Proper holomorphic mappings of Stein manifolds

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Dedication and Abstract

This talk is dedicated to the 80th anniversary of Otto Forster, with my admiration and very best wishes.

Forster belongs to the famous German school of complex analysis centered around Heinrich Behnke, Karl Stein, Hans Grauert, Reinhold Remmert, and Friedrich Hirzebruch. A considerable part of his work is focused on problems of complex analytic geometry, in particular, the Oka-Grauert theory, its extensions and applications.

This talk is motivated by his paper **Plongements des variétés de Stein**, **Comment. Math. Helv. 45 (1970) 170–184**.

In this paper, Forster treated the classical problem of the embedding dimension of Stein manifolds. While the time was not ripe yet for proving the optimal result, he made the correct conjecture which was established in 1992 by Yakov Eliashberg and Mischa Gromov.

The methods introduced by Grauert and Forster, and further developed by several other mathematicians, had a major impact on complex geometry. My aim today is to survey some of these developments.

Stein manifolds (Karl Stein, 1951)

A complex manifold X is said to be a **Stein manifold** if it admits many holomorphic functions in the following sense:

- holomorphic functions on X separate points:

$$x, x' \in X,\ x \neq x' \Longrightarrow f(x) \neq f(x') \text{ for some } f \in \mathcal{O}(X);$$

- X is **holomorphically convex**: for every discrete sequence $a_j \in X$ there exists a holomorphic function f on X such that

$$\lim_{j \to \infty} |f(a_j)| = +\infty.$$

A **Stein space** is a complex space satisfying these axioms.

Examples of Stein manifolds

- Domains in $\mathbb{C}$, open Riemann surfaces (**Behnke & Stein 1949**).
- $\mathbb{C}^n$, and domains of holomorphy in $\mathbb{C}^n$ (**Cartan & Thullen 1932**).
- A closed complex submanifold of a Stein manifold is Stein. In particular, closed complex submanifolds of $\mathbb{C}^N$ are Stein.
- If $E \to X$ is a holomorphic vector bundle and the base X is Stein, then the total space E is Stein.

And a few examples of non-Stein manifolds:

- Any compact complex manifold. Indeed, a Stein manifold does not contain any compact complex subvarieties of positive dimension.
- Quotients of Stein manifolds need not be Stein.
- There exists a fibre bundle $E \to \mathbb{C}$ with fibre $\mathbb{C}^2$ and nonlinear structure group $\Gamma \subset \mathrm{Aut}\,\mathbb{C}^2$ such that E is non-Stein.

  Examples by **Skoda, Demailly, Rosay**.

Theorems of Grauert and Lefschetz-Milnor

Grauert 1958: A complex manifold X is Stein if and only if it admits a *strongly plurisubharmonic exhaustion function* $\rho\colon X \to \mathbb{R}$:

$$dd^c\rho = i\partial\bar{\partial}\rho > 0 \quad \text{(a Kähler form).}$$

In any local coordinate system, such a function is **strongly subharmonic** on each complex line L:

$$\triangle(\rho|_L) > 0.$$

Critical points of such ρ have Morse index $\leq n = \dim_{\mathbb{C}} X$.

This implies the theorem of **Lefschetz and Milnor**:

A Stein manifold X of complex dimension n is homotopy equivalent to a CW-complex of dimension at most n.

In particular, an open Riemann surface is homotopy equivalent to a bouquet of circles.

