Mergelyan's and Arakelian's theorems for manifold-valued maps

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Abstract

I will present some recent developments in holomorphic approximation theory of **Mergelyan, Arakelian, and Carleman type** for manifold-valued maps. They are treated in the following papers:

F. Forstnerič: Mergelyan's and Arakelian's theorems for manifold-valued maps. Moscow Math. J. 19:3 (2019) 465–484.
https://arxiv.org/abs/1801.04773

B. Chenoweth: Carleman Approximation of Maps into Oka Manifolds. Proc. Amer. Math. Soc., https://arxiv.org/abs/1804.10680

A more comprehensive discussion of these topics is available in the survey
J.-E. Fornæss, F. Forstnerič, E.F. Wold:
Holomorphic approximation: the legacy of Weierstrass, Runge, Oka-Weil, and Mergelyan. To appear in the volume "Advancements in Complex Analysis" by Springer-Verlag. https://arxiv.org/abs/1802.03924

Classical approximation theorems on $\mathbb{C}$

C. Runge 1885 If K is a compact set with connected complement in $\mathbb{C}$ then every $f \in \mathcal{O}(K)$ is a uniform limit of holomorphic polynomials.

S. N. Mergelyan 1951 If $K \subset \mathbb{C}$ is as in Runge's theorem then every function in $\mathscr{A}(K) = \mathscr{C}(K) \cap \mathcal{O}(\mathring{K})$ is a uniform limit of polynomials.

In view of Runge's theorem, Mergelyan's theorem is equivalent to

The Mergelyan property (MP): $\mathscr{A}(K) = \overline{\mathcal{O}}(K) = \mathscr{R}(K)$.

A. Vitushkin 1966 Characterization of MP for general compacts in terms of continuous capacity. MP is also called the **Vitushkin property**.

N. U. Arakelian 1964–1971 The following conditions are equivalent for a **closed** (not necessarily compact) set E in a domain $X \subset \mathbb{C}$:

(a) Every function in $\mathscr{A}(E)$ is a uniform limit of functions in $\mathcal{O}(X)$.

(b) **Arakelian's condition:** The complement $X^* \setminus E$ of E in the one point compactification $X^* = X \cup \{*\}$ of X is connected and locally connected at $*$.

Approximation on more general domains

K. Oka 1936, A. Weil 1935 Runge's theorem holds on any compact $\mathcal{O}(X)$-convex set K in a Stein manifold X. Here,

$$\widehat{K} = \{p \in \mathbb{C}^n : |f(p)| \le \max_K |f| \ \ \forall f \in \mathscr{O}(X)\},$$

and K is $\mathcal{O}(X)$-convex iff $K = \widehat{K}$. Modern proof uses Hörmander's L^2-technique for solving the $\bar{\partial}$-equation with weights.

H. Behnke and K. Stein 1949 Runge's theorem holds for any compact set K without holes in an open Riemann surface X.

E. Bishop 1958 (localization theorem) Let K be a compact set in a Riemann surface X. If every point $p \in K$ has a compact neighborhood $D_p \subset X$ such that $K \cap D_p$ has MP, then K has MP. In particular, a compact set without holes in an open Riemann surface has MP.

P. M. Gauthier and W. Hengartner 1975, **S. Scheinberg 1978** Arakelian's condition is necessary for uniform approximation of functions on a closed subset E in an arbitrary connected open Riemann surface X, but is not sufficient in general. Many sufficient conditions are known.

Oka Manifolds

How can one extend Runge's theorem to manifold-valued maps?
The natural source manifolds are **Stein manifolds**, i.e., closed complex submanifolds of Euclidean spaces $\mathbb{C}^N$ **(K. Stein, 1951)**.

One-dimensional Stein manifolds are open Riemann surfaces.

What are good target manifolds for Runge approximation?
Obviously bad are **Kobayashi hyperbolic manifolds**, the simplest one being $Y = \mathbb{C} \setminus \{0,1\}$. By **Picard's theorem**, every holomorphic map $\mathbb{C} \to \mathbb{C} \setminus \{0,1\}$ is constant. Hence, we cannot approximate nonconstant holomorphic maps $\mathbb{D} = \{|z| < 1\} \to \mathbb{C} \setminus \{0,1\}$ by entire maps.

