Non-orientable minimal surfaces in $\mathbb{R}^n$

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Abstract

The purpose of this lecture is to show how complex analytic methods can be used for constructions of orientable and also non-orientable minimal surfaces in $\mathbb{R}^n$ for any $n \geq 3$. In particular, we obtain

- the Mergelyan approximation theorem and the Oka principle for conformal minimal surfaces in $\mathbb{R}^n$;
- (complete) proper minimal surfaces in convex domains in $\mathbb{R}^n$ ($n \geq 3$) and in minimally convex domains in $\mathbb{R}^3$;
- complete minimal surfaces in $\mathbb{R}^n$ bounded by Jordan curves.

Based on joint work with

- Antonio Alarcón and Francisco J. López, University of Granada
- Barbara Drinovec Drnovšek, University of Ljubljana
A (very) brief history of minimal surface theory

1744 Euler The only area minimizing surfaces of rotation in $\mathbb{R}^3$ are planes and catenoids.

1760 Lagrange: A graph $z = f(x, y)$ is area minimizing if and only if

$$\text{div} \left( \frac{\nabla f}{\sqrt{1 + |\nabla f|^2}} \right) = 0.$$ 

1776 Meusnier A smooth surface $S \subset \mathbb{R}^3$ satisfies locally the above equation iff its mean curvature function $H$ vanishes identically. The helicoid is a minimal surface.

1873 Plateau Minimal surfaces can be obtained as soap films.

1932 Douglas, Radó Every Jordan curve in $\mathbb{R}^3$ spans a minimal surface.

1965 Calabi’s Conjecture: Every complete minimal surface in $\mathbb{R}^3$ is unbounded. (Complete: every divergent curve has infinite length.) This conjecture, which is wrong as stated, opened a major direction.

Conformal minimal $=$ conformal harmonic

Theorem (Classical)

Let $M$ be a surface endowed with a conformal structure. The following are equivalent for a conformal immersion $X : M \rightarrow \mathbb{R}^n$ ($n \geq 3$):

- $X$ is minimal (a stationary point of the area functional).
- $X$ has identically vanishing mean curvature vector: $H = 0$.
- $X$ is harmonic: $\triangle X = 0$.

Indeed, we have $\triangle X = 2\bar{\zeta}H$ where $\bar{\zeta} = |X_u|^2 = |X_v|^2$ and $\zeta = u + iv$ be a local holomorphic coordinate on $M$. 

We emphasize the difference between minimal surfaces: these are stationary points of the area functional, and are locally area minimizing; and area-minimizing surfaces: these are surfaces which globally minimize the area among all nearby surfaces with the same boundary. Minimal graphs $z = f(x, y)$ are area minimizing.
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Let $M$ be an open Riemann surface and $X = (X_1, \ldots, X_n): M \to \mathbb{R}^n$ a smooth immersion. Fix a nonvanishing holomorphic 1-form $\theta$ on $M$. 
Weierstrass representation of orientable minimal surfaces

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$$A_* = \{ z = (z_1, \ldots, z_n) \in \mathbb{C}^n \setminus \{0\} : \sum_{j=1}^n z_j^2 = 0 \} \quad (\text{null quadric}).$$
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Since $\bar{\partial}\partial \mathbf{X} = \bar{\partial} f \wedge \theta$, $\mathbf{X}$ is harmonic iff $f = \partial \mathbf{X}/\theta$ is holomorphic.
Weierstrass representation of orientable minimal surfaces

Let $M$ be an open Riemann surface and $X = (X_1, \ldots, X_n) : M \to \mathbb{R}^n$ a smooth immersion. Fix a nonvanishing holomorphic $1$-form $\theta$ on $M$.

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**Conclusion:** Every conformal minimal immersion $M \to \mathbb{R}^n$ is of the form

$$X(p) = X(p_0) + 2 \int_{p_0}^p \Re (f \theta), \quad p_0, p \in M,$$

where $f : M \to \mathcal{A}_*$ is holomorphic and the real periods of $f \theta$ vanish.
Connection with holomorphic null curves

The flux homomorphism $\text{Flux}(\mathbf{X}) : H_1(M, \mathbb{Z}) \to \mathbb{R}^n$:

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If $\text{Flux}(X) = 0$, then

$$Z(p) = \int_p^p f \theta \in \mathbb{C}^n, \quad p \in M$$

is a **holomorphic null curve** $Z = (Z_1, \ldots, Z_n) : M \to \mathbb{C}^n$, i.e.,

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The real and the imaginary part of a holomorphic null curve \( Z = X + iY: M \to \mathbb{C}^n \) are conformal minimal immersions \( X, Y: M \to \mathbb{R}^n \). The converse holds on the disk \( \mathbb{D} = \{ \zeta \in \mathbb{C}: |\zeta| < 1 \} \).
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**Theorem**

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\[ \mathcal{A}_* = \{ z = (z_1, \ldots, z_n) \in \mathbb{C}^n \setminus \{0\} : \sum_{j=1}^{n} z_j^2 = 0 \} \]

is an **Oka manifold**, i.e., maps \( M \to \mathcal{A}_* \) from any Stein manifold \( M \) (in particular, from any open Riemann surface) satisfy all forms of the Oka principle (with approximation, interpolation, parametric, \ldots).
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**Proof:** The holomorphic vector fields on \( \mathbb{C}^n \),

\[ V_{j,k}(z) = z_j \frac{\partial}{\partial z_k} - z_k \frac{\partial}{\partial z_j}, \quad 1 \leq j, k \leq n \]

are linear and hence \( \mathbb{C} \)-complete, their flows preserve \( \mathcal{A}_* \), and they span the tangent space of \( \mathcal{A}_* \) at every point. Thus, \( \mathcal{A}_* \) is elliptic in the sense of Gromov, and hence an Oka manifold. **M. Gromov, JAMS 2 (1989).**
A survey of our main results, 2013–2015

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- **Runge approximation theorem:** If $K$ is a compact Runge subset of $M$, then every conformal minimal immersion $K \rightarrow \mathbb{R}^n$ can be approximated by proper conformal minimal immersions $M \rightarrow \mathbb{R}^n$. The analogous result for null curves.

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- **Isotopies:** Every conformal minimal immersion $M \to \mathbb{R}^n$ is regularly homotopic (in the space of of conformal minimal immersions) to the real part of a holomorphic null curve $M \to \mathbb{C}^n$.