Embedding Stein manifolds into Euclidean spaces

Remmert 1956, Narasimhan 1960, Bishop 1961:

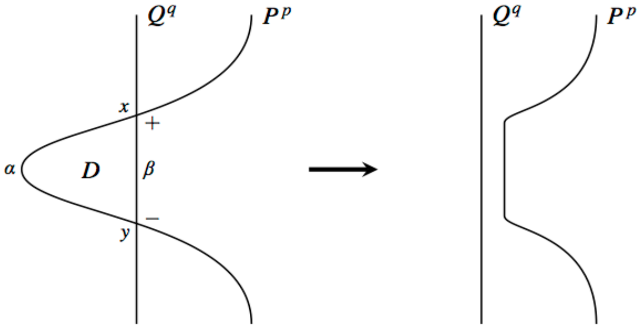
A Stein manifold X of dimension n admits

- an almost proper holomorphic map $X \to \mathbb{C}^n$;
- a proper holomorphic map $X \to \mathbb{C}^{n+1}$;
- a proper holomorphic immersion $X \to \mathbb{C}^{2n}$, and
- a proper holomorphic embedding $X \hookrightarrow \mathbb{C}^{2n+1}$.

Results of this type which depend on general position arguments are often, but not always, optimal.

For example, a smooth manifold of dimension n embeds into $\mathbb{R}^{2n}$ and, if $n > 1$, it immerses into $\mathbb{R}^{2n-1}$ **(Whitney 1944)**. These dimensions are optimal at least for certain values of n.

The Whitney trick



The illustration shows the **Whitney trick** for removing a pair of intersections of complementary dimensions and opposite orientation; in particular, a pair of double points of an immersed n-manifold in $\mathbb{R}^{2n}$.

This type of modification is impossible in the complex world.

A theorem of Forster

We have already mentioned that a Stein manifold X is topologically relatively simple — a CW-complex of dimension at most $n = \dim_{\mathbb{C}} X$. Hence, one expects that X should embed into $\mathbb{C}^N$ for $N \leq 2n$.

What is the smallest $N = N(n)$ that works for all Stein X^n?

Theorem (Forster, Comment. Math. Helv. 1970)

A Stein manifold X^n admits a proper holomorphic immersion into $\mathbb{C}^{2n-1}$, and a proper holomorphic embedding into $\mathbb{C}^{2n}$ if $n > 1$. If $n \geq 6$, then X^n embeds into $\mathbb{C}^N$ with

$$N = 2n - \left[\frac{n-2}{3}\right] \approx \frac{5n+2}{3}.$$

A topological obstruction

Write $X \times \mathbb{C}^N = \mathbb{C}^N_X$. Since a Stein manifold X^n is a CW-complex of dimension $\leq n$, a transversality argument shows the existence of $[(n+1)/2]$ pointwise linearly independent sections of TX; hence

$$TX \cong \mathbb{C}_X^{[(n+1)/2]} \oplus E, \quad \mathrm{rank}\, E = [n/2].$$

By the **Oka-Grauert principle** and **Cartan's Theorem B**, this splitting holds holomorphically. If X immerses into $\mathbb{C}^N$ with $N = n + q$, then

$$TX \oplus E' = \mathbb{C}^N_X, \quad \mathrm{rank}\, E' = q$$

where $E' \to X$ is the complex normal bundle of the immersion. Hence

$$E \oplus E' = \mathbb{C}_X^{q+[n/2]} \implies c(E) \smile c(E') = 1.$$

Choosing X (and hence E) suitably complicated, this requires

$$q = \mathrm{rank}\, E' \geq \mathrm{rank}\, E = [n/2] \implies N = n + q \geq [3n/2].$$

A similar argument gives $N \geq [3n/2] + 1$ for proper embeddings.

Forster's example

Example (Forster, 1970; Proposition 3)

Let Y be the Stein surface

$$Y = \{[x:y:z] \in \mathbb{CP}^2 : x^2 + y^2 + z^2 \neq 0\}.$$

Given an integer $n \geq 2$, the Stein manifold

$$X^n = \begin{cases} Y^m, & \textit{if} \quad \text{n=2m;} \\ Y^m \times \mathbb{C}, & \textit{if} \quad \text{n=2m+1.} \end{cases}$$

does not admit a proper holomorphic embedding into $\mathbb{C}^{[3n/2]}$ or a holomorphic immersion into $\mathbb{C}^{[3n/2]-1}$.

