WHAT IS AN OKA MANIFOLD?

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Dedication

I wish to dedicate this talk to the memory of Hans Grauert.

He was one of the founders and main contributors to modern complex analysis and geometry,

and the principal creator of the Oka-Grauert theory.

Oka theory...

...is about a tight relationship between homotopy theory and complex geometry involving **Stein manifolds** and **Oka manifolds**.

It has a long and rich history, beginning with Kiyoshi Oka around 1940, continued by Hans Grauert in late 1950's, revitalized by Mikhael Gromov in 1989, and leading to an introduction and systematic study of *Oka manifolds* and *Oka maps*.

The Oka Principle: There are only topological obstructions to solving certain complex-analytic problems on Stein manifolds and Stein spaces.

This is a complex-analytic analogue of the Hirsch-Smale-Gromov **h-principle** in smooth geometry.

Stein manifolds (Karl Stein, 1951)

A complex manifold S is said to be a **Stein manifold** if

- holomorphic functions on S separate points:

$$x, x' \in S,\ x \neq x' \Longrightarrow f(x) \neq f(x') \text{ for some } f \in \mathcal{O}(S), \text{ and}$$

- S is **holomorphically convex**: For every compact set $K \subset S$, its $\mathcal{O}(S)$-convex hull $\widehat{K}$ is also compact:

$$\widehat{K} = \{x \in S \colon |f(x)| \leq \sup_K |f|,\ \forall f \in \mathcal{O}(S)\}$$

Equivalently, for every discrete sequence $a_j \in S$ there exists a holomorphic function f on S such that $|f(a_j)| \to +\infty$ as $j \to \infty$.

A **Stein space** is a complex space with singularities satisfying these two axioms.

Examples of Stein manifolds

- Domains in $\mathbb{C}$, open Riemann surfaces (Behnke & Stein).
- $\mathbb{C}^n$, and domains of holomorphy in $\mathbb{C}^n$ (Cartan & Thullen).
- Closed complex submanifolds of $\mathbb{C}^N$.
- A closed complex submanifold of a Stein manifold is Stein.
- If $E \to S$ is a holomorphic vector bundle and S is Stein, then E is Stein.

And a few examples of non-Stein manifolds:

- A Stein manifold does not contain any compact complex subvariety of positive dimension.
- Quotients of Stein manifolds need not be Stein.
- There exists a fiber bundle $E \to \mathbb{C}$ with fiber $\mathbb{C}^2$ and nonlinear structure group $\Gamma \subset \mathrm{Aut}\,\mathbb{C}^2$ such that E is non-Stein.

Cartan's Theorem B

Cartan-Serre Theorem B (1951-56): A complex manifold S is Stein iff for every coherent analytic sheaf $\mathscr{F}$ over S we have

$$H^k(S, \mathscr{F}) = 0 \quad \text{for all } k = 1, 2, \ldots.$$

Hence an analytic problem on a Stein manifold whose obstruction lies in such a cohomology group is solvable.

Applying this with sheaves Ω^p of holomorphic p-forms gives

Every $\overline{\partial}$-problem on a Stein manifold is solvable: For every differential form f with $\overline{\partial} f = 0$ there exists a form u solving

$$\overline{\partial} u = f.$$

These results can be considered as a form of **Oka principle for linear complex-analytic problems** on Stein manifolds.

Theorems of Grauert and Lefschetz-Milnor

Grauert, 1958; Narasimhan, 1962: A complex manifold S is Stein iff it admits a *strongly plurisubharmonic exhaustion function* $\rho\colon S \to \mathbb{R}$,

$$\mathrm{d}\mathrm{d}^c\rho = \mathrm{i}\partial\overline{\partial}\rho > 0 \quad \text{(a Kähler form)}.$$

In any local coordinate system on S, such a function is *strongly subharmonic* on each complex line:

$$\triangle(\rho|_L) > 0.$$

Critical points of such ρ have Morse index $\leq n = \dim_{\mathbb{C}} S$.