General position theorems

- Every null curve $M \to \mathbb{C}^n$ ($n \geq 3$) can be approximated uniformly on compacts by \textit{(properly) embedded null curves} $M \hookrightarrow \mathbb{C}^n$.

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- Open Problems: Does every Riemann surface admit a proper conformal minimal embedding in $\mathbb{R}^4$?
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- Open Problems: Does every Riemann surface admit a proper conformal minimal embedding in $\mathbb{R}^4$?
  Does it admit a proper holomorphic embedding in $\mathbb{C}^2$?

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**Complete minimal surfaces with Jordan boundaries:**

Every conformal minimal immersion $X_0: M \to \mathbb{R}^n$ ($n \geq 3$) can be approximated, uniformly on $M$, by continuous maps $X: M \to \mathbb{R}^n$ such that $X: \hat{M} \to \mathbb{R}^n$ is a complete conformal minimal immersion and $X: bM \to \mathbb{R}^n$ is a topological embedding. If $n \geq 5$ then $X: M \to \mathbb{R}^n$ can be chosen a topological embedding.
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**Complete proper minimal surfaces in convex domains:**
Let $D$ be a convex domain in $\mathbb{R}^n$ ($n \geq 3$). Every conformal minimal immersion $X_0: M \to D$ can be approximated, uniformly on compacts in $\hat{M}$, by proper complete conformal minimal immersions $X: \hat{M} \to D$. If $D$ is bounded and strongly convex, then $X$ can be chosen continuous on $M$ (mapping $\partial M$ to $\partial D$).
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Minimal surfaces in minimally convex domains in \( \mathbb{R}^3 \)

A domain \( D \subset \mathbb{R}^3 \) is \textbf{minimally convex} if it admits a smooth exhaustion function \( \rho \): \( D \to \mathbb{R} \) such that for every point \( x \in D \), the sum of the smallest two eigenvalues of \( \text{Hess}_\rho(x) \) is positive.

A. Alarcón, B. Drinovec Drnovšek, F. Forstnerič, F.J. López, Minimal surfaces in minimally convex domains. arxiv:1510.04006
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A domain $D$ with $C^2$ boundary is minimally convex (also called **mean-convex**) if and only if $\kappa_1(x) + \kappa_2(x) \geq 0$ for each point $x \in \partial D$, where $\kappa_1(x), \kappa_2(x)$ are the principal curvatures of $\partial D$ at $x$. 

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- If $D \subset \mathbb{R}^3$ is minimally convex, then every conformal minimal immersion $X : M \to D$ can be approximated, uniformly on compacts in $\hat{M}$, by proper complete conformal minimal immersions $\tilde{X} : \hat{M} \to D$. If $\partial D$ is smooth and $\kappa_1(x) + \kappa_2(x) > 0$ for each point $x \in \partial D$ (such $D$ is called a **strongly mean convex domain**), then $\tilde{X}$ can be chosen continuous on $M$. 

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What about non-orientable minimal surfaces?

Assume that $N$ is a non-orientable surface with a conformal structure.
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There is a 2-sheeted covering $\pi: M \to N$ by a Riemann surface $M$ and a fixed-point-free antiholomorphic involution $\mathcal{I}: M \to M$ (the deck transformation of $\pi$) such that $N = M/\mathcal{I}$. 

Every conformal minimal immersion $Y: N \to \mathbb{R}^n$ lifts to an $\mathcal{I}$-invariant conformal minimal immersion $X: M \to \mathbb{R}^n$, i.e., $X \circ \mathcal{I} = X$.

Conversely, a $\mathcal{I}$-invariant conformal minimal immersion $X: M \to \mathbb{R}^n$ descends to a conformal minimal immersion $Y: N \to \mathbb{R}^n$. 
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Main theorem for non-orientable minimal surfaces

Theorem (Alarcón, F., López; in preparation)

Let $M$ be an open Riemann surface (or a bordered Riemann surface) with a fixed-point-free antiholomorphic involution $\mathcal{I}$. Then, all results mentioned above hold for $\mathcal{I}$-invariant conformal minimal immersions $M \rightarrow \mathbb{R}^n$. Hence, all mentioned results also hold for conformal minimal immersions $N \rightarrow \mathbb{R}^n$ from any non-orientable surface $N$ endowed with a conformal structure (without having to change the conformal structure).
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A holomorphic 1-form $\phi$ on $M$ is $\mathcal{J}$-invariant if

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**Notation:** $\mathcal{O}_\mathcal{J}(M), \Omega_\mathcal{J}(M)$. 

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**Notation:** \( \mathcal{O}_\mathcal{I}(M), \Omega_\mathcal{I}(M) \).

Clearly, a function \( f = u + iv : M \to \mathbb{C} \) belongs to \( \mathcal{O}_\mathcal{I}(M) \) iff \( u, v : M \to \mathbb{R} \) are conjugate harmonic functions satisfying

\[
u \circ \mathcal{I} = -v.
\]
Definition
A holomorphic function $f \in \mathcal{O}(M)$ is $\mathcal{J}$-invariant if
\[ f \circ \mathcal{J} = \bar{f}. \]
A holomorphic 1-form $\phi$ on $M$ is $\mathcal{J}$-invariant if
\[ \mathcal{J}^* \phi = \bar{\phi}. \]

Notation: $\mathcal{O}_\mathcal{J}(M), \Omega_\mathcal{J}(M)$.

Clearly, a function $f = u + iv : M \to \mathbb{C}$ belongs to $\mathcal{O}_\mathcal{J}(M)$ iff $u, v : M \to \mathbb{R}$ are conjugate harmonic functions satisfying
\[ u \circ \mathcal{J} = u, \quad v \circ \mathcal{J} = -v. \]

For every $f \in \mathcal{O}(M)$ we have that $\bar{f} \circ \mathcal{J} \in \mathcal{O}(M)$ and
\[ f + \bar{f} \circ \mathcal{J} \in \mathcal{O}_\mathcal{J}(M). \]
Invariant sprays

Let $\mathbb{B}^N \subset \mathbb{C}^N$ be the unit ball and $r > 0$. 
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Let $\mathbb{B}^N \subset \mathbb{C}^N$ be the unit ball and $r > 0$.