In 2009, I introduced the following class of manifolds and proved the next theorem, thereby answering a question of **M. Gromov (1989)**:

Definition

A complex manifold Y is an **Oka manifold** if for every compact convex set $K \subset \mathbb{C}^n$ $(n \in \mathbb{N})$ and holomorphic map $f : U \to Y$ on a neighbourhood $U \subset \mathbb{C}^n$ of K, we can approximate f uniformly on K by entire maps $\mathbb{C}^n \to Y$.

Runge's theorem for maps to Oka manifolds

Theorem (F., 2005–2009)

If X is Stein, $K \subset X$ is compact $\mathscr{O}(X)$-convex, Y is an Oka manifold, and $f : X \to Y$ is a continuous map that is holomorphic on a neighbourhoood of K, then f can be approximated uniformly on K by holomorphic maps $F : X \to Y$ homotopic to X. (Many other conditions can be fulfilled: interpolation on a subvariety of X, parametric case, section of fibre bundles with Oka fibres over a Stein base.)

Many sufficient conditions are known for a complex manifold to be Oka:

- **Oka & Weil 1935** $\mathbb{C}^N$ is an Oka manifold.
- **Oka 1939** $\mathbb{C}^*$ is an Oka manifold.
- **Grauert 1958** Every complex homogeneous manifold is Oka.
- **Gromov 1989** If the tangent bundle of a complex manifold Y is spanned by $\mathbb{C}$-complete holomorphic vector fields, then Y is Oka.
- **F 2006** If $E \to B$ is a fibre bundle with an Oka fibre Y, then E is Oka if and only if B is Oka.
- **Kusakabe 2019** If Y is Zariski locally Oka, then Y is Oka.

Mergelyan's theorem holds universally

Theorem (Mergelyan's theorem for manifold-valued maps)

Let K be a compact set with the Mergelyan property in a Riemann surface X, and let Y be an arbitrary complex manifold. Then, every map $f : K \to Y$ of class $\mathscr{A}(K, Y)$ can be approximated uniformly on K by maps in $\mathscr{O}(K, Y)$ (i.e., holomorphic in open neighbourhoods of K).

Sketch of proof:

- By the converse to Bishop's localization theorem, due to **A. Boivin and B. Jiang 2004**, K has the local Mergelyan property.
- By **Poletsky 2013**, it follows that the graph of f on K has a basis of open Stein neighbourhoods $\Omega \subset X \times Y$.
- Choose a holomorphic embedding $\iota : \Omega \hookrightarrow Z \subset \mathbb{C}^N$. There is a holomorphic retraction $\rho : V \to Z$ from a neighbourhood $V \subset \mathbb{C}^N$.
- Approximate the map $\iota \circ (\mathrm{Id}_K, f) : K \to \mathbb{C}^N$ by a holomorphic map $F : U \to V$ from a neighbourhood $U \subset X$ of K. The map

  $$\tilde{f} = pr_Y \circ \rho \circ F : U \to Y$$

  then approximates f on K.

Bishop-Mergelyan theorem for maps to Oka manifolds

Corollary

If K is a compact set without holes in an open Riemann surface X and Y is an Oka manifold, then any continuous map $K \to Y$ which is holomorphic in $\mathring{K}$ is approximable by entire holomorphic maps $X \to Y$.

Proof.

We have already seen that any map $f \in \mathscr{A}(K, Y)$ can be approximated by maps $\tilde{f} \in \mathscr{O}(K, Y)$. (This holds for any complex manifold Y.)

If Y is an Oka manifold, then (since an open Riemann surface X is a Stein manifold and a set K without holes in X is $\mathscr{O}(X)$-convex), any map $\tilde{f} \in \mathscr{O}(K, Y)$ can be approximated uniformly on K by entire maps $F \in \mathscr{O}(X, Y)$ according to the previous theorem. □

Arakelian's theorem for maps to compact homogeneous manifolds

Theorem (F., Moscow Math. J. 2019)

Assume that Y is a compact complex homogeneous manifold.

