Non-orientable minimal surfaces in $\mathbb{R}^n$

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Abstract

The purpose of this lecture is to show how **complex analytic methods** can be used for constructions of **orientable** and also **non-orientable minimal surfaces in $\mathbb{R}^n$** for any ≥ 3. In particular, we obtain

- the Mergelyan approximation theorem and the Oka principle for conformal minimal surfaces in $\mathbb{R}^n$;
- (complete) proper minimal surfaces in convex domains in $\mathbb{R}^n$ $(n \geq 3)$ and in minimally convex domains in $\mathbb{R}^3$;
- complete minimal surfaces in $\mathbb{R}^n$ bounded by Jordan curves.

Based on joint work with

- **Antonio Alarcón and Francisco J. López, University of Granada**
- **Barbara Drinovec Drnovšek, University of Ljubljana**

A (very) brief history of minimal surface theory

1744 **Euler** The only area minimizing surfaces of rotation in $\mathbb{R}^3$ are planes and catenoids.

1760 **Lagrange**: A graph $z = f(x, y)$ is area minimizing if and only if

$$\mathrm{div}\left(\frac{\nabla f}{\sqrt{1+|\nabla f|^2}}\right) = 0.$$

1776 **Meusnier** A smooth surface $S \subset \mathbb{R}^3$ satisfies locally the above equation iff its mean curvature function H vanishes identically. The helicoid is a minimal surface.

1873 **Plateau** Minimal surfaces can be obtained as soap films.

1932 **Douglas, Radó** Every Jordan curve in $\mathbb{R}^3$ spans a minimal surface.

1965 **Calabi's Conjecture:** Every complete minimal surface in $\mathbb{R}^3$ is unbounded. (Complete: every divergent curve has infinite length.) This conjecture, which is wrong as stated, opened a major direction.

2000 **S.-T. Yau: The Calabi-Yau Problem**.

Conformal minimal = conformal harmonic

Theorem (Classical)

Let M be a surface endowed with a conformal structure. The following are equivalent for a **conformal** *immersion $\mathbf{X} : M \to \mathbb{R}^n$ $(n \geq 3)$:*

- $\mathbf{X}$ *is minimal (a stationary point of the area functional).*
- $\mathbf{X}$ *has identically vanishing mean curvature vector: $\mathbf{H} = 0$.*
- $\mathbf{X}$ *is harmonic: $\triangle \mathbf{X} = 0$.*

Indeed, we have $\triangle \mathbf{X} = 2\xi \mathbf{H}$ where $\xi = |\mathbf{X}_u|^2 = |\mathbf{X}_v|^2$ and $\zeta = u + \imath v$ be a local holomorphic coordinate on M.

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- **minimal surfaces:** these are stationary points of the area functional, and are locally area minimizing; and
- **area-minimizing surfaces:** these are surfaces which globally minimize the area among all nearby surfaces with the same boundary. Minimal graphs $z = f(x, y)$ are area minimizing.

Weierstrass representation of orientable minimal surfaces

Let M be an open Riemann surface and $\mathbf{X} = (X_1, \ldots, X_n) \colon M \to \mathbb{R}^n$ a smooth immersion. Fix a nonvanishing holomorphic 1-form θ on M.

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$$\mathcal{A}_* = \{z = (z_1, \ldots, z_n) \in \mathbb{C}^n \setminus \{0\} \colon \sum_{j=1}^n z_j^2 = 0\} \quad \textbf{(null quadric)}.$$

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Since $\bar\partial \partial \mathbf{X} = \bar\partial \mathbf{f} \wedge \theta$, $\mathbf{X}$ is harmonic iff $\mathbf{f} = \partial \mathbf{X}/\theta$ is holomorphic.

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Conclusion: Every conformal minimal immersion $M \to \mathbb{R}^n$ is of the form

$$\mathbf{X}(p) = \mathbf{X}(p_0) + 2\int_{p_0}^{p} \Re\,(\mathbf{f}\theta)\,, \quad p_0, p \in M,$$

where $\mathbf{f}\colon M \to \mathcal{A}_*$ is holomorphic and the real periods of $\mathbf{f}\theta$ vanish.