**Definition**

A holomorphic spray of maps $F : M \times r\mathbb{B}^N \to \mathbb{C}^n$ is $\mathcal{J}$-invariant if

$$F(\mathcal{J}p, \bar{z}) = \overline{F(p, z)}, \quad p \in M, \ z \in r\mathbb{B}^N.$$
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Note that $F(\cdot, z) : M \rightarrow C^n$ is $\mathcal{I}$-invariant if $z \in \mathbb{R}^N \subset C^N$. 

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**Example**

If $V$ is a holomorphic vector field on $\mathbb{C}^n$ which is real on $\mathbb{R}^n$, then its flow satisfies $\phi_t(\bar{z}) = \overline{\phi_t(z)}$. Let $V_1, \ldots, V_N$ be holomorphic vector fields on $\mathbb{C}^n$ which are real on $\mathbb{R}^n$, and let $\phi^j_t$ denote the flow of $V_j$. Given a $\mathcal{J}$-invariant holomorphic map $X : M \to \mathbb{C}^n$, the map

$$F(p, t_1, \ldots, t_N) = \phi^1_{t_1} \circ \cdots \circ \phi^N_{t_N} (X(p))$$

is a $\mathcal{J}$-invariant holomorphic spray of maps $M \to \mathbb{C}^n$. 
Lemma

Let \((M, \mathcal{I})\) be a bordered Riemann surface with a fixed-point-free involution \(\mathcal{I}: M \to M\). The exists a Runge homology basis \(\mathcal{B} = \mathcal{B}^+ \cup \mathcal{B}^-\) for \(H_1(M, \mathbb{Z})\), where

\[
\mathcal{B}^+ = \{\delta_1, \ldots, \delta_l\}, \quad \mathcal{B}^- = \{\mathcal{I}(\delta_2), \ldots, \mathcal{I}(\delta_l)\}, \quad \mathcal{I}_*\delta_1 = \delta_1.
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Denote by \(E\) the union of supports of the curves in \(\mathcal{B}\). The Runge property means that \(M \subset E\) has no relatively compact connected components. This guarantees Mergelyan approximation on \(E\).
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Let $\mathcal{P}^+ = (\mathcal{P}^+_1, \ldots, \mathcal{P}^+_l): \mathcal{O}(M) \to \mathbb{C}^l$ denote the period map given by

$$\mathcal{P}^+_j(f) = \int_{\delta_j} f\theta, \quad f \in \mathcal{O}(M), \ j = 1, \ldots, l.$$

Similarly, we define $\mathcal{P}^+ (\phi) = (\int_{\delta_j} \phi)_{j=1,\ldots,l}$ for a holomorphic 1-form $\phi$. 
Lemma (1)

Let $\phi$ be a $\mathcal{I}$-invariant holomorphic 1-form on $M$. Then:

(a) $\phi$ is exact if and only if $\mathcal{P}^+(\phi) = 0$.
(b) $\Re \phi$ is exact if and only if $\Re \mathcal{P}^+(\phi) = 0$. 
Lemma (1)

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(b) \( \Re \phi \) is exact if and only if \( \Re \mathcal{P}^+(\phi) = 0 \).

Proof. (a) By \( \mathcal{I} \)-invariance of \( \phi \) we have

\[
\int_{\mathcal{I}_*\delta_j} \phi = \int_{\delta_j} \mathcal{I}^*\phi = \int_{\delta_j} \overline{\phi}, \quad j = 1, \ldots, l.
\]

Therefore, \( \mathcal{P}^+(\phi) = 0 \) implies that \( \phi \) has vanishing periods over all curves in \( \mathcal{B} = \mathcal{B}^+ \cup \mathcal{B}^- \), and hence it is exact. The converse is obvious.
Lemma (1)

Let $\phi$ be a $\mathcal{I}$-invariant holomorphic 1-form on $M$. Then:

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Proof. (a) By $\mathcal{I}$-invariance of $\phi$ we have

$$\int_{\mathcal{I}_* \delta_j} \phi = \int_{\delta_j} \mathcal{I}^* \phi = \int_{\delta_j} \bar{\phi}, \quad j = 1, \ldots, l.$$ 

Therefore, $\mathcal{P}^+(\phi) = 0$ implies that $\phi$ has vanishing periods over all curves in $\mathcal{B} = \mathcal{B}^+ \cup \mathcal{B}^-$, and hence it is exact. The converse is obvious.

(b) Likewise, $\mathcal{R} \mathcal{P}^+(\phi) = 0$ implies that $\Re \phi$ is exact. The imaginary periods $\Im \mathcal{P}^+(\phi)$ (the flux of $\phi$) can be arbitrary subject to the conditions

$$\int_{\mathcal{I}_* \delta_j} \Im \phi = - \int_{\delta_j} \Im \phi, \quad j = 1, \ldots, l.$$ 

In particular, $\int_{\delta_1} \Im \phi = 0$ since $\mathcal{I}_* \delta_1 = \delta_1$. 
Lemma (2)

Let $\mathcal{B} = \mathcal{B}^+ \cup \mathcal{B}^-$ be a homology basis of $H_1(M, \mathbb{Z})$ as above, and let $\mathcal{P}^+: \mathcal{A}(M, \mathbb{C}^n) \to (\mathbb{C}^n)^l$ denote the period map associated to $\mathcal{B}^+$:

$$\mathcal{P}^+(f) = \left(\int_{\gamma_i} f \theta\right)_{i=1,\ldots,l} \in (\mathbb{C}^n)^l.$$
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$$\mathcal{P}^+(f) = \left( \int_{\gamma_i} f \theta \right)_{i=1,...,l} \in (\mathbb{C}^n)^l.$$ 

For every nonflat, $\mathcal{I}$-invariant map $f: M \rightarrow \mathcal{A}_*$ of class $\mathcal{A}(M)$ there exists a dominating $\mathcal{I}$-invariant spray $F: M \times \mathbb{R}^N \rightarrow \mathcal{A}_*$ of class $\mathcal{A}(M)$ which is also period dominating, in the sense that the differential

$$\frac{\partial}{\partial \zeta} \bigg|_{\zeta=0} \mathcal{P}^+(F(\cdot, \zeta)): \mathbb{C}^N \rightarrow (\mathbb{C}^n)^l$$

maps $\mathbb{R}^N$ (the real part of $\mathbb{C}^N$) surjective onto $\mathbb{R}^n \times (\mathbb{C}^n)^{l-1}$.
Definition

Let $M$ be an open Riemann surface with a fixed-point-free antiholomorphic involution $\mathcal{I}: M \to M$. 