The main point is that $c_1(TY)$ is the nonzero element of $H^2(Y;\mathbb{Z}) \cong H^2(\mathbb{RP}^2;\mathbb{Z}) \cong \mathbb{Z}_2$, so Y is not parallelizable. Hence, Y does not admit a proper holomorphic embedding into $\mathbb{C}^3$ or a holomorphic immersion into $\mathbb{C}^2$. Moreover, for any $n \geq 2$ the top Chern class $c_m(TX)$ is the nonzero element of $H^{2m}(X;\mathbb{Z}) = \mathbb{Z}_2$.

Forster's Conjecture

Forster: *La proposition 3 montre que nos résultats sur les plongements et les immersions des variétés de Stein de dimension 2 sont les meilleurs possible. Pour les autres dimensions on peut conjecturer que toute variété de Stein de dimension n se plonge dans $\mathbb{C}^{N+1}$ et s'immerge dans $\mathbb{C}^N$, où $N = n + [n/2]$.*

Conjecture: A Stein manifold X^n embeds properly into $\mathbb{C}^N$ with

$$N = \left[\frac{3n}{2}\right] + 1 = n + \left[\frac{n}{2}\right] + 1,$$

and it immerses properly holomorphically into $\mathbb{C}^M$ with $M = \left[\frac{3n}{2}\right]$.

What is known 47 years later?

Eliashberg & Gromov 1992; Schürmann 1997:

Forster's conjecture is correct for $n > 1$.

Theorem

A Stein manifold X^n immerses properly holomorphically into $\mathbb{C}^{\left[\frac{3n+1}{2}\right]}$, and if $n > 1$ then it embeds properly holomorphically into $\mathbb{C}^{\left[\frac{3n}{2}\right]+1}$.

Schürmann also proved a precise embedding result for Stein spaces of bounded embedding dimension.

What about $n = 1$? Does every open Riemann surface embed holomorphically into $\mathbb{C}^2$? (It embeds properly into $\mathbb{C}^3$ and immerses properly into $\mathbb{C}^2$ by the Bishop-Narasimhan theorem.)

This question is still very much open and will be discussed later.

Methods to prove the embedding theorem

Some of the ideas and techniques originate in Forster's work, and they also use the **Oka-Grauert theory**. They were further developed by **Gromov 1989**, **Eliashberg & Gromov (1992)**, with extension to Stein spaces given by **Schürmann 1997**.

We begin by choosing a generic almost proper holomorphic map $h\colon X^n \to \mathbb{C}^n$ (**Bishop 1961**) and look for a holomorphic map $g\colon X \to \mathbb{C}^q$ such that $f = (h, g) : X \to \mathbb{C}^{n+q}$ is a proper holomorphic embedding.

$$\begin{tikzcd}
 & \mathbb{C}^{n+q} \arrow[d] \\
X \arrow[ur, "{f=(h,g)}"] \arrow[r, "h"] & \mathbb{C}^n
\end{tikzcd}$$

Such g is found by an induction on strata in suitable stratifications of X and $\mathbb{C}^n$ which are **equisingular** with respect to $h\colon X \to \mathbb{C}^n$.

Equisingular stratifications

Definition

Let X, Y be Stein manifolds and $h\colon X \to Y$ be a proper holomorphic map. Let S be a locally closed connected complex submanifold of Y and $\widetilde{S} = h^{-1}(S) \subset X$. We say that S is ***h*-equisingular** if

(i) the map $h|_{\widetilde{S}}\colon \widetilde{S} \to S$ is a (necessarily proper) immersion, hence a finite holomorphic covering,

(ii) $\dim \ker dh_x$ is constant on every connected component of $\widetilde{S}$, and

(iii) $\ker dh_x \subset T_x X$ is transverse to $T_x \widetilde{S}$ at every point $x \in \widetilde{S}$.