Connection with holomorphic null curves

The **flux homomorphism** $\mathrm{Flux}(\mathbf{X})\colon H_1(M,\mathbb{Z}) \to \mathbb{R}^n$:

$$\mathrm{Flux}(\mathbf{X})(\gamma) = \int_\gamma d^c\mathbf{X} = 2\int_\gamma \Im\left(\mathbf{f}\theta\right), \quad [\gamma] \in H_1(M,\mathbb{Z}).$$

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If $\mathrm{Flux}(\mathbf{X}) = 0$, then

$$\mathbf{Z}(p) = \int_\cdot^p \mathbf{f}\theta \in \mathbb{C}^n, \qquad p \in M$$

is a **holomorphic null curve** $\mathbf{Z} = (Z_1,\ldots,Z_n)\colon M \to \mathbb{C}^n$, i.e.,

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The real and the imaginary part of a holomorphic null curve $\mathbf{Z} = \mathbf{X} + \imath\mathbf{Y}\colon M \to \mathbb{C}^n$ are conformal minimal immersions $\mathbf{X}, \mathbf{Y}\colon M \to \mathbb{R}^n$. The converse holds on the disk $\mathbb{D} = \{\zeta \in \mathbb{C}\colon |\zeta| < 1\}$.

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$$\mathcal{A}_* = \{z = (z_1, \ldots, z_n) \in \mathbb{C}^n \setminus \{0\} \colon \sum_{j=1}^{n} z_j^2 = 0\}$$

is an **Oka manifold***, i.e., maps $M \to \mathcal{A}_*$ from any Stein manifold M (in particular, from any open Riemann surface) satisfy all forms of the Oka principle (with approximation, interpolation, parametric, ...).*

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Proof: The holomorphic vector fields on $\mathbb{C}^n$,

$$V_{j,k}(z) = z_j \frac{\partial}{\partial z_k} - z_k \frac{\partial}{\partial z_j}, \quad 1 \le j, k \le n$$

are linear and hence $\mathbb{C}$-complete, their flows preserve $\mathcal{A}_*$, and they span the tangent space of $\mathcal{A}_*$ at every point. Thus, $\mathcal{A}_*$ is elliptic in the sense of Gromov, and hence an Oka manifold. **M. Gromov, JAMS 2 (1989)**.

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- **Oka principle:** Every continuous map $\mathbf{f}_0 \colon M \to \mathcal{A}_*$ is homotopic to a holomorphic map $\mathbf{f} \colon M \to \mathcal{A}_*$ such that $\mathbf{f}\theta$ has vanishing periods, hence it defines a holomorphic null curve $\int \mathbf{f}\theta \colon M \to \mathbb{C}^n$.

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- **Isotopies:** Every conformal minimal immersion $M \to \mathbb{R}^n$ is regularly homotopic (in the space of of conformal minimal immersions) to the real part of a holomorphic null curve $M \to \mathbb{C}^n$.

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General position theorems

- Every null curve $M \to \mathbb{C}^n$ $(n \geq 3)$ can be approximated uniformly on compacts by **(properly) embedded null curves** $M \hookrightarrow \mathbb{C}^n$.

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  Does it admit a proper holomorphic embedding in $\mathbb{C}^2$?

  The Bell-Narasimhan Conjecture; Encyclopaedia Math. Sci. 69, 1–38, Springer (1990)

Complete proper minimal surfaces

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- **Complete minimal surfaces with Jordan boundaries:**

  Every conformal minimal immersion $\mathbf{X}_0 \colon M \to \mathbb{R}^n$ $(n \geq 3)$ can be approximated, uniformly on M, by continuous maps $\mathbf{X} \colon M \to \mathbb{R}^n$ such that $\mathbf{X} \colon \mathring{M} \to \mathbb{R}^n$ is a **complete conformal minimal immersion** and $\mathbf{X} \colon bM \to \mathbb{R}^n$ is a **topological embedding**.
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- **Complete proper minimal surfaces in convex domains:**

  Let D be a convex domain in $\mathbb{R}^n$ $(n \geq 3)$. Every conformal minimal immersion $\mathbf{X}_0 \colon M \to D$ can be approximated, uniformly on compacts in $\mathring{M}$, by proper complete conformal minimal immersions $\mathbf{X} \colon \mathring{M} \to D$. If D is bounded and strongly convex, then $\mathbf{X}$ can be chosen continuous on M (mapping bM to bD).

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A. Alarcón, B. Drinovec Drnovšek, F. Forstnerič, F.J. López, Proc. London Math. Soc. (3) 111 (2015)

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A domain D with $\mathscr{C}^2$ boundary is minimally convex (also called **mean-convex**) if and only if $\kappa_1(\mathbf{x}) + \kappa_2(\mathbf{x}) \geq 0$ for each point $\mathbf{x} \in bD$, where $\kappa_1(\mathbf{x}), \kappa_2(\mathbf{x})$ are the principal curvatures of bD at $\mathbf{x}$.