If E is a closed Arakelian set in a domain $X \subset \mathbb{C}$, then every continuous map $X \to Y$ which is holomorphic in the interior of E can be approximated uniformly on E by holomorphic maps $X \to Y$.

Generalizations

The same result holds in any open Riemann surface X such that for every compact set K in X there exists a bounded linear operator

$$T_K : \mathscr{C}^{0,1}(K) \to \mathscr{C}_b(X)$$

satisfying the following two conditions.

1. For every $g \in \mathscr{C}^{0,1}(K)$ we have $\bar{\partial}\, T_K(g) = g$ in the sense of distributions, where we take $g = 0$ on $X \setminus K$.
2. T_K is a **holomorphic operator**: if $g(\cdot\,, w) \in \mathscr{C}^{0,1}(K)$ is a family depending holomorphically on a parameter $w \in W \subset \mathbb{C}^n$, then $T_K(g(\cdot\,, w)) \in \mathscr{C}_b(X)$ also depends holomorphically on w.

On $\mathbb{C}$, one can use the **Cauchy-Green operator**

$$T_K(g)(z) = \frac{1}{\pi} \int_K \frac{g(\zeta)}{z - \zeta}\, du\, dv, \qquad z \in \mathbb{C},\ \zeta = u + \mathrm{i}v.$$

Recall that for any $g \in L^p(K)$, $p > 2$, $T_K(g)$ is a bounded continuous function on $\mathbb{C}$ that is holomorphic on $\mathbb{C} \setminus K$, vanishes at infinity, and satisfies the uniform Hölder condition with exponent $\alpha = 1 - 2/p$; furthermore, $T_K \colon L^p(K) \to \mathscr{C}^{\alpha}(\mathbb{C})$ is a continuous linear operator.

Outline of proof

We follow **Rosay-Rudin's proof (1989)** for the case of functions, adding techniques from nonlinear approximation theory (the Oka theory). For simplicity, we assume that $X = \mathbb{C}$.

Since the set $E \subset \mathbb{C}$ is Arakelian, we can find a sequence of closed discs $\Delta_1 \subset \Delta_2 \subset \cdots \subset \bigcup_{i=1}^{\infty} \Delta_i = \mathbb{C}$ such that, letting

$$H_i = H_{E \cup \Delta_i} = \text{the union of holes of } E \cup \Delta_i,$$

we have that

$$\Delta_i \cup \overline{H}_i \subset \mathring{\Delta}_{i+1}, \quad i = 1, 2, \ldots.$$

Set

$$E_0 = E, \qquad E_i = E \cup \Delta_i \cup H_i \ \text{ for } \ i = 1, 2, \ldots.$$

Note that E_i is a closed Arakelian set in $\mathbb{C}$ for each i and

$$E_i \subset E_{i+1}, \qquad \overline{E_i \setminus E} \subset \mathring{\Delta}_{i+1}, \qquad \bigcup_{i=0}^{\infty} E_i = \mathbb{C}.$$

Sets in the induction step

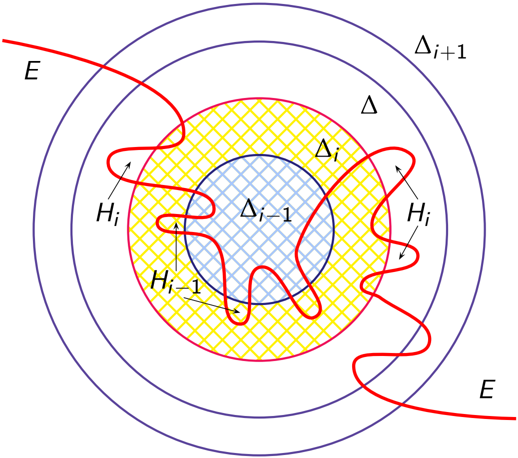

Figure: $E_i = E \cup \Delta_i \cup H_i$

Proof - continued

Let $f_0 : \mathbb{C} \to Y$ be a continuous map such that $f_0|_E \in \mathscr{A}(E)$.