This implies the theorem of *Lefschetz and Milnor*:

A Stein manifold of complex dimension n is homotopy equivalent to a CW-complex of dimension at most n.

Embedding Stein manifolds in Euclidean spaces

Remmert, Bishop, Narasimhan, 1956-60: A complex manifold S of dimension n is Stein iff it is embeddable as a closed complex submanifold of some $\mathbb{C}^N$; one can take $N = 2n + 1$.

Thus Stein manifolds are holomorphic analogues of affine algebraic manifolds. In fact, every relatively compact domain in a Stein manifold is biholomorphic to a domain in an affine algebraic manifold (Stout, 1984).

Eliashberg and Gromov, 1992: If $n = \dim S > 1$ then S is embeddable in $\mathbb{C}^N$ with $N = \left[\frac{3n}{2}\right] + 1$.
Forster, 1971: This N is optimal.

The case $n = 1$, $N = 2$ remains a difficult open problem:

Is every open Riemann surface biholomorphic to some closed nonsingular complex curve in $\mathbb{C}^2$?

Recent advances in this direction: E. F. Wold & F.

The main theme of Oka theory...

... are properties of a complex manifold X which say that there exist many holomorphic maps $S \to X$ from any Stein manifold S.

Oka properties indicate that X is *holomorphically large* or *flexible*. They are modeled upon the known results for holomorphic functions $S \to X = \mathbb{C}$.

In many interesting cases we prove

The weak homotopy equivalence principle: The inclusion $\mathcal{O}(S,X) \hookrightarrow \mathcal{C}(S,X)$ of the space of holomorphic maps into the space of continuouos maps induces isomorphisms of all homotopy groups (a *weak homotopy equivalence*):

$$\pi_k(\mathcal{O}(S,X)) \cong \pi_k(\mathcal{C}(S,X)), \quad \forall k = 0, 1, 2, \ldots$$

Obstruction to Oka properties: holomorphic rigidity

A main feature distinguishing complex analysis and geometry from smooth geometry is the phenomenon of *holomorphic rigidity*.

- *Schwarz lemma:* Let $\mathbb{D} = \{z \in \mathbb{C} \colon |z| < 1\}$, and let $f \colon \mathbb{D} \to \mathbb{D}$ be a holomorphic function. Then

$$\frac{|\mathrm{d}f(z)|}{1 - |f(z)|^2} \leq \frac{|\mathrm{d}z|}{1 - |z|^2}, \quad z \in \mathbb{D}.$$

  This says that holomorphic self-mappings of the disc $\mathbb{D}$ are distance decreasing in the Poincaré metric on $\mathbb{D}$.
- *Picard's theorem*: Every holomorphic function $f \colon \mathbb{C} \to \mathbb{C} \setminus \{0, 1\}$ is constant.
- *Green's theorem*: Every holomorphic map $\mathbb{C} \to \mathbb{CP}^2 \setminus \{5 \text{ lines}\}$ is constant.

None of these examples satisfies any Oka property.

Brody and Kobayashi hyperbolicity

These classical holomorphic rigidity theorems lead to the notion of **hyperbolic manifolds**. A complex manifold X is

- *Brody hyperbolic* if every holomorphic map $\mathbb{C} \to X$ is constant.
- *Brody volume hyperbolic* if every holomorphic map $\mathbb{C}^n \to X$ ($n = \dim X$) has rank $< n$ at each point (a degenerate map).
- *Kobayashi (complete) hyperbolic* if the Kobayashi pseudometric k_X is a (complete) metric. k_X is the integrated form of the infinitesimal metric defined on $v \in \mathrm{T}_x X$ by

$$|v| = \inf \left\{ \frac{1}{|\lambda|} : \; f : \mathbb{D} \to X \text{ holo.}, \; f(0) = x, \; f'(0) = \lambda v \right\}.$$

A compact complex manifold is Brody hyp. iff it is Kobayashi hyp. A majority of complex manifolds are close to hyperbolic. Every compact complex manifold of maximal Kodaira dimension is volume hyperbolic, and is conjecturally 'almost' hyperbolic.