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- If $D \subset \mathbb{R}^3$ is minimally convex, then every conformal minimal immersion $\mathbf{X} \colon M \to D$ can be approximated, uniformly on compacts in $\mathring{M}$, by proper complete conformal minimal immersions $\widetilde{\mathbf{X}} \colon \mathring{M} \to D$. If bD is smooth and $\kappa_1(\mathbf{x}) + \kappa_2(\mathbf{x}) > 0$ for each point $\mathbf{x} \in bD$ (such D is called a **strongly mean convex domain**), then $\widetilde{\mathbf{X}}$ can be chosen continuous on M.

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A. Alarcón, B. Drinovec Drnovšek, F. Forstnerič, F.J. López, Minimal surfaces in minimally convex domains. arxiv:1510.04006

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Every conformal minimal immersion $\mathbf{Y}\colon N \to \mathbb{R}^n$ lifts to a $\mathfrak{I}$-invariant conformal minimal immersion $\mathbf{X}\colon M \to \mathbb{R}^n$, i.e.,

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Conversely, a $\mathfrak{I}$-invariant conformal minimal immersion $\mathbf{X}\colon M \to \mathbb{R}^n$ descends to a conformal minimal immersion $\mathbf{Y}\colon N \to \mathbb{R}^n$.

Main theorem for non-orientable minimal surfaces

Theorem (Alarcón, F., López; in preparation)

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Then, all results mentioned above hold for $\mathfrak{I}$-invariant conformal minimal immersions $M \to \mathbb{R}^n$.

Main theorem for non-orientable minimal surfaces

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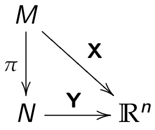
Hence, all mentioned results also hold for conformal minimal immersions $N \to \mathbb{R}^n$ from any non-orientable surface N endowed with a conformal structure (without having to change the conformal structure).



Tools: $\mathfrak{I}$-invariant functions, 1-forms, and sprays

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Clearly, a function $f = u + \imath v \colon M \to \mathbb{C}$ belongs to $\mathscr{O}_{\mathfrak{I}}(M)$ iff $u, v \colon M \to \mathbb{R}$ are conjugate harmonic functions satisfying

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For every $f \in \mathscr{O}(M)$ we have that $\overline{f \circ \mathfrak{I}} \in \mathscr{O}(M)$ and

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Example

If V is a holomorphic vector field on $\mathbb{C}^n$ which is real on $\mathbb{R}^n$, then its flow satisfies $\phi_{\bar{t}}(\bar{z}) = \overline{\phi_t(z)}$. Let $V_1, \ldots, V_N$ be holomorphic vector fields on $\mathbb{C}^n$ which are real on $\mathbb{R}^n$, and let ϕ_t^j denote the flow of V_j. Given a $\mathfrak{I}$-invariant holomorphic map $\mathbf{X}\colon M \to \mathbb{C}^n$, the map

$$F(p, t_1, \ldots, t_N) = \phi_{t_1}^1 \circ \cdots \circ \phi_{t_N}^N(\mathbf{X}(p))$$

is a $\mathfrak{I}$-invariant holomorphic spray of maps $M \to \mathbb{C}^n$.

$\mathfrak{I}$-invariant homology basis and period map

Lemma

Let $(M, \mathfrak{I})$ be a bordered Riemann surface with a fixed-point-free involution $\mathfrak{I}\colon M \to M$. The exists a **Runge homology basis** *$\mathcal{B} = \mathcal{B}^+ \cup \mathcal{B}^-$ for $H_1(M, \mathbb{Z})$, where*

$$\mathcal{B}^+ = \{\delta_1, \ldots, \delta_l\}, \quad \mathcal{B}^- = \{\mathfrak{I}(\delta_2), \ldots, \mathfrak{I}(\delta_l)\}, \quad \mathfrak{I}_*\delta_1 = \delta_1.$$

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Denote by E the union of supports of the curves in $\mathcal{B}$. The Runge property means that $M \subset E$ has no relatively compact connected components. This guarantees Mergelyan approximation on E.

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Let $\mathcal{P}^+ = (\mathcal{P}_1^+, \ldots, \mathcal{P}_l^+)\colon \mathscr{O}(M) \to \mathbb{C}^l$ denote the **period map** given by

$$\mathcal{P}_j^+(f) = \int_{\delta_j} f\theta, \qquad f \in \mathscr{O}(M),\ j = 1, \ldots, l.$$

Similarly, we define $\mathcal{P}^+(\phi) = \left(\int_{\delta_j} \phi\right)_{j=1,\ldots,l}$ for a holomorphic 1-form ϕ.