Fix a number $\epsilon > 0$. We inductively construct a sequence of continuous maps $f_i : \mathbb{C} \to Y$ such that for all $i \in \mathbb{N}$ we have

$$f_i|_{E_i} \in \mathscr{A}(E_i, Y)$$

and

$$\mathrm{dist}_Y(f_i(z), f_{i-1}(z)) < 2^{-i}\epsilon, \qquad z \in E_{i-1}.$$

Since the sets E_i exhaust $\mathbb{C}$, the sequence f_i clearly converges uniformly on compacts in $\mathbb{C}$ to a holomorphic map

$$F = \lim_{i \to \infty} f_i : \mathbb{C} \to Y$$

satisfying

$$\sup_{z \in E} \mathrm{dist}_Y(f(z), F(z)) < \epsilon.$$

The induction step $i-1 \to i$

Assume that $f_{i-1} : \mathbb{C} \to Y$ is continuous and $f_{i-1}|_{E_{i-1}} \in \mathscr{A}(E_{i-1}, Y)$. Recall that $\Delta_i \cup \overline{H}_i \subset \mathring{\Delta}_{i+1}$.

Pick a closed disc $\Delta \subset \mathbb{C}$ such that

$$\Delta_i \cup \overline{H}_i \subset \Delta \subset \mathring{\Delta}_{i+1}.$$

Since $E_i = E \cup \Delta_i \cup H_i$, it follows that

$$E_i \setminus \Delta = E_{i-1} \setminus \Delta = E \setminus \Delta.$$

Since E_{i-1} has no holes, $E_{i-1} \cap \Delta_{i+1}$ has no holes either.

As Y is complex homogeneous and hence Oka, Mergelyan's theorem furnishes for any $c > 0$ a holomorphic map $g : \mathbb{C} \to Y$ satisfying

$$\mathrm{dist}_Y(f_{i-1}(z), g(z)) < c, \qquad z \in E_{i-1} \cap \Delta_{i+1}.$$

Unbounded Cartan pairs in Riemann surfaces

To complete the induction step, we glue f_{i-i} and g into $f_i \in \mathscr{A}(E_i, Y)$. We shall now outline the relevant gluing (fusion) method.

The following is a variation of the standard definition of a **Cartan pair**. The main difference is that the closed sets A and B may now be unbounded (non-compact).

Definition

Let X be an open Riemann surface. A pair of closed subsets (A, B) of X is a *Cartan pair* if it satisfies the following two conditions.

(a) The set $K = A \cap B$ is compact.

(b) $\overline{A \setminus B} \cap \overline{B \setminus A} = \varnothing$.

The notion of a Cartan pair in a higher dimensional manifold also requires that the sets $A \cap B$ and $A \cup B$ be Stein compacta (and, in some applications, even compact strongly pseudoconvex domains).

Gluing f_{i-i} and h into $f_i \in \mathscr{A}(E_i, Y)$

Consider the following Cartan pair decomposition (A, B) of the unbounded closed set $E_i = A \cup B$:

$$A = \overline{E_i \setminus \Delta} = E \setminus \mathring{\Delta}, \qquad B = E_i \cap \Delta_{i+1}.$$

We have that

$$K := A \cap B = E_i \cap (\Delta_{i+1} \setminus \mathring{\Delta}) = E \cap (\Delta_{i+1} \setminus \mathring{\Delta}).$$

To conclude the induction step, we glue the pair of maps $f_{i-1} \in \mathscr{A}(A, Y)$ and $g \in \mathscr{A}(B, Y)$ over $K = A \cap B$ to get the next map $f_i \in \mathscr{A}(A \cup B, Y) = \mathscr{A}(E_i, Y)$ in the sequence.

When $Y = \mathbb{C}$, we solve a **Cousin-I problem with uniform bounds** (very classical). The nonlinear case is obtained by gluing pairs of **dominating sprays** of maps. A crucial ingredient in both cases is the existence of a sup-norm bounded solution operators for the $\bar{\partial}$-equation.