Properties opposite to hyperbolicity

Call a complex manifold X *holomorphically flexible* if X admits many holomorphic maps $\mathbb{C}^n \to X$ for every $n \in \mathbb{N}$.

Since every Stein manifold S embeds in some $\mathbb{C}^n$, we expect that there also exist many holomorphic maps $S \to X$.

What is a good way to interpret 'many maps'?

Start with two classical 19th century theorems:

Weierstrass Theorem. On a discrete subset of a domain $\Omega \subset \mathbb{C}$ we can prescribe the values of a holomorphic function $f \in \mathcal{O}(\Omega)$.

Runge Theorem. If $K \subset \mathbb{C}$ is a compact set with no holes, then every holomorphic function $K \to \mathbb{C}$ can be approximated uniformly on K by entire functions.

Higher-dimensional analogues, 1950's

Cartan Extension Theorem. If T is a closed complex subvariety of a Stein manifold S, then every holomorphic function $T \to \mathbb{C}$ extends to a holomorphic function $S \to \mathbb{C}$.

Oka-Weil Approximation Theorem. If $K = \widehat{K}$ is a compact holomorphically convex set in a Stein manifold S, then every holomorphic function $K \to \mathbb{C}$ can be approximated uniformly on K by holomorphic functions $S \to \mathbb{C}$.

These are fundamental properties of Stein manifolds.

A twist of philosophy: We can also view them as properties of the target manifold, the complex number field $\mathbb{C}$. We shall now formulate them as properties of an arbitrary target X.

The Basic Oka Property (BOP)

A complex manifold X enjoys BOP if the following holds:

Given a Stein inclusion $T \hookrightarrow S$ and a compact $\mathcal{O}(S)$-convex set $K = \widehat{K} \subset S$, every continuous map $f\colon S \to X$ that is holomorphic on $K \cup T$ can be deformed to a holomorphic map $\widetilde{f}\colon S \to X$.

The deformation (homotopy) can be kept fixed on T and holomorphic on K (approximating f on K).

$$\begin{tikzcd} K \cup T \arrow[r, "f"] \arrow[d] & X \\ S \arrow[ur, dotted, "\widetilde{f}"'] & \end{tikzcd}$$

By Oka-Weil and Cartan, $\mathbb{C}$, and hence $\mathbb{C}^n$, satisfy BOP.

The Parametric Oka Property (POP)

Let $Q \subset P$ be compacts in some $\mathbb{R}^m$. (It suffices to consider polyhedra.) Consider continuous maps $f \colon P \times S \to X$ such that

- $f(p, \cdot)|_T$ is holomorphic on T for every $p \in P$, and
- $f(p, \cdot)$ is holomorphic on S for every $p \in Q$.

A complex manifold X enjoys POP if every such f can be deformed to a map $\widetilde{f} \colon P \times S \to X$ such that

- $\widetilde{f}(p, \cdot)$ is holomorphic on S for all $p \in P$, and
- the homotopy is fixed on $(P \times T) \cup (Q \times S)$.

Applying POP with parameter pairs $\emptyset \subset S^k$ and $S^k \subset B^{k+1}$ gives the *weak homotopy equivalence principle*:

$$\pi_k(\mathcal{O}(S, X)) \cong \pi_k(\mathcal{C}(S, X)), \quad \forall k \in \mathbb{N}.$$

The Oka-Grauert Principle

Good candidates for having Oka properties are complex manifolds X with sufficiently many holomorphic maps $\mathbb{C}^n \to X$.

Example: Let G be a complex Lie group, acting holomorphically and transitively on a complex manifold X. Let $\mathfrak{g} = \mathrm{T}_1 G \cong \mathbb{C}^n$ be the Lie algebra of G. For every point $x \in X$ we have a holomorphic dominating map

$$s_x \colon \mathfrak{g} \cong \mathbb{C}^n \to X, \quad s_x(v) = \mathrm{e}^v \cdot x$$

such that $s_x(0) = x$ and $\mathrm{d}_0 s_x$ is surjective.