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Lemma (1)

Let ϕ be a $\mathfrak{I}$-invariant holomorphic 1-form on M. Then:

(a) *ϕ is exact if and only if $\mathcal{P}^+(\phi) = 0$.*

(b) *$\Re\phi$ is exact if and only if $\Re\mathcal{P}^+(\phi) = 0$.*

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Proof. (a) By $\mathfrak{I}$-invariance of ϕ we have

$$\int_{\mathfrak{I}_*\delta_j} \phi = \int_{\delta_j} \mathfrak{I}^*\phi = \int_{\delta_j} \overline{\phi}, \qquad j = 1, \dots, l.$$

Therefore, $\mathcal{P}^+(\phi) = 0$ implies that ϕ has vanishing periods over all curves in $\mathcal{B} = \mathcal{B}^+ \cup \mathcal{B}^-$, and hence it is exact. The converse is obvious.

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(b) Likewise, $\Re\mathcal{P}^+(\phi) = 0$ implies that $\Re\phi$ is exact. The imaginary periods $\Im\mathcal{P}^+(\phi)$ (the flux of ϕ) can be arbitrary subject to the conditions

$$\int_{\mathfrak{I}_*\delta_j} \Im\phi = -\int_{\delta_j} \Im\phi, \qquad j = 1, \ldots, l.$$

In particular, $\int_{\delta_1} \Im\phi = 0$ since $\mathfrak{I}_*\delta_1 = \delta_1$.

$\mathfrak{I}$-invariant period dominating sprays

Lemma (2)

Let $\mathcal{B} = \mathcal{B}^+ \cup \mathcal{B}^-$ *be a homology basis of* $H_1(M, \mathbb{Z})$ *as above, and let* $\mathcal{P}^+ \colon \mathscr{A}(M, \mathbb{C}^n) \to (\mathbb{C}^n)^l$ *denote the period map associated to* $\mathcal{B}^+$*:*

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For every nonflat, $\mathfrak{I}$-invariant map $f \colon M \to \mathcal{A}_*$ of class $\mathscr{A}(M)$ there exists a dominating $\mathfrak{I}$-invariant spray $F \colon M \times r\mathbb{B}^N \to \mathcal{A}_*$ of class $\mathscr{A}(M)$ which is also **period dominating**, *in the sense that the differential*

$$\frac{\partial}{\partial \zeta}\bigg|_{\zeta=0} \mathcal{P}^+(F(\cdot\,,\zeta)) \colon \mathbb{C}^N \to (\mathbb{C}^n)^l$$

maps $\mathbb{R}^N$ (the real part of $\mathbb{C}^N$) surjective onto $\mathbb{R}^n \times (\mathbb{C}^n)^{l-1}$.

(Very) special Cartan pairs

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A special Cartan pair (A, B) is **very special** if the sets B' and $A \cap B'$ are discs. (Of course $\mathfrak{I}(B')$ and $A \cap \mathfrak{I}(B')$ are then also discs.)

Gluing pairs of $\mathfrak{I}$-invariant sprays

Lemma (3)

Let $(M, \mathfrak{I})$ *be as above. Assume that*

- (A, B) *is a special* $\mathfrak{I}$*-invariant Cartan pair in* M*,*
- $\epsilon > 0$ *and* $r > 0$ *are real number, and*
- $F\colon A \times r\mathbb{B}^N \to \mathcal{A}_*$ *is a* $\mathfrak{I}$*-invariant spray of class* $\mathscr{A}(A)$ *which is dominating over the set* $C = A \cap B$*.*

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Then, there exist $\delta > 0$ and $r' \in (0, r)$ such that for every $\mathfrak{I}$-invariant spray $G\colon B \times r\mathbb{B}^N \to \mathcal{A}_*$ of class $\mathscr{A}(B)$ satisfying

$$||F - G||_{0, C \times r\mathbb{B}^N} < \delta$$

there is a $\mathfrak{I}$-invariant spray $H\colon (A \cup B) \times r'\mathbb{B}^N \to \mathcal{A}_*$ of class $\mathscr{A}(A \cup B)$ satisfying

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Basic approximation lemma for $\mathfrak{I}$-invariant maps

Lemma (4)

Let $(M, \mathfrak{I})$ be as above, and let (A, B) be a **very special $\mathfrak{I}$-invariant Cartan pair** *in M. Let $\mathcal{P}^+$ denote the period map on A (Lemma 2).*

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Every $\mathfrak{I}$-invariant map $f\colon A \to \mathcal{A}_*$ of class $\mathscr{A}(A)$ can be approximated uniformly on A by $\mathfrak{I}$-invariant holomorphic maps $\tilde{f}\colon A \cup B \to \mathcal{A}_*$ satisfying $\mathcal{P}^+(\tilde{f}) = \mathcal{P}^+(f)$.