The proof of the embedding theorem consists of a repeated application of the following two basic operations:

(A) **separations of points over a stratum**, and

(B) **elimination of the kernel of the differential over a stratum.**

In this talk, I will concentrate in (A); analysis of (B) is similar.

Separation of points over a stratum

Let Y be a Stein manifold (in our case, $Y = \mathbb{C}^n$).

Let $Y_1 \subset Y_0$ be closed complex subvarieties of Y such that $S = Y_0 \setminus Y_1$ is a connected complex submanifold that is equisingular for $h \colon X \to Y$.

Let $X_j = h^{-1}(Y_j)$ for $j = 0, 1$; then $\widetilde{S} = X_0 \setminus X_1$ is smooth and $h \colon \widetilde{S} \to S$ is a finite holomorphic covering map.

Assume that $g' \colon X \to \mathbb{C}^q$ is a holomorphic map such that $f' = (h, g') \colon X \to Y \times \mathbb{C}^q$ is an embedding over $X_1 = h^{-1}(Y_1)$.

Lemma (Eliashberg & Gromov 1992)

If $q \geq 2$ and $2q > \dim Y_0$, then there exists a holomorphic map $g \colon X \to \mathbb{C}^q$ such that

(i) $f = (h, g) \colon X \to Y \times \mathbb{C}^q$ *is injective on* $\widetilde{S} = X_0 \setminus X_1$, *and*

(ii) $g - g'$ *vanishes to the second order along the subvariety* X_1.

How to find such g?

We seek g in the form

$$g(x) = g_\alpha(x) = g'(x) + \sum_{j=1}^{N} \alpha_j(h(x))\psi_j(x), \quad x \in X,$$

where $\alpha_j \in \mathcal{O}(Y)$ are to be determined and $\psi_1, \dots, \psi_N \colon X \to \mathbb{C}^q$ $(N \geq dq)$ are holomorphic maps chosen such that

(a) each ψ_j vanishes to the second order along the subvariety X_1, and

(b) for every $y \in S$ let $h^{-1}(y) = \{x_1, \dots, x_d\} \in \widetilde{S}$ (distinct points); then the following $dq \times N$ matrix has maximal rank dq:

$$\begin{pmatrix} \psi_1(x_1) & \psi_2(x_1) & \dots & \psi_N(x_1) \\ \psi_1(x_2) & \psi_2(x_2) & \dots & \psi_N(x_2) \\ \dots & \dots & \dots & \dots \\ \psi_1(x_d) & \psi_2(x_d) & \dots & \psi_N(x_d) \end{pmatrix}$$

Avoiding the bad set

Condition (b) implies that for every fixed $y \in S$ the linear map $\Phi_y \colon \mathbb{C}^N \to (\mathbb{C}^q)^d$ given by

$$\mathbb{C}^N \ni \alpha = (\alpha_1, \ldots, \alpha_N) \xmapsto{\Phi_y} \left(g'(x_k) + \sum_{j=1}^{N} \alpha_j \psi_j(x_k) \right)_{k=1,\ldots,d}$$

is surjective. Let $\Sigma_y \subset \mathbb{C}^N$ be the "bad set" consisting of all $\alpha = (\alpha_1, \ldots, \alpha_N) \in \mathbb{C}^N$ such that for some $1 \le i \ne k \le d$ we have

$$g'(x_i) + \sum_{j=1}^{N} \alpha_j \psi_j(x_i) = g'(x_k) + \sum_{j=1}^{N} \alpha_j \psi_j(x_k)$$

It then suffices to find a holomorphic map $\alpha \colon Y \to \mathbb{C}^N$ such that

$$(*) \qquad \alpha(y) \notin \Sigma_y \quad \text{for all } y \in Y.$$

For such α, the map $f = (h, g_\alpha) \colon X \to Y \times \mathbb{C}^q$ is an embedding of X_0.