Gluing – the linear case

Let (A, B) be a Cartan pair. We wish to glue $f \in \mathscr{A}(A, \mathbb{C})$ and $g \in \mathscr{A}(B, \mathbb{C})$ into $F \in \mathscr{A}(A \cup B, \mathbb{C})$ which is close to f on A and is close to g on B.

Pick a smooth function $\chi : \mathbb{C} \to [0, 1]$ which equals 0 on a neighbourhood of $\overline{A \setminus B}$ and equals 1 on a neighbourhood of $\overline{B \setminus A}$. Let T_K be the Cauchy-Green operator. Set $c = f|_K - g|_K \in \mathscr{A}(K)$ and

$$\begin{aligned} \mathcal{A}c &= c\chi - T_K(c\bar{\partial}\chi) \in \mathscr{A}(A), \\ \mathcal{B}c &= c(\chi - 1) - T_K(c\bar{\partial}\chi) \in \mathscr{A}(B). \end{aligned}$$

Then,

$$\mathcal{A}c - \mathcal{B}c = c \;\text{ on } K,$$

and we obtain a solution $F \in \mathscr{A}(A \cup B, \mathbb{C})$ by setting

$$F = f - \mathcal{A}c \;\text{ on } A, \quad F = g - \mathcal{B}c \;\text{ on } B.$$

Note that $\|\mathcal{A}c\|_A \leq const. \|f - g\|_K$ and $\|\mathcal{B}c\|_B \leq const. \|f - g\|_K$.

A gluing (fusion) lemma for sprays of maps

We now consider the general case when $f \in \mathscr{A}(A, Y)$ and $g \in \mathscr{A}(B, Y)$. The following lemma will be used.

Lemma

Let (A, B) be a Cartan pair in a domain $X \subset \mathbb{C}$. Set $K = A \cap B$.

Given a bounded open convex set $0 \in W \subset \mathbb{C}^n$ for some $n \in \mathbb{N}$ and a number $r \in (0, 1)$, there is a $\delta > 0$ satisfying the following property.

For every map $\gamma \colon K \times W \to K \times \mathbb{C}^n$ of the form

$$\gamma(z, w) = \big(z, \psi(z, w)\big), \quad z \in K, \ w \in W,$$

and of class $\mathscr{A}(K \times W)$, with $\mathrm{dist}_{K \times W}(\gamma, \mathrm{Id}) < \delta$, there exist maps

$$\alpha_\gamma : A \times rW \to A \times \mathbb{C}^n, \qquad \beta_\gamma : B \times rW \to B \times \mathbb{C}^n,$$

of the same form and of class $\mathscr{A}(A \times rW)$ and $\mathscr{A}(B \times rW)$, respectively, depending smoothly on γ, such that $\alpha_{\mathrm{Id}} = \mathrm{Id}$, $\beta_{\mathrm{Id}} = \mathrm{Id}$, and

$$\gamma \circ \alpha_\gamma = \beta_\gamma \quad \text{holds on } K \times rW.$$

Applying the gluing lemma

Recall that Y is compact homogeneous. Let $\phi_t^1, \dots, \phi_t^n$ be flows of holomorphic vector fields on Y spanning TY. This gives sprays of maps

$$\tilde{f} \in \mathscr{A}(A \times \mathbb{C}^n, Y), \qquad \tilde{g} \in \mathscr{A}(B \times \mathbb{C}^n, Y),$$

by taking

$$\tilde{f}(z, w) = \phi_{w_1}^1 \circ \phi_{w_2}^2 \circ \cdots \circ \phi_{w_n}^n (f(z)),$$

and similarly for $\tilde{g}$. Note that $\tilde{f}(\cdot\,, 0) = f$, $\tilde{g}(\cdot\,, 0) = g$, and both maps are submersive with respect to $w \in \mathbb{C}^n$ at $w = 0$. Furthermore, given $\epsilon > 0$ there is an open set $0 \in W_0 \Subset \mathbb{C}^n$ such that

$$\sup_{z \in A,\ w \in W_0} \mathrm{dist}_Y(\tilde{f}(z, w), f(z)) < \epsilon, \qquad \sup_{z \in B,\ w \in W_0} \mathrm{dist}_Y(\tilde{g}(z, w), g(z)) < \epsilon.$$

Here we use compactness of Y!