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Proof. By Lemma 2, there exists a $\mathfrak{J}$-invariant dominating and period dominating spray $F\colon A \times r\mathbb{B}^N \to \mathcal{A}_*$ of class $\mathscr{A}(A)$ with $F(\cdot\,, 0) = f$.

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Pick a number $r' \in (0, r)$.
Since $\mathcal{A}_*$ is an Oka manifold, it is possible to approximate F, uniformly on $C' \times r'\mathbb{B}^N$, by a holomorphic spray $G\colon B' \times r'\mathbb{B}^N \to \mathcal{A}_*$.

Proof of Lemma 4

We extend G to $\mathfrak{I}(B') \times r'\mathbb{B}^N$ by symmetrizing:

$$G(p, \zeta) = G(\mathfrak{I}(p), \bar{\zeta}) \quad \text{for } p \in \mathfrak{I}(B') \text{ and } \zeta \in r'\mathbb{B}^N.$$

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By Lemma 3, there is $r'' \in (0, r')$ such that, if G is sufficiently close to F on $(A \cap B) \times r'\mathbb{B}^N$, then there is a $\mathfrak{I}$-invariant spray $H\colon (A \cup B) \times r''\mathbb{B}^N \to \mathcal{A}_*$ which approximates F on $A \times r''\mathbb{B}^N$.

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If the approximation is sufficiently close, the period domination property of F implies that there exists $\zeta \in r''\mathbb{B}^N \cap \mathbb{R}^N$ such that the $\mathfrak{I}$-invariant map $\tilde{f} = H(\cdot\,, \zeta)\colon A \cup B \to \mathcal{A}_*$ satisfies the condition $\mathcal{P}^+(f) = \mathcal{P}^+(\tilde{f})$.

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This proves Lemma 4. $\square$

Change of topology of the domain

Attach to a domain $A \subset M$ a smooth arc E (or a couple of arcs $E = E_1 \cup E_2$ with $\mathfrak{I}(E_1) = E_2$ and $E_1 \cap E_2 = \emptyset$) and proceed as follows.

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1. Extend the derivative $f = 2\partial X/\theta$ from A to a map $f\colon A \cup E \to \mathcal{A}_*$ such that $f \circ \mathfrak{I} = \overline{f}$ and $\int_E \Re(f\theta)$ has a correct value (to satisfy the period vanishing condition if E closes to a nontrivial loop in $A \cup E$).

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3. Approximate F by a holomorphic spray $\widetilde{F}\colon D \times r\mathbb{B}^N \to \mathcal{A}_*$ on a neighborhood D of $A \cup E$. The period domination of F ensures that there is a value $\zeta_0 \in r\mathbb{B}^N \cap \mathbb{R}^N$ such that $\widetilde{F}(\cdot\,, \zeta_0)$ integrates to a conformal minimal immersion $D \to \mathbb{R}^n$.

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1. Extend the derivative $f = 2\partial X/\theta$ from A to a map $f\colon A \cup E \to \mathcal{A}_*$ such that $f \circ \mathfrak{I} = \overline{f}$ and $\int_E \Re(f\theta)$ has a correct value (to satisfy the period vanishing condition if E closes to a nontrivial loop in $A \cup E$).
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3. Approximate F by a holomorphic spray $\widetilde{F}\colon D \times r\mathbb{B}^N \to \mathcal{A}_*$ on a neighborhood D of $A \cup E$. The period domination of F ensures that there is a value $\zeta_0 \in r\mathbb{B}^N \cap \mathbb{R}^N$ such that $\widetilde{F}(\cdot\,, \zeta_0)$ integrates to a conformal minimal immersion $D \to \mathbb{R}^n$.
4. In the non-orientable case, choose a neighborhood V_1 of the arc E_1, approximate F over $(A \cup E) \cap V_1$ by a spray G over V_1, define G on $\mathfrak{I}(V_1) \supset E_2$ by $G(p, \zeta) = G(\mathfrak{I}(p), \bar{\zeta})$, and glue F and G into an $\mathfrak{I}$-invariant spray over a neighborhood of $A \cup E$. Finish as before.

Conclusion

The **Runge-Mergelyan approximation theorem** and the **Oka principle for conformal minimal immersions** (both in the orientable and non-orientable case) are proved by using Lemmas 1–4, together with the analysis at critical points of a strongly subharmonic exhaustion function on M as explained above.

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The **Runge-Mergelyan approximation theorem** and the **Oka principle for conformal minimal immersions** (both in the orientable and non-orientable case) are proved by using Lemmas 1–4, together with the analysis at critical points of a strongly subharmonic exhaustion function on M as explained above.