Finding a section avoiding a subvariety

Introducing the codimension q linear subspaces (the diagonals)

$$\Lambda_{i,k} = \left\{ (b_1, \ldots, b_d) \in (\mathbb{C}^q)^d : b_i = b_k \right\}$$

we see that

$$\Sigma_y = \bigcup_{1 \le i \ne k \le d} \Phi_y^{-1}(\Lambda_{ik}) \subset \mathbb{C}^N$$

is a union of $\binom{d}{2}$ affine linear subspaces of codimension q in $\mathbb{C}^N$, depending holomorphically on $y \in S$ and diverging to ∞ as $y \to Y_1$. Hence, they are the fibres of a subvariety $\Sigma \subset Y \times \mathbb{C}^N$ projecting to S.

For points y in a neighborhood of Y_1 the condition (*) holds for $\alpha = 0$.

The main topological point: Since Y_0 is Stein, the assumption

$$2q = \mathrm{codim}_{\mathbb{R}}(\Sigma_y, \mathbb{C}^N) > \dim_{\mathbb{C}} Y_0$$

ensures the existence of a continuous map $Y_0 \to \mathbb{C}^N$ satisfying (*).
Hence, condition $2q > n$ suffices for all strata arising in the proof.

Gromov's Oka principle

The proof is now completed by applying the following result.

Theorem (Gromov 1989; Prezelj & F., 2000-2002)

Assume that $\pi\colon E \to Y$ is a holomorphic vector bundle over a Stein base Y and $\Sigma \subset E$ is a closed analytic subset whose fibres Σ_y $(y \in Y)$ are algebraic subvarieties of codimension ≥ 2 in $E_y \cong \mathbb{C}^N$ satisfying a local uniform tameness condition. Then, sections $Y \to Z := E \setminus \Sigma$ satisfy the parametric Oka principle; in particular, every continuous section is homotopic to a holomorphic section.

The special case when $\pi\colon Z \to Y$ is a **holomorphic fibre bundle with complex homogeneous fibre** is due to **Grauert and Kerner 1963**, with an important extension due to **Forster and Ramspott 1963** (Okasche Paare von Garben nicht-abelscher Gruppen, Invent. Math. 1, 1966).

Gromov proved the Oka principle for sections of holomorphic submersions $Z \to Y$ which admit a **dominating holomorphic fibre-spray** over a small neighborhood of each point $y_0 \in Y$ (the **elliptic submersions**).

Oka manifolds

Definition (F., 2009)

A complex manifold Z is an **Oka manifold** if every holomorphic map $D \to Z$ from an open convex set $D \subset \mathbb{C}^n$ (for any $n \in \mathbb{N}$) can be approximated uniformly on compacts by entire maps $\mathbb{C}^n \to Z$.

Theorem (F., 2005–10)

Maps $X \to Z$ from any reduced Stein space X to Z satisfy (all forms of) the Oka principle if and only if Z is an Oka manifold.

Furthermore, the Oka principle holds for sections of any stratified holomorphic fibre bundle $\pi\colon Z \to X$ with Oka fibres over a Stein space.

This answers a question of Gromov. Oka manifolds have been studied intensively in recent years; my monograph *Stein Manifolds and Holomorphic Mappings* (Springer, 2011 & 2017) is devoted to this subject. It is easy to see that the complement of an algebraic subvariety of codimension ≥ 2 in $\mathbb{C}^N$, or in $\mathbb{CP}^N$, is an Oka manifold.

Holomorphic automorphisms enter the picture

The proof of the embedding theorem fails when trying to embed an open Riemann surface X properly holomorphically into $\mathbb{C}^2$. In this case, $q = 1$ and hence $\Sigma_y \subset \mathbb{C}^N$ are hypersurfaces; **the Oka principle fails**.

Completely different methods have been developed, based on the theory of holomorphic automorphisms of complex Euclidean spaces.