Grauert, 1957-58: **Every complex Lie group and, more generally, every homogeneous manifold, enjoys POP.**

The same holds for sections $S \to Z$ of holomorphic G-bundles $\pi \colon Z \to S$ (G a complex Lie group) over a Stein space S.

Classification of principal G-bundles

A principal G-bundle is a fiber bundle with fiber G and with the transition maps given by (left or right) multiplications by G.

It is well known (and easily seen) that every isomorphism $E \xrightarrow{\cong} E'$ between principal G-bundles over S is a section of an associated G-bundle $Z \to S$. Hence Grauert's theorem implies:

For any complex Lie group G, the topological and the holomorphic isomorphism classes of principal G-bundles over any Stein space S are in one-to-one correspondence:

$$\mathcal{O}^G \hookrightarrow \mathcal{C}^G \text{ induces an isomorphism } H^1(S, \mathcal{O}^G) \cong H^1(S, \mathcal{C}^G).$$

The same is true for the associated fiber bundles with G-homogeneous fibers; in particular, for *complex vector bundles* (take the group $G = GL_k(\mathbb{C})$).

Special case: Oka's theorem for line bundles

K. Oka, 1939: For complex line bundles over a Stein S, the holomorphic and the topological classifications agree:

$$\mathrm{Pic}(S) = H^1(S, \mathcal{O}^*) \cong H^1(S, \mathcal{C}^*) \cong H^2(X, \mathbb{Z}).$$

Oka treated this as a *second Cousin problem*:
Given an open cover $\mathcal{U} = \{U_j\}$ of S and a collection of nonvanishing holomorphic functions $f_{ij} \colon U_{ij} \to \mathbb{C} \setminus \{0\}$ such that

$$f_{ii} = 1, \quad f_{ij} f_{ji} = 1, \quad f_{ij} f_{jk} f_{jk} = 1,$$

find nonvanishing functions $f_i \colon U_i \to \mathbb{C} \setminus \{0\}$ such that

$$f_i = f_{ij} f_j \quad \text{on } U_{ij}.$$

Oka: *If the problem can be solved by continuous functions, then it can also be solved by holomorphic functions.*

Cohomological proof of Oka's theorem

Let $\sigma(f) = \mathrm{e}^{2\pi \mathrm{i} f}$. The *exponential sheaf sequence*:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathcal{O} & \xrightarrow{\sigma} & \mathcal{O}^* & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathcal{C} & \xrightarrow{\sigma} & \mathcal{C}^* & \longrightarrow & 1 \end{array}$$

Long exact sequence on cohomology:

$$\begin{array}{ccccccc} H^1(S,\mathcal{O}) & \longrightarrow & H^1(S,\mathcal{O}^*) & \xrightarrow{c_1} & H^2(S,\mathbb{Z}) & \longrightarrow & H^2(S,\mathcal{O}) \\ \downarrow & & \downarrow & & \| & & \downarrow \\ 0 & \longrightarrow & H^1(S,\mathcal{C}^*) & \xrightarrow{c_1} & H^2(S,\mathbb{Z}) & \longrightarrow & 0 \end{array}$$

If S is Stein then $H^1(S,\mathcal{O}) = 0 = H^2(S,\mathcal{O})$, and hence $H^1(S,\mathcal{O}^*) \cong H^1(S,\mathcal{C}^*) \cong H^2(S,\mathbb{Z})$.

Main analytic ingredients of Grauert's proof

This cohomological proof fails for nonabelian Lie groups. Grauert's proof is constructive and uses two main ingredients:

• *Homotopy version of Runge approximation theorem* for maps of Stein manifolds S to homogeneous manifolds X:

Given a pair $K \subset L$ of compact $\mathcal{O}(S)$-convex sets and a homotopy of holomorphic maps $f_t \colon K \to X$ ($t \in [0,1]$) such that f_0 is holomorphic on L, $\{f_t\}$ is approximable uniformly on K by holomorphic homotopy $\widetilde{f}_t \colon L \to X$ ($t \in [0,1]$) with $\widetilde{f}_0 = f_0$.