To obtain **complete bounded conformal minimal surfaces** in $\mathbb{R}^n$ (including those with Jordan boundaries), and **proper conformal minimal surfaces** in (minimally) convex domains, we also use approximate solutions to the **Riemann-Hilbert boundary value problem** for conformal minimal surfaces and holomorphic null curves.

The Riemann-Hilbert problem for minimal surfaces

Assume that M is a compact bordered Riemann surface and $\mathbf{X}\colon M \to \mathbb{R}^n$ $(n \geq 3)$ is a conformal minimal immersion. Let $I \subset bM$ be an arc.

Let $\mathbf{Y}\colon bM \times \overline{\mathbb{D}} \to \mathbb{R}^n$ be a continuous map of the form

$$\mathbf{Y}(p,\xi) = \mathbf{X}(p) + f(p,\xi)\mathbf{u} + g(p,\xi)\mathbf{v}, \quad p \in I,\ \xi \in \overline{\mathbb{D}},$$

where $\mathbf{u},\mathbf{v} \in \mathbb{R}^n$ is an orthonormal pair, $F(p,\cdot\,) = f(p,\cdot\,) + \mathrm{i}g(p,\cdot\,)$ is a holomorphic immersion for each $p \in I$, and $F(p,\cdot\,) = 0$ for $p \in bM \setminus I$. Then, we can find conformal minimal immersions $\widetilde{\mathbf{X}}\colon M \to \mathbb{R}^n$ such that

- $\widetilde{\mathbf{X}}$ approximates $\mathbf{X}$ outside a small neighbourhood of I in M;
- $\widetilde{\mathbf{X}}(M)$ lies close to $\mathbf{X}(M) \cup \bigcup_{p\in I} \mathbf{Y}(p,\overline{\mathbb{D}})$;
- $\widetilde{\mathbf{X}}(p)$ lies close to the curve $\mathbf{Y}(p,b\mathbb{D})$ for every $p \in I$;
- $\mathrm{Flux}(\widetilde{\mathbf{X}}) = \mathrm{Flux}(\mathbf{X})$.

A. Alarcón, B. Drinovec Drnovšek, F. Forstnerič, F.J. López, Proc. London Math. Soc. (3) 111 (2015)

THANK YOU FOR YOUR ATTENTION

Appendix A: Curvature of surfaces in $\mathbb{R}^n$

Assume that D is a domain in $\mathbb{R}^2_{(u_1,u_2)}$ and $\mathbf{X} = (X_1, \ldots, X_n)\colon D \to \mathbb{R}^n$ is a $\mathscr{C}^2$ immersion. Let $S = \mathbf{X}(D) \subset \mathbb{R}^n$, a parametrized surface in $\mathbb{R}^n$.

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Consider a smooth embedded curve in S,

$$\lambda(t) = \mathbf{X}(u_1(t), u_2(t)) \in S.$$

Let $s = s(t)$ denote the arc length on λ. The number

$$\kappa(\mathbf{T}, \mathbf{N}) := \frac{d^2\lambda}{ds^2} \cdot \mathbf{N} = \sum_{i,j=1}^{2} \left(\mathbf{X}_{u_i u_j} \cdot \mathbf{N} \right) \frac{du_i}{ds} \frac{du_j}{ds}$$

is the **normal curvature** of S at $p = \lambda(t) \in S$ in the tangent direction $\mathbf{T} = \lambda'(s) \in T_p S$ with respect to the normal vector $\mathbf{N} \in N_p S$.

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In terms of t-derivatives we get

$$\kappa(\mathbf{T}, \mathbf{N}) = \frac{\sum_{i,j=1}^{2} \left(\mathbf{X}_{u_i u_j} \cdot \mathbf{N} \right) \dot{u}_i \dot{u}_j}{\sum_{i,j=1}^{2} g_{i,j} \dot{u}_i \dot{u}_j} = \frac{\text{second fundamental form}}{\text{first fundamental form}}$$

The principal curvature and the mean curvature

Fix a normal vector $\mathbf{N} \in N_pS$ and vary the unit tangent vector $\mathbf{T} \in T_pS$. The **principal curvatures** of S at p in direction $\mathbf{N}$ are the numbers

$$\kappa_1(\mathbf{N}) = \max_{\mathbf{T}} \kappa(\mathbf{T}, \mathbf{N}), \qquad \kappa_2(\mathbf{N}) = \min_{\mathbf{T}} \kappa(\mathbf{T}, \mathbf{N}).$$

Their average

$$\mathrm{H}(\mathbf{N}) = \frac{\kappa_1(\mathbf{N}) + \kappa_2(\mathbf{N})}{2} \in \mathbb{R}$$