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(Very) special Cartan pairs
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Let $M$ be an open Riemann surface with a fixed-point-free antiholomorphic involution $\mathcal{I}: M \to M$.

A pair $(A, B)$ of compact sets in $M$ is a $\mathcal{I}$-invariant Cartan pair if

- the sets $A$, $B$, $A \cap B$, and $A \cup B$ are $\mathcal{I}$-invariant with $C^1$ boundaries;
- $A \setminus B \cap B \setminus A = \emptyset$ (the separation property).

A $\mathcal{I}$-invariant Cartan pair $(A, B)$ is special if $B = B' \cup \mathcal{I}(B')$, where $B'$ is a compact set with $C^1$ boundary in $M$ and $B' \cap \mathcal{I}(B') = \emptyset$.

A special Cartan pair $(A, B)$ is very special if the sets $B'$ and $A \cap B'$ are discs. (Of course $\mathcal{I}(B')$ and $A \cap \mathcal{I}(B')$ are then also discs.)
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Lemma (3)

Let \((M, \mathcal{I})\) be as above. Assume that

- \((A, B)\) is a special \(\mathcal{I}\)-invariant Cartan pair in \(M\),
- \(\epsilon > 0\) and \(r > 0\) are real number, and
- \(F : A \times rB^N \to A_*\) is a \(\mathcal{I}\)-invariant spray of class \(\mathcal{A}(A)\) which is dominating over the set \(C = A \cap B\).
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Then, there exist \(\delta > 0\) and \(r' \in (0, r)\) such that for every \(\mathcal{I}\)-invariant spray \(G : B \times r\mathbb{B}^N \rightarrow \mathcal{A}_*\) of class \(\mathcal{A}(B)\) satisfying

\[
\|F - G\|_{0, C \times r\mathbb{B}^N} < \delta
\]

there is a \(\mathcal{I}\)-invariant spray \(H : (A \cup B) \times r'\mathbb{B}^N \rightarrow \mathcal{A}_*\) of class \(\mathcal{A}(A \cup B)\) satisfying

\[
\|H - F\|_{0, A \times r'\mathbb{B}^N} < \epsilon.
\]
Lemma (4)

Let \((M, \mathcal{I})\) be as above, and let \((A, B)\) be a very special \(\mathcal{I}\)-invariant Cartan pair in \(M\). Let \(\mathcal{P}^+\) denote the period map on \(A\) (Lemma 2).
Lemma (4)

Let \((M, \mathcal{I})\) be as above, and let \((A, B)\) be a very special \(\mathcal{I}\)-invariant Cartan pair in \(M\). Let \(\mathcal{P}^+\) denote the period map on \(A\) (Lemma 2).

Every \(\mathcal{I}\)-invariant map \(f : A \to A_*\) of class \(\mathcal{A}(A)\) can be approximated uniformly on \(A\) by \(\mathcal{I}\)-invariant holomorphic maps \(\tilde{f} : A \cup B \to A_*\) satisfying \(\mathcal{P}^+(\tilde{f}) = \mathcal{P}^+(f)\).
Basic approximation lemma for $\mathcal{I}$-invariant maps

Lemma (4)

Let $(M,\mathcal{I})$ be as above, and let $(A, B)$ be a **very special** $\mathcal{I}$-invariant Cartan pair in $M$. Let $\mathcal{P}^+$ denote the period map on $A$ (Lemma 2).

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**Proof.** By Lemma 2, there exists a $\mathcal{I}$-invariant dominating and period dominating spray $F : A \times r\mathbb{B}^N \to A_*$ of class $\mathcal{A}(A)$ with $F(\cdot, 0) = f$. 
Lemma (4)

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Proof. By Lemma 2, there exists a \(\mathcal{I}\)-invariant dominating and period dominating spray \(F: A \times rB^n \to A_*\) of class \(\mathcal{A}(A)\) with \(F(\cdot, 0) = f\).

By the definition of a very special Cartan pair, \(B = B' \cup \mathcal{I}(B')\) is the union of two disjoint disks, and \(C' = A \cap B' \subset B'\) is a disk.
Basic approximation lemma for $\mathcal{I}$-invariant maps

**Lemma (4)**

Let $(M, \mathcal{I})$ be as above, and let $(A, B)$ be a very special $\mathcal{I}$-invariant Cartan pair in $M$. Let $\mathcal{P}^+$ denote the period map on $A$ (Lemma 2).

Every $\mathcal{I}$-invariant map $f : A \to A_*$ of class $\mathcal{A}(A)$ can be approximated uniformly on $A$ by $\mathcal{I}$-invariant holomorphic maps $\tilde{f} : A \cup B \to A_*$ satisfying $\mathcal{P}^+(\tilde{f}) = \mathcal{P}^+(f)$.

**Proof.** By Lemma 2, there exists a $\mathcal{I}$-invariant dominating and period dominating spray $F : A \times r\mathbb{B}^N \to A_*$ of class $\mathcal{A}(A)$ with $F(\cdot, 0) = f$.

By the definition of a very special Cartan pair, $B = B' \cup \mathcal{I}(B')$ is the union of two disjoint disks, and $C' = A \cap B' \subset B'$ is a disk.

Pick a number $r' \in (0, r)$.

Since $A_*$ is an Oka manifold, it is possible to approximate $F$, uniformly on $C' \times r'\mathbb{B}^N$, by a holomorphic spray $G : B' \times r'\mathbb{B}^N \to A_*$. 


Proof of Lemma 4

We extend $G$ to $\mathcal{I}(B') \times r'B^N$ by symmetrizing:

$$G(p, \zeta) = G(\mathcal{I}(p), \bar{\zeta}) \quad \text{for } p \in \mathcal{I}(B') \text{ and } \zeta \in r'B^N.$$ 

It follows that $G$ is an $\mathcal{I}$-invariant spray on $B \times r'B^N$ with values in $A_\ast$. 
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It follows that $G$ is an $\mathcal{I}$-invariant spray on $B \times r' B^N$ with values in $A_*$. By Lemma 3, there is $r'' \in (0, r')$ such that, if $G$ is sufficiently close to $F$ on $(A \cap B) \times r' B^N$, then there is a $\mathcal{I}$-invariant spray $H: (A \cup B) \times r'' B^N \to A_*$ which approximates $F$ on $A \times r'' B^N$. 