Theorem (Andersén-Lempert 1992, Forstnerič-Rosay 1993)

Assume that $\Omega_0 \subset \mathbb{C}^n$ $(n > 1)$ *is a Runge domain and*

$$F_t \colon \Omega_0 \to \Omega_t \subset \mathbb{C}^n \quad (t \in [0,1])$$

is a smooth isotopy of biholomorphic maps between Runge domains, with $F_0 = \mathrm{Id}|_{\Omega_0}$*. Then,* $F_1 \colon \Omega_0 \to \Omega_1$ *is a locally uniform limit of holomorphic automorphisms of* $\mathbb{C}^n$*.*

The main point is that **every holomorphic vector field on $\mathbb{C}^n$ can be approximated by finite sums of complete holomorphic vector fields.**

Embedding Riemann surfaces into $\mathbb{C}^2$

By using this result and a technique for exposing boundary points of bordered Riemann surface, E.F. Wold and myself proved the following.

Theorem (Wold & F. 2009, 2013)

(a) *If a compact bordered Riemann surface M embeds holomorphically into $\mathbb{C}^2$, then $\mathring{M}$ embeds* **properly** *holomorphically into $\mathbb{C}^2$.*

(b) *Every circled domain in $\mathbb{C}$, such that all but finitely many punctures are limits of complementary discs, embeds properly into $\mathbb{C}^2$.*

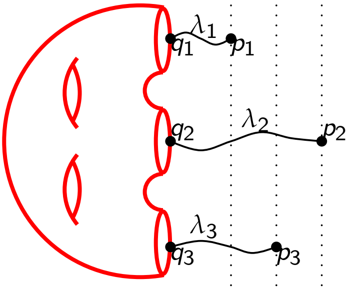
The same result holds for domains in tori.

He & Schramm 1993, 1995 Every domain in $\mathbb{CP}^1$ with at most countably many complementary components is conformally equivalent to a circled domain.

Earlier results: **Kasahara & Nishino, Laufer, Alexander, 1970–76**; **Globevnik and Stensønes 1995**

Problem: Does the complement $\mathbb{C} \setminus K$ of each Cantor set K embed properly holomorphically into $\mathbb{C}^2$? (Example by **Orevkov 2008**.)

Idea of proof



We attach to $M \subset \mathbb{C}^2$ smooth curves λ_i with exposed boundary points p_i. We can deform M by stretching it inside a thin tube along each of the curves λ_i such that q_i goes to p_i; these become exposed points of the new embedding of M into $\mathbb{C}^2$. We then send these exposed points to infinity by a rational shear, thereby opening each of the boundary curves of M. Finally, we inductively push the (noncompact!) boundary curves to infinity by using automorphisms of $\mathbb{C}^2$.

Stein manifolds with the density property

Varolin 2000 The Andersén-Lempert theorem holds if $\mathbb{C}^n$ is replaced by any Stein manifold Z with the **holomorphic density property (DP)**:

Every holomorphic vector field on Z can be approximated by finite sums of $\mathbb{C}$-complete holomorphic vector fields.

Every such manifold is an Oka manifold. A great majority of complex Lie groups and homogeneous manifolds enjoy DP. It was recently proved that these manifolds are universal models for proper holomorphic immersions and embeddings of Stein manifolds.

Theorem (Andrist, F., Ritter, Wold 2016; F. 2017)

Assume that X is a Stein manifold, and let Z be a Stein manifold with the density property.

