ such that $F|_{\hat{M}} : \hat{M} \rightarrow \mathbb{C}^{2k+1}$ is a complete Legendrian embedding.
Comments about the proof

Consider $\mathbb{C}^3_{(x,y,z)}$ with the contact form $\alpha = dz + xdy$. A Legendrian curve $(x, y, z): M \to \mathbb{C}^3$ is a holomorphic map such that $xdy$ is an exact 1-form and $z = -\int xdy$.

In an approximation problem on a Runge domain $D \subset M$, first create a period dominating spray $(x(\cdot, \zeta), y(\cdot, \zeta)): D \to \mathbb{C}^2$ depending holomorphically on $\zeta \in \mathbb{C}^\ell$, $\ell = \text{rank} \, H_1(M; \mathbb{Z})$. The approximated spray $(\tilde{x}(\cdot, \zeta), \tilde{y}(\cdot, \zeta)): M \to \mathbb{C}^2$ then contains an element for which $\tilde{x}(\cdot, \zeta) d\tilde{y}(\cdot, \zeta)$ is exact on $D$.

**Change of topology**: extend $x, y$ smoothly to the arc $E$ attached to $D \subset M$ such that $\int_E xdy$ has the correct value. In particular, ensure that $\int_C xdy = 0$ over the new cycle $C$ formed in part by $E$. Use period dominating sprays and Mergelyan approximation.

**The Riemann-Hilbert lemma holds for Legendrian curves**: if the central curve $f: M \to \mathbb{C}^3$ and all attached boundary discs $g(p, \cdot): \overline{D} \to \mathbb{C}^3$ ($p \in bM$) are Legendrian, we can choose a Legendrian approximate solution $h: M \to \mathbb{C}^3$ to the Riemann-Hilbert problem.
A hyperbolic contact system on $\mathbb{C}^{2k+1}$

**Theorem (F., 2016)**

For any $k \geq 1$ there exists a holomorphic contact system $\xi$ on $\mathbb{C}^{2k+1}$ which is **Kobayashi hyperbolic**; in particular, every holomorphic Legendrian curve $C \to (\mathbb{C}^{2k+1}, \xi)$ is constant.

**Idea of proof:** We take $\alpha = \Phi^* \alpha_0$ where $\alpha_0 = dz + \sum_{j=1}^k x_j dy_j$ is the standard contact form on $\mathbb{C}^{2k+1}$ and $\Phi: \mathbb{C}^{2k+1} \to \Omega \subset \mathbb{C}^{2k+1}$ is a **Fatou-Bieberbach map** whose image $\Omega$ avoids the union of cylinders

$$K = \bigcup_{N=1}^{\infty} 2^{N-1} b \overline{D}_{(x,y)}^{2k} \times C_N \overline{D}_z.$$

If $C_N \geq k 2^{3N+1}$ for all $N \in \mathbb{N}$, then $\mathbb{C}^{2k+1} \setminus K$ is $\alpha_0$-hyperbolic; hence $(\mathbb{C}^{2k+1}, \alpha)$ is hyperbolic.
Darboux charts around immersed Legendrian curves

Let \((X, \xi)\) be an arbitrary contact complex manifold.

**Theorem (Alarcón & F. 2017)**

Let \(R\) be an open Riemann surface with a nowhere vanishing holomorphic 1-form \(\theta\), and let \(f : R \rightarrow (X, \xi)\) be a holomorphic Legendrian immersion. Then, every compact set in \(R\) has a neighborhood \(U \subset R\) and a holomorphic immersion \(F : U \times \mathbb{B}^{2k} \rightarrow X\) such that the contact structure \(F^*\xi\) is given by \((x \in U, \text{the other coordinates Euclidean})\)

\[
\alpha = dz - y\theta(x) - \sum_{j=2}^{k} y_j dx_j. \quad \text{Darboux chart}
\]

**Corollary**

Let \(M \subset R\) be a compact bordered Riemann surface. Then \(f|_M\) can be uniformly approximated by topological embeddings \(F : M \rightarrow X\) such that \(F|_\hat{M} : \hat{M} \rightarrow X\) is a complete Legendrian embedding.
Another classical example of a directed system are null holomorphic curves in $\mathbb{C}^n$ and minimal surfaces, in $\mathbb{R}^n$.

Let $M$ be an open Riemann surface.

A **null holomorphic curve** is a holomorphic immersion $Z = (Z_1, \ldots, Z_n): M \rightarrow \mathbb{C}^n \ (n \geq 3)$ whose derivative satisfies

$$(dZ_1)^2 + \cdots + (dZ_n)^2 = 0.$$ 

An immersion $X = (X_1, \ldots, X_n): M \rightarrow \mathbb{R}^n$ is a **conformal minimal (harmonic) immersion**, abbreviated **CMI**, iff $\partial X = (\partial X_1, \ldots, \partial X_n)$ is a **holomorphic 1-form** on $M$ satisfying the same equation:

$$(\partial X_1)^2 + \cdots + (\partial X_n)^2 = 0.$$ 

The real part $X = \Re Z$ of a null curve is a CMI; converse holds on simply connected domains.
Fix a nowhere vanishing holomorphic 1-form $\theta$ on $M$. Then every conformal minimal immersion $X : M \to \mathbb{R}^n$ is of the form

$$X(p) = X(p_0) + \int_{p_0}^p \Re(f\theta), \quad p, p_0 \in M,$$

where $f : M \to A^{n-1} \setminus \{0\}$ is a holomorphic map into the null quadric

$$A^{n-1} = \{(z_1, \ldots, z_n) \in \mathbb{C}^n : z_1^2 + z_2^2 + \cdots + z_n^2 = 0\}$$

such that the $\mathbb{C}^n$-valued 1-form $f\theta$ has vanishing real periods.