Conclusion of the proof

We may assume that g is uniformly as close as desired to f on K. It follows that $\tilde{g}$ is uniformly as close as desired to $\tilde{f}$ on $K \times W_0$.

Since $w \mapsto \tilde{f}(\cdot\,, w)$ is dominating at $w = 0$, there are a smaller neighbourhood $0 \in W \Subset W_0$ of the origin in the parameter space and a number $\eta_0 > 0$ such that for every g satisfying

$$\sup_{z \in K} \mathrm{dist}(f(z), g(z)) < \eta \le \eta_0$$

there exists a map $\gamma \colon K \times W \to K \times \mathbb{C}^n$ of the form

$$\gamma(z, w) = \big(z, \psi(z, w)\big), \quad z \in K,\ w \in W,$$

and of class $\mathscr{A}(K \times W)$, with

$$\mathrm{dist}_{K \times W}(\gamma, \mathrm{Id}) < \delta = const_f \cdot \eta,$$

such that

$$\tilde{f} = \tilde{g} \circ \gamma \ \text{ on } K \times W.$$

Conclusion of the proof

Choose a number $0 < r < 1$. Assuming as we may that η (and hence δ) are small enough, the splitting lemma gives a decomposition

$$\gamma \circ \alpha = \beta \quad \text{on } K \times rW$$

for some holomorphic maps

$$\alpha : A \times rW \to A \times W, \qquad \beta : B \times rW \to B \times W,$$

of the same form as γ and of class $\mathscr{A}(A \times rW)$ and $\mathscr{A}(B \times rW)$, respectively, which are close to the identity on the respective sets (depending on the proximity of γ to the identity, which in turn depends on how close is g to f on K).
Recall that

$$\tilde{f} = \tilde{g} \circ \gamma \ \text{ on } K \times W.$$

Precomposing the equation with α, it follows that

$$\tilde{f} \circ \alpha = \tilde{g} \circ \beta \ \text{ on } K \times rW.$$

Taking $w = 0 \in \mathbb{C}^n$ gives a map $F \in \mathscr{A}(A \cup B, Y)$ satisfying

$$\sup \operatorname{dist}(F(z), f(z)) < \epsilon, \qquad \sup \operatorname{dist}(F(z), g(z)) < \epsilon.$$

Carleman approximation on totally real manifolds

Let X be a Stein manifold and $E \subset X$ be a closed subset. Set

$$\widehat{E}_{\mathcal{O}(X)} = \widehat{E} = \bigcup_{j=1}^{\infty} \widehat{E}_j$$

where E_j is a normal exhaustion of E by compacts.

The set E has **bounded exhaustion hulls (BEH)** if for any compact $K \subset X$ there is a bigger compact $K' \subset X$ such that

$$\widehat{K \cup E} \subset E \cup K'.$$

For a closed set E in an open Riemann surface X, this is equivalent to the classical **Arakelian condition**.

Manne 1993 If X is Stein and $E \subset X$ is a closed $\mathscr{C}^k$ totally real submanifold that is $\mathcal{O}(X)$-convex and has BEH, then E admits $\mathscr{C}^k$-Carleman approximation by entire functions $X \to \mathbb{C}$.

Magnusson & Wold 2016 If a closed holomorphically convex set E in a Stein manifold X admits $\mathscr{C}^0$ Carleman approximation, then E has BEH.

Carleman approximation of maps to Oka manifolds

Theorem (B. Chenoweth 2018, to appear in Proc. AMS)

Let X be a Stein manifold and Y be an Oka manifold. Assume that $K \subset X$ is a compact $\mathscr{O}(X)$-convex subset and $M \subset X$ is a closed totally real submanifold of class $\mathscr{C}^k$ $(k \in \mathbb{N})$ which is $\mathscr{O}(X)$-convex, has bounded exhaustion hulls, and $E = K \cup M$ is $\mathscr{O}(X)$-convex.