(Proof: Using the exponential map $\mathfrak{g} \times X \to X$, $(v, x) \mapsto \mathrm{e}^v x$, pull back $\{f_t\}_{t\in[0,1]}$ to a homotopy of sections of a vector bundle over S, apply the Oka-Weil theorem, then push back to X.)

• *Cartan's lemma*: Given a suitable pair of compact sets $K = K_0 \cup K_1$ in S, every holomorphic maps $f \colon K_0 \cap K_1 \to G$ splits as a product $f = f_0 \cdot f_1$, where $f_j \colon K_j \to G$ is holomorphic for $j = 0, 1$. (This is used for gluing holomorphic sections.)

Generalizations and applications of Grauert's theorem

- *Forster and Ramspott*, 1964–1970:
  - Generalization to certain stratified bundles with homogeneous fibers, and to Oka pairs of sheaves.
  - Optimal estimates for the number of holomorphic sections needed to generate a holomorphic vector bundle, or a coherent analytic sheaf over a Stein space.
  - The Oka principle for complete intersection subvarieties.
  - Embedding theorems for Stein manifolds.

- *Forster & Ohsawa*, 1984: Complete intersections in $\mathbb{C}^n$ for entire functions of finite order.
- *Heinzner and Kutzschebauch*, 1995: An equivariant version of Grauert's Oka principle.
- *Henkin and Leiterer*, 1986 and 1998: Different proof and an extension to 1-convex manifolds.
- *Leiterer and Vâjâitu, 2003:* Banach-valued bundles.

Sprays and elliptic manifolds

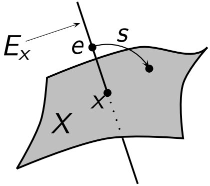
The main problem after 1960's was how to extend the Oka principle beyond the realm of complex homogeneous manifolds. The need was evident in many complex-geometric problems.

The first such extensions were obtained by M. Gromov in 1989.

A **dominating spray** on X is a holomorphic map $s\colon E \to X$, defined on the total space of a holomorphic vector bundle E over X, such that for every $x \in X$ we have

- $s(0_x) = x$, and
- $\mathrm{d}s(0_x)\colon E_x \cong \mathbb{C}^n \to \mathrm{T}_x X$ is onto for every $x \in X$.

A complex manifold with a dominating spray is said to be **elliptic**.



Gromov's Oka principle

Gromov, 1989: **Every elliptic manifold satisfies POP.**

Dominating sprays serve as a replacement of the exponential map on a Lie group to linearize the approximation and gluing problems which arise in the construction of holomorphic maps.

A detailed proof: Prezelj and F., 2000.

Generalization: A complex manifold is **subelliptic** if there exist finitely many sprays $s_j \colon E_j \to X$ such that

$$\sum_j \mathrm{d}s_j(E_{j,x}) = \mathrm{T}_x X, \quad \forall x \in X.$$

F., 2002: **Every subelliptic manifold satisfies POP.**

Examples of (sub-) elliptic manifolds

- A homogeneous X is elliptic: $X \times \mathfrak{g} \xrightarrow{s} X$, $(x, v) \mapsto \mathrm{e}^v \cdot x$.
- Assume that X admits $\mathbb{C}$-complete holomorphic vector fields $v_1, \ldots, v_k$ that span $\mathrm{T}_x X$ at every point. Let ϕ_t^j denote the flow of v_j for time $t \in \mathbb{C}$. Then the map $s \colon E = X \times \mathbb{C}^k \to X$,

$$s(x, t_1, \ldots, t_k) = \phi_{t_1}^1 \circ \cdots \circ \phi_{t_k}^k(x)$$

  is a dominating spray on X.
- A spray of this type exists on $X = \mathbb{C}^n \setminus A$, where A is algebraic subvariety with $\dim A \le n - 2$.
  Use vector fields $f(\pi(z))v$ ($v \in \mathbb{C}^n$, $\pi \colon \mathbb{C}^n \to \mathbb{C}^{n-1}$ linear projection, $\pi(v) = 0$) that vanish on A: $f = 0$ on $\pi(A) \subset \mathbb{C}^{n-1}$.
- $\mathbb{P}^n \setminus A$ is subelliptic if A is a subvariety of codimension ≥ 2.