Similarly, every holomorphic null curve is of the form

$$Z(p) = Z(p_0) + \int_{p_0}^p f\theta, \quad p \in M$$

where $f$ is as above and $f\theta$ has vanishing periods.
Example: the catenoid and the helicoid

Example
Consider the null curve

\[ Z(\zeta) = (\cos \zeta, \sin \zeta, -i\zeta) \in \mathbb{C}^3, \quad \zeta = u + iv \in \mathbb{C}, \]

\[ \partial Z = (-\sin \zeta, \cos \zeta, -i)d\zeta, \quad \sin^2 \zeta + \cos^2 \zeta + (-i)^2 = 0, \]

and the associated family of minimal surfaces in \( \mathbb{R}^3 \) for \( t \in \mathbb{R} \):

\[ X_t(\zeta) = \Re \left( e^{it}Z(\zeta) \right) \]

\[ = \cos t \begin{pmatrix} \cos u \cdot \cosh v \\ \sin u \cdot \cosh v \\ v \end{pmatrix} + \sin t \begin{pmatrix} \sin u \cdot \sinh v \\ -\cos u \cdot \sinh v \\ u \end{pmatrix} \]

At \( t = 0 \) we have a **catenoid** and at \( t = \pm \pi/2 \) a **helicoid**.
The catenoid and the helicoid
The family of minimal surfaces \( X_t(\zeta) = \Re \left( e^{it} Z(\zeta) \right) \), \( \zeta \in \mathbb{C} \), \( t \in \mathbb{R} \):
The Calabi-Yau problem for minimal surfaces

**Calabi 1965 Conjecture:** every complete minimal surface in $\mathbb{R}^3$ is unbounded.

**Osserman, Jorge and Meeks 1983** A complete conformal minimal surface in $\mathbb{R}^3$ of finite total Gauss curvature (FTC) is proper; its conformal type is a finitely punctured compact Riemann surface.

**On the other hand, omitting FTC leads to counterexamples:**

**Jorge & Xavier 1980** There exists a complete minimal surface in $\mathbb{R}^3$ with a bounded coordinate function. (*Calabi was somewhat wrong.*)

**Nadirashvili 1996** The disc is a complete bounded immersed minimal surface in $\mathbb{R}^3$. **Ferrer, Martin, Meeks 2012** There exist complete bounded immersed minimal surfaces in $\mathbb{R}^3$ with arbitrary topology. (*Calabi was completely wrong.*)

**S.T. Yau 2000**: Review of geometry and analysis (the Millenium Lecture). **Calabi-Yau problem**: When is Calabi’s conjecture true?
Embedded minimal surfaces in $\mathbb{R}^3$

**Colding & Minicozzi 2008** A complete embedded minimal surface $M$ with finite topology in $\mathbb{R}^3$ is proper. (Calabi was right for embedded surfaces with finite topology.)

**Meeks-Rosenberg 2005** The helicoid (conformal type C) is the only nonflat, properly embedded, simply connected minimal surface in $\mathbb{R}^3$.

**Costa 1984** Besides the plane, the helicoid, and the catenoid, Costa’s surface was the first example of a complete, properly embedded parabolic minimal surface in $\mathbb{R}^3$.

It is of finite total curvature and has three ends, two catenoidal ones at the top and the bottom (as all FTC properly embedded minimal surfaces besides the plane have) and a planar end in the middle.
Costa’s surface
Complete minimal surfaces with Jordan boundaries

**Theorem (Alarcón, F., 2015)**

*Every bordered Riemann surface* $M$ *admits a complete proper conformal minimal immersion into the ball of* $\mathbb{R}^3$.

**Theorem (Alarcón, Drinovec, F., López, 2016)**

*Let* $M$ *be a compact bordered Riemann surface, and let* $n \geq 3$. *Every conformal minimal immersion* $f : M \to \mathbb{R}^n$ *can be approximated, uniformly on* $M$, *by continuous maps* $F : M \to \mathbb{R}^n$ *such that* $F|_{\partial M} : \partial M \to \mathbb{R}^n$ *is a topological embedding and* $F|_{\tilde{M}} : \tilde{M} \to \mathbb{R}^n$ *is a complete immersed conformal minimal surface (embedded if* $n \geq 5$).

Our surfaces don’t have FTC, but we have a complete control of both the conformal structure (any bordered Riemann surface) and of the boundary (a union of Jordan curves).
Catenoidal cloud over the Sierra Nevada (Granada)

~ Thank you for your attention ~

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