(a) *If $\dim Z > 2\dim X$, then every continuous map $X \to Z$ is homotopic to a proper holomorphic embedding $X \hookrightarrow Z$.*

(b) *If $\dim Z = 2\dim X$, then every continuous map $X \to Z$ is homotopic to a proper holomorphic immersion with simple double points.*

Examples of manifolds with DP or VDP

- $\mathbb{C}^n$ for $n \geq 1$ satisfies VDP for $dz_1 \wedge \cdots \wedge dz_n$ (Andersén).
- $\mathbb{C}^n$ for any $n > 1$ satisfies DP (Andersén and Lempert).
- $(\mathbb{C}^*)^n$ with the volume form $\frac{dz_1}{z_1} \wedge \cdots \wedge \frac{dz_n}{z_n}$ satisfies VDP (Varolin). It is not known whether DP holds when $n > 1$.
- If G is a linear algebraic group and $H \subset G$ is a closed proper reductive subgroup, then $X = G/H$ is a Stein manifold with the density property, except when $X = \mathbb{C}, (\mathbb{C}^*)^n$, or a $\mathbb{Q}$-homology plane with fundamental group $\mathbb{Z}_2$ (Kaliman, Donzelli & Dvorsky).
- In particular, a linear algebraic group with connected components different from $\mathbb{C}$ or $(\mathbb{C}^*)^n$ has DP (Kaliman and Kutzschebauch).
- If $p : \mathbb{C}^n \to \mathbb{C}$ is a holomorphic function with smooth reduced zero fibre, then $X = \{xy = p(z)\}$ has DP (K&K). The same is true if the source $\mathbb{C}^n$ of p is an arbitrary Stein manifold with DP.
- A Cartesian product $X_1 \times X_2$ of two Stein manifolds X_1, X_2 with DP also has DP. The analogous result holds for VDP (K&K).

Gizatullin surfaces and the Koras-Russell cubic

- **Andrist 2017** A generic **Gizatullin surface** has the density property. A smooth affine algebraic surface X is a Gizatullin surface if $\mathrm{Aut}_{\mathrm{alg}}(X)$ acts transitively on X up to finitely many points. Every such surface X admits a fibration $\pi\colon X \to \mathbb{C}$ whose generic fiber equals $\mathbb{C}$ and there is only one exceptional fiber. If this exceptional fiber is reduced, then X has the density property.
- **Leuenberger 2016** established DP for a family of hypersurfaces

$$X = \{(x, y, z) \in \mathbb{C}^{n+3} : x^2 y = a(z) + x b(z)\},$$

where $x, y \in \mathbb{C}$ and $a, b \in \mathbb{C}[z]$ are polynomials in $z \in \mathbb{C}^{n+1}$. This family includes the **Koras-Russell cubic threefold**

$$C = \{(x, y, z_0, z_1) \in \mathbb{C}^4 : x^2 y + x + z_0^2 + z_1^3 = 0\}.$$

This threefold is diffeomorphic to $\mathbb{R}^6$, but is not algebraically isomorphic to $\mathbb{C}^3$ (**Makar-Limanov, Dubouloz**).

It remains an open question whether C is biholomorphic to $\mathbb{C}^3$.

The soft Oka principle for embeddings

Every Stein manifold X embeds properly holomorphically into any other Stein manifold Y satisfying $\dim Y \geq 2 \dim X + 1$ if we allow a homotopic deformation of the Stein structure on the source manifold.

Theorem (Slapar & F., 2007; Drinovec D. & F., 2010)

(a) Let (X, J_0) and Y be Stein manifolds with $\dim Y \geq 2 \dim X + 1$. Given a continuous map $f_0 \colon X \to Y$, we can homotopically deform J_0 to another Stein structure J on X, and we can deform f_0 to a proper embedding $f \colon X \to Y$ which is holomorphic as a map from (X, J) to Y.

(b) If X is a strongly pseudoconvex domain then no deformation of J_0 is necessary, i.e., every holomorphic map $\overline{X} \to Y$ is homotopic to a proper holomorphic embedding $X \to Y$.

In part (a), the point is to make the complex structure on X "smaller" in order to fit into Y. This is related to **Eliashberg's construction (1990)** of integrable Stein structures of smooth orientable manifolds X^{2n} with the homotopy type of an n-dimensional CW complex.

THANK YOU FOR YOUR ATTENTION

AND HAPPY BIRTHDAY

TO PROFESSOR OTTO FORSTER