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If the approximation is sufficiently close, the period domination property of $F$ implies that there exists $\zeta \in r''\mathbb{B}^N \cap \mathbb{R}^N$ such that the $\mathcal{I}$-invariant map $\tilde{f} = H(\cdot, \zeta): A \cup B \to A_*$ satisfies the condition $\mathcal{P}^+(f) = \mathcal{P}^+(\tilde{f})$. 
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It follows that $G$ is an $\mathcal{I}$-invariant spray on $B \times r'\mathbb{B}^N$ with values in $\mathcal{A}_*$. By Lemma 3, there is $r'' \in (0, r')$ such that, if $G$ is sufficiently close to $F$ on $(A \cap B) \times r'\mathbb{B}^N$, then there is a $\mathcal{I}$-invariant spray $H: (A \cup B) \times r''\mathbb{B}^N \to \mathcal{A}_*$ which approximates $F$ on $A \times r''\mathbb{B}^N$.

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This proves Lemma 4. $\square$
Change of topology of the domain

Attach to a domain $A \subset M$ a smooth arc $E$ (or a couple of arcs $E = E_1 \cup E_2$ with $\mathcal{J}(E_1) = E_2$ and $E_1 \cap E_2 = \emptyset$) and proceed as follows.

1. Extend the derivative $f = \partial_X/\theta$ from $A$ to a map $f : A \cup E \to A^*$ such that $f \circ \mathcal{J} = f$ and $\int_E \mathcal{R}(f \theta)$ has a correct value (to satisfy the period vanishing condition if $E$ closes to a nontrivial loop in $A \cup E$).

2. Embed $f$ into a period-dominating ($\mathcal{J}$-invariant) holomorphic spray $F : (A \cup E) \times \mathbb{B}_N \to A^*$ with $F(\cdot, 0) = f$.

3. Approximate $F$ by a holomorphic spray $\tilde{F} : D \times \mathbb{B}_N \to A^*$ on a neighborhood $D$ of $A \cup E$. The period domination of $F$ ensures that there is a value $\zeta_0 \in \mathbb{B}_N \cap \mathbb{R}_N$ such that $\tilde{F}(\cdot, \zeta_0)$ integrates to a conformal minimal immersion $D \to \mathbb{R}^n$.

4. In the non-orientable case, choose a neighborhood $V_1$ of the arc $E_1$, approximate $F$ over $(A \cup E) \cap V_1$ by a spray $G$ over $V_1$, define $G$ on $\mathcal{J}(V_1) \supset E_2$ by $G(p, \zeta) = G(\mathcal{J}(p), \bar{\zeta})$, and glue $F$ and $G$ into an $\mathcal{J}$-invariant spray over a neighborhood of $A \cup E$. Finish as before.
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Attach to a domain $A \subset M$ a smooth arc $E$ (or a couple of arcs $E = E_1 \cup E_2$ with $\mathcal{I}(E_1) = E_2$ and $E_1 \cap E_2 = \emptyset$) and proceed as follows.

1. Extend the derivative $f = 2\partial X/\theta$ from $A$ to a map $f : A \cup E \to \mathcal{A}_*$ such that $f \circ \mathcal{I} = \overline{f}$ and $\int_E \mathcal{R}(f \theta)$ has a correct value (to satisfy the period vanishing condition if $E$ closes to a nontrivial loop in $A \cup E$).
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2. Embed $f$ into a period-dominating ($\mathcal{I}$-invariant) holomorphic spray $F : (A \cup E) \times rB^N \to \mathcal{A}_*$ with $F(\cdot, 0) = f$. 
Attach to a domain $A \subset M$ a smooth arc $E$ (or a couple of arcs $E = E_1 \cup E_2$ with $\mathcal{I}(E_1) = E_2$ and $E_1 \cap E_2 = \emptyset$) and proceed as follows.

1. Extend the derivative $f = 2\partial X / \theta$ from $A$ to a map $f: A \cup E \to A_*$ such that $f \circ \mathcal{I} = \tilde{f}$ and $\int_E \mathcal{R}(f \theta)$ has a correct value (to satisfy the period vanishing condition if $E$ closes to a nontrivial loop in $A \cup E$).

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2. Embed $f$ into a period-dominating ($\mathcal{I}$-invariant) holomorphic spray $F : (A \cup E) \times r\mathbb{B}^N \to A^*$ with $F(\cdot, 0) = f$.

3. Approximate $F$ by a holomorphic spray $\tilde{F} : D \times r\mathbb{B}^N \to A^*$ on a neighborhood $D$ of $A \cup E$. The period domination of $F$ ensures that there is a value $\zeta_0 \in r\mathbb{B}^N \cap \mathbb{R}^N$ such that $\tilde{F}(\cdot, \zeta_0)$ integrates to a conformal minimal immersion $D \to \mathbb{R}^n$.

4. In the non-orientable case, choose a neighborhood $V_1$ of the arc $E_1$, approximate $F$ over $(A \cup E) \cap V_1$ by a spray $G$ over $V_1$, define $G$ on $\mathcal{I}(V_1) \supset E_2$ by $G(p, \zeta) = G(\mathcal{I}(p), \tilde{\zeta})$, and glue $F$ and $G$ into an $\mathcal{I}$-invariant spray over a neighborhood of $A \cup E$. Finish as before.
The **Runge-Mergelyan approximation theorem** and the **Oka principle for conformal minimal immersions** (both in the orientable and non-orientable case) are proved by using Lemmas 1–4, together with the analysis at critical points of a strongly subharmonic exhaustion function on $M$ as explained above.
The Runge-Mergelyan approximation theorem and the Oka principle for conformal minimal immersions (both in the orientable and non-orientable case) are proved by using Lemmas 1–4, together with the analysis at critical points of a strongly subharmonic exhaustion function on $M$ as explained above.

To obtain complete bounded conformal minimal surfaces in $\mathbb{R}^n$ (including those with Jordan boundaries), and proper conformal minimal surfaces in (minimally) convex domains, we also use approximate solutions to the Riemann-Hilbert boundary value problem for conformal minimal surfaces and holomorphic null curves.
Assume that $M$ is a compact bordered Riemann surface and $X: M \to \mathbb{R}^n$ ($n \geq 3$) is a conformal minimal immersion. Let $I \subset bM$ be an arc.

