Stein Manifolds and Holomorphic Mappings: Errata

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Abstract This note contains errata for my book *Stein Manifolds and Holomorphic Mappings (The Homotopy Principle in Complex Analysis)*, Second Edition, Springer, Cham, 2017. A survey of main new developments since the book's publication can be found in the paper entitled *Developments in Oka theory since 2017* which is also posted on this webpage.

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1. Errata

page 28: in the penultimate item, ρ must be replaced by u.

page 74: In Proposition 3.3.2, the complex space Z must be Stein, but the base space X need not be Stein.

page 75: Theorem 3.3.5 is stated incorrectly. A counterexample is the projection $\pi : Z = \mathbb{C}^2 \to \mathbb{C} = X$, $\pi(x, y) = x$, and $S = \{(x, y) : x = y^2\}$. The theorem holds (with the proof given in [202, Proposition 3.2]) under the following additional condition:

For every point $z_0 \in S$ there are a neighbourhood $U \subset Z$ and a holomorphic map $\psi = (\psi_1, \ldots, \psi_d) : U \to \mathbb{C}^d$, where d is the codimension of the fibres S_x in Z_x ($x \in X$), such that $S \cap U = \{\psi = 0\}$ and the vertical differential $VD(\psi)$ has maximal rank d.

Equivalently, S can be locally straightened by a biholomorphic fibre preserving map on a neighbourhood of z_0 in Z. (Since $\pi : Z \to X$ is a submersion, the fibres Z_x are complex manifolds.) This condition implies that S admits a normal bundle in Z which is a holomorphic subbundle of the vertical tangent bundle $VT(Z)|_S$ restricted to S. This obviously fails in the above example: for any holomorphic function f on $\mathbb{C}^2$ near $(0, 0)$ vanishing on S, the function $f(0, \cdot\,)$ has a zero of order at least 2 at the origin.

Applications of Theorem 3.3.5 in the book are consistent with this additional condition.

page 156, Theorem 4.11.13: the correct attribution is [322, Theorem 6].

page 225: a part of the argument in the second paragraph of the proof of Theorem 5.6.5 is incomplete since we do not have a Stein domain in E which is needed to apply Theorem 3.3.5. The problem is resolved by passing to the Stein graphs of the respective maps f, h and using the following result [194, Lemma 3.4].

Lemma 1. *Let $\pi : \Omega \to Y$ be a holomorphic submersion of a Stein manifold Ω onto a complex manifold Y. Then there are an open Stein domain $W \subset Y \times \Omega$ containing the submanifold $S := \{(y, z) \in Y \times \Omega : \pi(z) = y\}$ and a holomorphic retraction $\widetilde{\rho} : W \to S$ of the form $\widetilde{\rho}(y, z) = (y, \rho(y, z))$ for every $(y, z) \in W$.*

It follows that for every $y \in Y$ the map $\rho_y = \rho(y, \cdot\,)$ is a holomorphic retraction of an open neighbourhood of the fibre $\Omega_y = \pi^{-1}(y)$ in Ω onto Ω_y, depending holomorphically on y. Note that Lemma 1 follows from Theorem 3.3.5 applied to the submersion $Y \times \Omega \to Y$, $(y, z) \mapsto y$, and the submanifold S defined in the lemma.

The proof of Theorem 5.6.5 can now be completed as follows. Consider the manifolds $\widetilde{X} = \mathbb{C}^n \times X$, $\widetilde{E} = \mathbb{C}^n \times E$ and the projection $\tilde{\pi} : \widetilde{E} \to \widetilde{X}$, $\tilde{\pi}(z, e) = (z, \pi(e))$. Recall that $U \subset \mathbb{C}^n$ is an open convex neighbourhood of the compact convex set $K \subset \mathbb{C}^n$ and $h : U \to E$ and $f = \pi \circ h : U \to X$ are holomorphic maps. Their graphs

$$\Gamma_f = \{(z, f(z)) : z \in U\} \subset \widetilde{X}, \qquad \Gamma_h = \{(z, h(z)) : z \in U\} \subset \widetilde{E}$$

are locally closed Stein submanifolds of $\widetilde{X}$ and $\widetilde{E}$, respectively, which therefore admits open Stein neighborhoods $Y \subset \widetilde{X}$ and $\Omega \subset \widetilde{E}$. These can be chosen such that $\tilde{\pi}|_\Omega : \Omega \to Y$ is a surjective holomorphic submersion. Let ρ_y for $y \in Y$ be a holomorphic retraction onto the fibre Ω_y (depending holomorphically on y), furnished by Lemma 1. If the approximating map $f_1 : V \to X$ from an open set $V \supset K$ in $\mathbb{C}^n$ (see the book) is sufficiently uniformly close to f on K, then the point $(z, h(z)) \in \Omega$ lies in the domain of the retraction $\rho_{(z,f_1(z))}$ for all z in some neighbourhood $U_1 \subset U$ of K. For every $z \in U_1$ let $h_1(z) \in E$ denote the projection of the point $\rho_{(z,f_1(z))}(z, h(z)) \in \Omega_{(z,f_1(z))} \subset \widetilde{E}$ to E. Then, the map $h_1 : U_1 \to E$ is holomorphic, uniformly close to h on K, and it satisfies $\pi \circ h_1(z) = f_1(z)$ for $z \in U_1$. The proof can now be completed as in the book.

page 233: the first statement in Proposition 5.6.23 is false; a counterexample is any compact hyperbolic manifold. Here is the correct statement supported by the proof.

Proposition 5.6.23 *A complex manifold Y with the density property whose tangent bundle is pointwise generated by holomorphic vector fields on Y is flexible in the sense of Arzhantsev et al., and hence an Oka manifold. In particular, a Stein manifold with the density property is elliptic.*

page 235, line 8: condition $A(r) \Subset D_1'$ should read $A(r) \Subset D_0'$.

page 252: Proposition 5.13.1 is immediate if the subspace P_0 of the parameter space P is a deformation retract of a neighbourhood of P_0 in P, since the retraction allows a suitable reparametrization of the given family of sections. This suffices for most applications.

page 285: The condition in the last line should read $f_{(p,0)}(\overline{V}) \subset D_j$ (replace K by $\overline{V}$).

page 287: Condition HAP is stated incorrectly in Definition 6.6.5: the same condition must hold for any local holomorphic spray of sections with parameter in a ball $\mathbb{B} \subset \mathbb{C}^n$. Equivalently, the stated condition must apply to the trivial extensions $Z \times \mathbb{B} \to X \times \mathbb{B}$. This holds for any subelliptic submersions $h : Z \to X$ in view of Theorem 6.6.2.

page 295: Proposition 6.10.1 gives a homotopy connecting a global continuous section to a complex of holomorphic sections of a holomorphic submersion. It was pointed out that the parametric case, which is the first step in the proof of Theorem 6.2.2, is not explained. This is true, but the proof is a simple application of partitions of unity on the parameter space. See Remark 3 in the paper *Developments in Oka theory since 2017*.

page 317, line 11: the statement $H^2(X, \mathbb{Z}) = H^2(M, \mathbb{Z})/(D)$ is not correct: we only have $H^2(X, \mathbb{Z}) \supset H^2(M, \mathbb{Z})/(D)$. This proper inclusion is enough for the argument.

page 379, line 3: Then a generic holomorphic *maps*... (replace by *map*)

page 488: formula (10.15) should read $g(C) = (d-1)(d-2)/2$.

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