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Let $G = (g_{i,j})$ and $h(\mathbf{N}) = (h_{i,j}(\mathbf{N})) = (\mathbf{X}_{u_i u_j} \cdot \mathbf{N})$ denote the matrices of the 1st and the 2nd fundamental form, respectively. The extremal values κ_1, κ_2 of $\kappa(\mathbf{T}, \mathbf{N})$ are roots of the equation

$$\det\big(h(\mathbf{N}) - \mu G\big) = 0$$

$$\det G \cdot \mu^2 - \big(g_{2,2} h_{1,1}(\mathbf{N}) + g_{1,1} h_{2,2}(\mathbf{N}) - 2 g_{1,2} h_{1,2}(\mathbf{N})\big)\mu + \det h(\mathbf{N}) = 0.$$

The mean curvature vector

The Vieta formula gives

$$\mathrm{H}(\mathbf{N}) = \frac{\kappa_1 + \kappa_2}{2} = \frac{g_{2,2}\mathbf{X}_{u_1 u_1} + g_{1,1}\mathbf{X}_{u_2 u_2} - 2g_{1,2}\mathbf{X}_{u_1 u_2}}{2 \det G} \cdot \mathbf{N}.$$

There is a unique normal vector $\mathbf{H} \in N_p S$ such that

$$\mathrm{H}(\mathbf{N}) = \mathbf{H} \cdot \mathbf{N} \quad \text{for all } \mathbf{N} \in N_p S.$$

This vector $\mathbf{H}$ is the **mean curvature vector** of the surface S at p.

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Assume now that we work in isothermal coordinates:

$$G = (g_{i,j}) = \xi I, \ \det G = \xi^2; \quad \xi = ||\mathbf{X}_{u_1}||^2 = ||\mathbf{X}_{u_2}||^2, \ \mathbf{X}_{u_1} \cdot \mathbf{X}_{u_2} = 0$$

Then:

$$\mathrm{H}(\mathbf{N}) = \frac{\mathbf{X}_{u_1 u_1} + \mathbf{X}_{u_2 u_2}}{2\xi} \cdot \mathbf{N} = \frac{\triangle \mathbf{X}}{2\xi} \cdot \mathbf{N}.$$

The main formula in isothermal coordinates

Claim: $\triangle \mathbf{X} = \mathbf{X}_{u_1 u_1} + \mathbf{X}_{u_2 u_2}$ is orthogonal to $S = \mathbf{X}(D)$.

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Proof: Conformality means that

$$\mathbf{X}_{u_1} \cdot \mathbf{X}_{u_1} = \mathbf{X}_{u_2} \cdot \mathbf{X}_{u_2}, \qquad \mathbf{X}_{u_1} \cdot \mathbf{X}_{u_2} = 0.$$

Differentiating the first identity on u_1 and the second one on u_2 yields

$$\mathbf{X}_{u_1 u_1} \cdot \mathbf{X}_{u_1} = \mathbf{X}_{u_1 u_2} \cdot \mathbf{X}_{u_2} = -\mathbf{X}_{u_2 u_2} \cdot \mathbf{X}_{u_1},$$

whence $\triangle \mathbf{X} \cdot \mathbf{X}_{u_1} = 0$. Similarly we get $\triangle \mathbf{X} \cdot \mathbf{X}_{u_2} = 0$ by differentiating the first identity on u_2 and the second one on u_1.

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Since $\mathrm{H}(\mathbf{N}) = \mathbf{H} \cdot \mathbf{N} = \frac{\triangle \mathbf{X}}{2\xi} \cdot \mathbf{N}$ and $\triangle \mathbf{X}$ is normal to S, we get

$$\triangle \mathbf{X} = 2\xi \mathbf{H}, \quad \xi = ||\mathbf{X}_{u_1}||^2 = ||\mathbf{X}_{u_2}||^2 \quad \text{(Main formula)}.$$

Lagrange's formula for the first variation of the area

The area of an immersed surface $\mathbf{X}\colon D \to \mathbb{R}^n$ equals

$$\mathcal{A}(\mathbf{X}) = \int_D \sqrt{\det G} \cdot du_1 du_2.$$

Let $\mathbf{N}\colon D \to \mathbb{R}^n$ be a *normal vector field* along $\mathbf{X}$ which vanishes on bD. Consider the 1-parameter family of maps $\mathbf{X}^t\colon D \to \mathbb{R}^n$:

$$\mathbf{X}^t(u) = \mathbf{X}(u) + t\,\mathbf{N}(u), \quad u \in D,\ t \in \mathbb{R}.$$