Let $Y: bM \times \overline{D} \to \mathbb{R}^n$ be a continuous map of the form

$$Y(p, \zeta) = X(p) + f(p, \zeta)u + g(p, \zeta)v, \quad p \in I, \quad \zeta \in \overline{D},$$

where $u, v \in \mathbb{R}^n$ is an orthonormal pair, $F(p, \cdot) = f(p, \cdot) + ig(p, \cdot)$ is a holomorphic immersion for each $p \in I$, and $F(p, \cdot) = 0$ for $p \in bM \setminus I$.

Then, we can find conformal minimal immersions $\tilde{X}: M \to \mathbb{R}^n$ such that

- $\tilde{X}$ approximates $X$ outside a small neighbourhood of $I$ in $M$;
- $\tilde{X}(M)$ lies close to $X(M) \cup \bigcup_{p \in I} Y(p, \overline{D})$;
- $\tilde{X}(p)$ lies close to the curve $Y(p, b\overline{D})$ for every $p \in I$;
- $\text{Flux}(\tilde{X}) = \text{Flux}(X)$.

THANK YOU FOR YOUR ATTENTION
Appendix A: Curvature of surfaces in $\mathbb{R}^n$

Assume that $D$ is a domain in $\mathbb{R}^2_{(u_1,u_2)}$ and $X = (X_1, \ldots, X_n): D \to \mathbb{R}^n$ is a $C^2$ immersion. Let $S = X(D) \subset \mathbb{R}^n$, a parametrized surface in $\mathbb{R}^n$. 

Consider a smooth embedded curve in $S$, $\lambda(t) = X(u_1(t), u_2(t)) \in S$. Let $s = s(t)$ denote the arc length on $\lambda$. The number $\kappa(T, N) = d^2\lambda ds^2 \cdot N$ is the normal curvature of $S$ at $p = \lambda(t) \in S$ in the tangent direction $T = \lambda'(s) \in T_p S$ with respect to the normal vector $N \in N_p S$. 

In terms of $t$-derivatives we get $\kappa(T, N) = \sum_{i,j=1}^2 (X_{u_i} u_j \cdot N) \frac{\dot{u}_i}{\dot{s}} \frac{\dot{u}_j}{\dot{s}} = \text{second fundamental form}$ $\sum_{i,j=1}^2 (g_{u_i} u_j) \frac{\dot{u}_i}{\dot{s}} \frac{\dot{u}_j}{\dot{s}} = \text{first fundamental form}$.
Appendix A: Curvature of surfaces in $\mathbb{R}^n$

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$$\lambda(t) = X(u_1(t), u_2(t)) \in S.$$ 

Let $s = s(t)$ denote the arc length on $\lambda$. The number

$$\kappa(T, N) := \frac{d^2\lambda}{ds^2} \cdot N = \sum_{i,j=1}^{2} \left( X_{u_i u_j} \cdot N \right) \frac{du_i}{ds} \frac{du_j}{ds}$$

is the normal curvature of $S$ at $p = \lambda(t) \in S$ in the tangent direction $T = \lambda'(s) \in T_p S$ with respect to the normal vector $N \in N_p S$. 

Appendix A: Curvature of surfaces in $\mathbb{R}^n$

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is the **normal curvature** of $S$ at $p = \lambda(t) \in S$ in the tangent direction $T = \lambda'(s) \in T_pS$ with respect to the normal vector $N \in N_pS$.

In terms of $t$-derivatives we get

$$\kappa(T, N) = \frac{\sum_{i, j=1}^{2} \left( X_{u_i u_j} \cdot N \right) \dot{u}_i \dot{u}_j}{\sum_{i, j=1}^{2} g_{i, j} \dot{u}_i \dot{u}_j} = \frac{\text{second fundamental form}}{\text{first fundamental form}}$$
The principal curvature and the mean curvature

Fix a normal vector $\mathbf{N} \in N_p S$ and vary the unit tangent vector $\mathbf{T} \in T_p S$. The **principal curvatures** of $S$ at $p$ in direction $\mathbf{N}$ are the numbers

$$
\kappa_1(\mathbf{N}) = \max_{\mathbf{T}} \kappa(\mathbf{T}, \mathbf{N}), \quad \kappa_2(\mathbf{N}) = \min_{\mathbf{T}} \kappa(\mathbf{T}, \mathbf{N}).
$$

Their average

$$
H(\mathbf{N}) = \frac{\kappa_1(\mathbf{N}) + \kappa_2(\mathbf{N})}{2} \in \mathbb{R}
$$

is the **mean curvature** of $S$ at $p$ in the normal direction $\mathbf{N} \in N_p S$. 

The principal curvature and the mean curvature

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is the **mean curvature** of $S$ at $p$ in the normal direction $\mathbf{N} \in N_p S$.

Let $G = (g_{i,j})$ and $h(\mathbf{N}) = (h_{i,j}(\mathbf{N})) = (X_{u_i u_j} \cdot \mathbf{N})$ denote the matrices of the 1st and the 2nd fundamental form, respectively. The extremal values $\kappa_1, \kappa_2$ of $\kappa(\mathbf{T}, \mathbf{N})$ are roots of the equation

$$\det(h(\mathbf{N}) - \mu G) = 0$$

$$\det G \cdot \mu^2 - (g_{2,2} h_{1,1}(\mathbf{N}) + g_{1,1} h_{2,2}(\mathbf{N}) - 2 g_{1,2} h_{1,2}(\mathbf{N})) \mu + \det h(\mathbf{N}) = 0.$$
The mean curvature vector

The Vieta formula gives

\[ H(N) = \frac{\kappa_1 + \kappa_2}{2} = \frac{g_{2,2}X_{u_1u_1} + g_{1,1}X_{u_2u_2} - 2g_{1,2}X_{u_1u_2}}{2 \det G} \cdot N. \]

There is a unique normal vector \( H \in N_pS \) such that

\[ H(N) = H \cdot N \quad \text{for all} \quad N \in N_pS. \]

This vector \( H \) is the mean curvature vector of the surface \( S \) at \( p \).
The mean curvature vector

The Vieta formula gives

\[ H(N) = \frac{\kappa_1 + \kappa_2}{2} = \frac{g_{2,2}X_{u_1u_1} + g_{1,1}X_{u_2u_2} - 2g_{1,2}X_{u_1u_2}}{2 \det G} \cdot N. \]

There is a unique normal vector \( H \in N_pS \) such that

\[ H(N) = H \cdot N \quad \text{for all} \quad N \in N_pS. \]

This vector \( H \) is the **mean curvature vector** of the surface \( S \) at \( p \).