Then, every map $f \in \mathscr{C}^k(X, Y)$ which is $\bar\partial$-flat to order k on E and is holomorphic on a neighbourhood of K (or just on $\mathring{K}$ if K is a compact strongly pseudoconvex domain) can be approximated in the fine $\mathscr{C}^k$ topology on E by holomorphic maps $F : X \to Y$.

The proof combines:

- Mergelyan approximation of functions on handlebodies,
- existence of Stein neighborhoods of graphs over handlebodies (this gives Mergelyan approximation of manifold-valued maps),
- gluing techniques.

Approximation theory in holomorphic directed systems

I wish to indicate a new direction in holomorphic approximation theory that I have become interested in recently.

A **holomorphic directed system** of first order on a complex manifolds X is given by a subset $\mathscr{G} \subset TX$ of its tangent bundle whose fibres $\mathscr{G}_x \subset T_x X$ $(x \in X)$ are complex cones. A real or complex submanifold $M \subset X$ is an **integral submanifold** of $\mathscr{G}$ if

$$T_x M \subset \mathscr{G}_x \ \text{ holds for all } x \in M.$$

More generally, one may consider subsets of higher order jet bundles. Approximation theory in such systems is of great interest.

Every holomorphic subbundle $E \subset TX$ determines a directed system. Of particular interest are **contact subbundles**, i.e., completely nonintegrable holomorphic hyperplane subbundles of TX. Such can only exist on odd dimensional manifolds X^{2n+1} and are given as the kernel $\xi = \ker \alpha$ of a (twisted) holomorphic 1-form satisfying the nondegeneracy condition

$$\alpha \wedge (d\alpha)^n \neq 0.$$

Approximation in holomorphic contact systems

The standard example on $\mathbb{C}^{2n+1}_{(x,y,z)}$ is

$$\alpha_{\rm std} = dz + xdy = dz + \sum_{i=1}^{n} x_i dy_i.$$

By **Darboux's theorem**, every contact system is locally of this kind. Integral complex submanifolds of a contact manifold (X^{2n+1}, ξ) are said to be **isotropic**, or **Legendrian** when they are of maximal dimension n.

Theorem (Alarcón, F., López, Compositio Math. 2017)

Every open Riemann surface is a properly embedded Legendrian curve in $(\mathbb{C}^3, \alpha_{\rm std})$. *Holomorphic integral curves in* $(\mathbb{C}^{2n+1}, \alpha_{\rm std})$ *satisfy the Runge approximation theorem, and also the Mergelyan and the Carleman approximation theorems on* admissible sets.

On the other hand, Runge's theorem fails in general for holomorphic Legenrdian curves in contact systems on $\mathbb{C}^3$. Indeed:

A hyperbolic contact system on $\mathbb{C}^3$

F. 2017 There exists an injective holomorphic map $\Phi : \mathbb{C}^3 \to \mathbb{C}^3$ (a Fatou-Bieberbach map) such that the holomorphic contact form $\alpha = \Phi^*(\alpha_{\rm std})$ on $\mathbb{C}^3$ is Kobayashi hyperbolic, i.e., it admits no nonconstant Legendrian lines $\mathbb{C} \to \mathbb{C}^3$.

Problem

1. *Does the local Mergelyan theorem for isotropic (Legendrian) curves hold in every complex contact manifold?*
2. *How many holomorphic contact structures are there on $\mathbb{C}^3$?*

   Eliashberg 1989, 1993 *On $\mathbb{R}^3$ there are countably many noncontactomorphic smooth contact structures.*
3. *Does there exist an* **algebraic contact form** α *on $\mathbb{C}^3$ (with polynomial coefficients) defining a hyperbolic contact structure? Is a generic algebraic 1-form of sufficiently big degree hyperbolic?*

   This is an analogue of **Kobayashi's conjecture** *for projective manifolds, solved in the affirmative by* **Demailly** *and* **Brotbek**.

THANK YOU

FOR YOUR ATTENTION