A calculation gives the formula for the first variation of the area:

$$\delta\mathcal{A}(\mathbf{X})\mathbf{N} = \frac{d}{dt}\bigg|_{t=0} \mathcal{A}(\mathbf{X}^t) = -2 \int_D \mathbf{H} \cdot \mathbf{N} \sqrt{\det G} \cdot du_1 du_2.$$

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It follows that $\delta\mathcal{A}(\mathbf{X}) = 0 \Longleftrightarrow \mathbf{H} = 0.$

Appendix B: Topological structure of non-orientable surfaces

Every compact non-orientable surface N without boundary is the connected sum $N = \mathbb{P}^2 \sharp \cdots \sharp \mathbb{P}^2$ of $g \geq 1$ copies of the real projective plane $\mathbb{P}^2$; the number g is the genus of N. (This is the maximal number of pairwise disjoint closed curves in N which reverse the orientation.)

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Furthermore, $\mathbb{K} = \mathbb{P}^2 \sharp \mathbb{P}^2$ is the Klein bottle, and for any non-orientable surface N we have $N \sharp \mathbb{K} = N \sharp \mathbb{T}$ where $\mathbb{T}$ is the torus.

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This gives the following dichotomy according to whether the genus g is even or odd:

(I) $g = 1 + 2k \geq 1$ is odd. In this case, $N = \mathbb{P}^2 \sharp \overbrace{\mathbb{T} \sharp \cdots \sharp \mathbb{T}}^{k}$.

(II) $g = 2 + 2k \geq 2$ is even. In this case, $N = \mathbb{P}^2 \sharp \mathbb{P}^2 \sharp \overbrace{\mathbb{T} \sharp \cdots \sharp \mathbb{T}}^{k}$.

Geometric model of 2-sheeted oriented covering

Let $\iota\colon M \to N$ be a 2-sheeted covering by a compact orientable surface $(M, \mathfrak{I})$. Then M has genus $g-1$. We construct an explicit geometric model for $(M, \mathfrak{I})$ in $\mathbb{R}^3$.

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Let $\mathbb{S}^2$ be the unit sphere in $\mathbb{R}^3$ centered at the origin, and let $\tau\colon \mathbb{R}^3 \to \mathbb{R}^3$ be the involution $\tau(\mathbf{x}) = -\mathbf{x}$.

Case (I): $N = \mathbb{P}^2 \sharp \overbrace{\mathbb{T} \sharp \cdots \sharp \mathbb{T}}^{k}$. We take M to be an embedded surface

$$(\mathbb{T}_1^- \sharp \cdots \sharp \mathbb{T}_k^-) \sharp \mathbb{S}^2 \sharp (\mathbb{T}_1^+ \sharp \cdots \sharp \mathbb{T}_k^+)$$

of genus $g-1 = 2k$ in $\mathbb{R}^3$ which is invariant by the symmetry with respect to the origin (i.e., $\tau(M) = M$), where $\mathbb{T}_j^-$, $\mathbb{T}_j^+$ are embedded tori in $\mathbb{R}^3$ with $\tau(\mathbb{T}_j^-) = \mathbb{T}_j^+$ for all j. Set $\mathfrak{I} = \tau|_M\colon M \to M$. (See Fig. 1.)

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We have $M = M^- \cup C \cup M^+$, where $C \subset \mathbb{S}^2$ is a closed $\mathfrak{I}$-invariant cylinder and M^- and M^+ are the closure of the two components of $M \setminus C$, both homeomorphic to the connected sum of k tori minus an open disk. Obviously $\mathfrak{I}(M^-) = M^+$ and $M^- \cap M^+ = \emptyset$.

Geometric model, Case II

Case (II): $N = \mathbb{P}^2 \sharp \mathbb{P}^2 \sharp \overbrace{\mathbb{T} \sharp \cdots \sharp \mathbb{T}}^{k} = \mathbb{K} \sharp \overbrace{\mathbb{T} \sharp \cdots \sharp \mathbb{T}}^{k}.$

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Let $\mathbb{T}_0 \subset \mathbb{R}^3$ be the standard revolution torus centered at the origin, i.e., invariant under the antipodal map τ. In this case we let M be an embedded τ-invariant surface

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in $\mathbb{R}^3$, where the tori $\mathbb{T}_j^\pm$ are as above, and set $\mathfrak{I} = \tau|_M$. (See Figure 2.)

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Write $M = M^- \cup K \cup M^+$, where $K \subset \mathbb{T}_0 \subset \mathbb{R}^3$ is a $\mathfrak{I}$-invariant torus minus two disjoint open disks, and M^- and M^+ are the closure of the two components of $M \setminus K$, both homeomorphic to the connected sum of k tori minus an open disk. Obviously $\mathfrak{I}(M^-) = M^+$ and $M^- \cap M^+ = \emptyset$.