Assume now that we work in isothermal coordinates:

\[ G = (g_{i,j}) = \zeta I, \quad \det G = \zeta^2; \quad \zeta = \|X_{u_1}\|^2 = \|X_{u_2}\|^2, \quad X_{u_1} \cdot X_{u_2} = 0 \]

Then:

\[ H(N) = \frac{X_{u_1u_1} + X_{u_2u_2}}{2\zeta} \cdot N = \frac{\triangle X}{2\zeta} \cdot N. \]
The main formula in isothermal coordinates

**Claim:** \( \Delta X = X_{u_1u_1} + X_{u_2u_2} \) is orthogonal to \( S = X(D) \).
The main formula in isothermal coordinates

Claim: $\triangle X = X_{u_1 u_1} + X_{u_2 u_2}$ is orthogonal to $S = X(D)$.

Proof: Conformality means that

$$X_{u_1} \cdot X_{u_1} = X_{u_2} \cdot X_{u_2}, \quad X_{u_1} \cdot X_{u_2} = 0.$$ 

Differentiating the first identity on $u_1$ and the second one on $u_2$ yields

$$X_{u_1 u_1} \cdot X_{u_1} = X_{u_1 u_2} \cdot X_{u_2} = -X_{u_2 u_2} \cdot X_{u_1},$$

whence $\triangle X \cdot X_{u_1} = 0$. Similarly we get $\triangle X \cdot X_{u_2} = 0$ by differentiating the first identity on $u_2$ and the second one on $u_1$. 
The main formula in isothermal coordinates

Claim: \( \Delta X = X_{u_1 u_1} + X_{u_2 u_2} \) is orthogonal to \( S = X(D) \).

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\]

whence \( \Delta X \cdot X_{u_1} = 0 \). Similarly we get \( \Delta X \cdot X_{u_2} = 0 \) by differentiating the first identity on \( u_2 \) and the second one on \( u_1 \).

Since \( H(N) = H \cdot N = \frac{\Delta X}{2\zeta} \cdot N \) and \( \Delta X \) is normal to \( S \), we get

\[
\Delta X = 2\zeta H, \quad \zeta = \|X_{u_1}\|^2 = \|X_{u_2}\|^2 \quad \text{(Main formula)}.
\]
Lagrange’s formula for the first variation of the area

The area of an immersed surface $\mathbf{X}: D \to \mathbb{R}^n$ equals

$$A(\mathbf{X}) = \int_D \sqrt{\det G} \cdot du_1 du_2.$$ 

Let $\mathbf{N}: D \to \mathbb{R}^n$ be a normal vector field along $\mathbf{X}$ which vanishes on $bD$. Consider the 1-parameter family of maps $\mathbf{X}^t: D \to \mathbb{R}^n$:

$$\mathbf{X}^t(u) = \mathbf{X}(u) + t \mathbf{N}(u), \quad u \in D, \ t \in \mathbb{R}.$$ 

A calculation gives the formula for the first variation of the area:

$$\delta A(\mathbf{X}) \mathbf{N} = \frac{d}{dt} \bigg|_{t=0} A(\mathbf{X}^t) = -2 \int_D \mathbf{H} \cdot \mathbf{N} \sqrt{\det G} \cdot du_1 du_2.$$ 

It follows that $\delta A(\mathbf{X}) \mathbf{N} = 0 \iff \mathbf{H} = 0.$
Lagrange’s formula for the first variation of the area

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$$A(\mathbf{X}) = \int_D \sqrt{\det G} \cdot \text{du}_1 \text{du}_2.$$

Let $\mathbf{N}: D \rightarrow \mathbb{R}^n$ be a *normal vector field* along $\mathbf{X}$ which vanishes on $bD$. Consider the 1-parameter family of maps $\mathbf{X}^t: D \rightarrow \mathbb{R}^n$:

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It follows that $\delta A(\mathbf{X}) = 0 \iff \mathbf{H} = 0$. 
Appendix B: Topological structure of non-orientable surfaces

Every compact non-orientable surface $N$ without boundary is the connected sum $N = \mathbb{P}^2 \# \cdots \# \mathbb{P}^2$ of $g \geq 1$ copies of the real projective plane $\mathbb{P}^2$; the number $g$ is the genus of $N$. (This is the maximal number of pairwise disjoint closed curves in $N$ which reverse the orientation.)
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Furthermore, $K = \mathbb{P}^2 \# \mathbb{P}^2$ is the Klein bottle, and for any non-orientable surface $N$ we have $N \# K = N \# T$ where $T$ is the torus.
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Furthermore, $K = \mathbb{P}^2 \# \mathbb{P}^2$ is the Klein bottle, and for any non-orientable surface $N$ we have $N \# K = N \# T$ where $T$ is the torus.

This gives the following dichotomy according to whether the genus $g$ is even or odd:

(I) $g = 1 + 2k \geq 1$ is odd. In this case, $N = \mathbb{P}^2 \# \underbrace{T \# \cdots \# T}_k$.

(II) $g = 2 + 2k \geq 2$ is even. In this case, $N = \mathbb{P}^2 \# \mathbb{P}^2 \# \underbrace{T \# \cdots \# T}_k$. 
Let $\iota: M \to N$ be a 2-sheeted covering by a compact orientable surface $(M, \mathcal{I})$. Then $M$ has genus $g - 1$. We construct an explicit geometric model for $(M, \mathcal{I})$ in $\mathbb{R}^3$. 

Let $S^2$ be the unit sphere in $\mathbb{R}^3$ centered at the origin, and let $\tau: \mathbb{R}^3 \to \mathbb{R}^3$ be the involution $\tau(x) = -x$.

Case (I): $N = P_2 \# k \mathbb{T} \# \cdots \# T \# \cdots \# T$. We take $M$ to be an embedded surface $(T_\pm 1 \# \cdots \# T \pm k)$ of genus $g = 2k$ in $\mathbb{R}^3$ which is invariant by the symmetry with respect to the origin (i.e., $\tau(M) = M$), where $T_\pm j$, $T_\pm j$ are embedded tori in $\mathbb{R}^3$ with $\tau(T_\pm j) = T_\pm j$ for all $j$. Set $I = \tau|_M: M \to M$. (See Fig. 1.)

We have $M = M^- \cup C \cup M^+$, where $C \subset S^2$ is a closed $\mathcal{I}$-invariant cylinder and $M^-$ and $M^+$ are the closure of the two components of $M \setminus C$, both homeomorphic to the connected sum of $k$ tori minus an open disk. Obviously $I(M^-) = M^+$ and $M^- \cap M^+ = \emptyset$. 

Geometric model of 2-sheeted oriented covering
Let $\iota: M \rightarrow N$ be a 2-sheeted covering by a compact orientable surface $(M, \mathcal{I})$. Then $M$ has genus $g - 1$. We construct an explicit geometric model for $(M, \mathcal{I})$ in $\mathbb{R}^3$.

Let $S^2$ be the unit sphere in $\mathbb{R}^3$ centered at the origin, and let $\tau: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be the involution $\tau(x) = -x$.

**Case (I):** $N = \mathbb{P}^2 \# T \# \cdots \# T$. We take $M$ to be an embedded surface

$$\left( T_1^- \# \cdots \# T_k^- \right) \# S^2 \# \left( T_1^+ \# \cdots \# T_k^+ \right)$$

of genus $g - 1 = 2k$ in $\mathbb{R}^3$ which is invariant by the symmetry with respect to the origin (i.e., $\tau(M) = M$), where $T_j^-, T_j^+$ are embedded tori in $\mathbb{R}^3$ with $\tau(T_j^-) = T_j^+$ for all $j$. Set $\mathcal{I} = \tau|_M: M \rightarrow M$. (See Fig. 1.)
Let $\iota: M \to N$ be a 2-sheeted covering by a compact orientable surface $(M, I)$. Then $M$ has genus $g - 1$. We construct an explicit geometric model for $(M, I)$ in $\mathbb{R}^3$.

Let $S^2$ be the unit sphere in $\mathbb{R}^3$ centered at the origin, and let $\tau: \mathbb{R}^3 \to \mathbb{R}^3$ be the involution $\tau(x) = -x$.

**Case (I):** $N = \mathbb{P}^2 \# \mathbb{T} \# \cdots \# \mathbb{T}$. We take $M$ to be an embedded surface

$$(\mathbb{T}_1^- \# \cdots \# \mathbb{T}_k^-) \# S^2 \# (\mathbb{T}_1^+ \# \cdots \# \mathbb{T}_k^+)$$

of genus $g - 1 = 2k$ in $\mathbb{R}^3$ which is invariant by the symmetry with respect to the origin (i.e., $\tau(M) = M$), where $\mathbb{T}_j^-$, $\mathbb{T}_j^+$ are embedded tori in $\mathbb{R}^3$ with $\tau(\mathbb{T}_j^-) = \mathbb{T}_j^+$ for all $j$. Set $I = \tau|_M: M \to M$. (See Fig. 1.)

We have $M = M^- \cup C \cup M^+$, where $C \subset S^2$ is a closed $I$-invariant cylinder and $M^-$ and $M^+$ are the closure of the two components of $M \setminus C$, both homeomorphic to the connected sum of $k$ tori minus an open disk. Obviously $I(M^-) = M^+$ and $M^- \cap M^+ = \emptyset$. 
Geometric model, Case II

Case (II): $N = \mathbb{P}^2 \# \mathbb{P}^2 \# \overbrace{T \# \cdots \# T}^{k} = K \# \overbrace{T \# \cdots \# T}^{k}$. 

Let $T_0 \subset \mathbb{R}^3$ be the standard revolution torus centered at the origin, i.e., invariant under the antipodal map $\tau$. In this case we let $M$ be an embedded $\tau$-invariant surface in $\mathbb{R}^3$, where the tori $T_{\pm j}$ are as above, and set $I = \tau |_M$. (See Figure 2.)

Write $M = M_- \cup K \cup M_+$, where $K \subset T_0 \subset \mathbb{R}^3$ is an $I$-invariant torus minus two disjoint open disks, and $M_-$ and $M_+$ are the closure of the two components of $M \setminus K$, both homeomorphic to the connected sum of $k$ tori minus an open disk. Obviously $I(M_-) = M_+$ and $M_- \cap M_+ = \emptyset$. 

Geometric model, Case II

**Case (II):** \( N = \mathbb{P}^2 \# \mathbb{P}^2 \# \overbrace{T \# \cdots \# T}^{k} = K \# \overbrace{T \# \cdots \# T}^{k}. \)

Let \( T_0 \subset \mathbb{R}^3 \) be the standard revolution torus centered at the origin, i.e., invariant under the antipodal map \( \tau \). In this case we let \( M \) be an embedded \( \tau \)-invariant surface

\[
(T_1^- \# \cdots \# T_k^-) \# T_0 \# (T_1^+ \# \cdots \# T_k^+)
\]

in \( \mathbb{R}^3 \), where the tori \( T_j^\pm \) are as above, and set \( \mathcal{I} = \tau|_M \). (See Figure 2.)
Geometric model, Case II

Case (II): \( N = P^2 \# P^2 \# \underbrace{T \# \cdots \# T}_k = K \# \underbrace{T \# \cdots \# T}_k \).

Let \( T_0 \subset \mathbb{R}^3 \) be the standard revolution torus centered at the origin, i.e., invariant under the antipodal map \( \tau \). In this case we let \( M \) be an embedded \( \tau \)-invariant surface

\[
(T_{-1}^{-} \# \cdots \# T_k^{-}) \# T_0 \# (T_{1}^{+} \# \cdots \# T_k^{+})
\]

in \( \mathbb{R}^3 \), where the tori \( T_{\pm}^j \) are as above, and set \( \mathcal{I} = \tau|_M \). (See Figure 2.)

Write \( M = M^- \cup K \cup M^+ \), where \( K \subset T_0 \subset \mathbb{R}^3 \) is a \( \mathcal{I} \)-invariant torus minus two disjoint open disks, and \( M^- \) and \( M^+ \) are the closure of the two components of \( M \setminus K \), both homeomorphic to the connected sum of \( k \) tori minus an open disk. Obviously \( \mathcal{I}(M^-) = M^+ \) and \( M^- \cap M^+ = \emptyset \).