Oka principle for sections of elliptic submersions

Gromov also obtained POP for sections of (non-locally trivial !) *elliptic holomorphic submersions* $Z \to S$ over a Stein S: Every point $x_0 \in S$ admits an open neighborhood $U \subset S$ and dominating sprays of the fibers Z_x, depending holomorphically on the point $x \in U$.

Details, generalizations: Prezelj & F., 2002–2010

Oka principle for sections avoiding complex subvarieties:

Let $E \to S$ be a holomorphic vector bundle (or a projective bundle), and let $\Sigma \subset E$ be a complex subvariety with algebraic fibers $\Sigma_x \subset E_x \in \{\mathbb{C}^n, \mathbb{CP}^n\}$ of codimension ≥ 2. Then sections $S \to E \setminus \Sigma$ avoiding Σ satisfy POP.

This result is crucial in the proof of the optimal embedding theorem for Stein manifolds:

$$S^n \hookrightarrow \mathbb{C}^{\left[\frac{3n}{2}\right]+1}.$$

Problems concerning ellipticity

Ellipticity is a useful geometric condition implying Oka properties. However, it also has several potential deficiencies:

- Is ellipticity necessary for Oka property?
- Lack of known functorial properties of ellipticity!
- In particular, does ellipticity behave well in fiber bundles?

Gromov's questions:

- Can one characterize Oka properties by a Runge approximation property for maps $\mathbb{C}^n \to X$?
- Does BOP imply POP for every manifold X?

Convex Approximation Property and Oka manifolds

Convex Approximation Property (CAP):
Every holomorphic map $K \to X$ from a compact (geometrically!) convex set $K \subset \mathbb{C}^n$ can be approximated uniformly on K by holomorphic maps $\mathbb{C}^n \to X$.

Observe that CAP equals BOP in the model case $S = \mathbb{C}^n$, K a convex set in $\mathbb{C}^n$, and $T = \emptyset$.

Theorem (F., 2005–2009): CAP $\Longrightarrow$ POP.

It easily follows that all Oka properties are equivalent.
A complex manifold satisfying any of these properties is called an

OKA MANIFOLD

A few properties of Oka manifolds

- If $\pi\colon E \to B$ is a holomorphic fiber bundle whose fiber $\pi^{-1}(b)$ is Oka, then B is Oka iff E is Oka.
- In particular, if $\pi\colon E \to B$ is a holomorphic covering projection then B is Oka iff E is Oka. Every holomorphic quotient of $\mathbb{C}^n$ and of $\mathbb{C}^n \setminus \{0\}$ is Oka (tori, Hopf manifolds).
- A Riemann surface is Oka iff it is non-hyperbolic; these are precisely the surfaces $\mathbb{CP}^1$, $\mathbb{C}$, $\mathbb{C} \setminus \{0\}$, and complex tori;
- No (volume) hyperbolic manifold X is Oka. Indeed, the Oka property can be seen as an answer to the question:

  What should it mean for a complex manifold to be anti-hyperbolic?
- A compact Oka manifold X^n is dominable by $\mathbb{C}^n$, and hence (Kodaira, Kobayashi-Ochiai) its Kodaira dimension satisfies $\kappa_X < n$. This means that

  No Oka manifold is of Kodaira general type.

Examples of Oka manifolds

- $\mathbb{C}^n, \mathbb{P}^n$, complex Lie groups and their homogeneous spaces;
- $\mathbb{C}^n \setminus A$, where A is an algebraic subvariety of codim. ≥ 2;
- $\mathbb{P}^n \setminus A$, where A is a subvariety of codim. ≥ 2;
- rational surfaces (in particular, all Hirzebruch surfaces; these are $\mathbb{P}^1$ bundles over $\mathbb{P}^1$);
- Hopf manifolds (quotients of $\mathbb{C}^n \setminus \{0\}$ by cyclic groups);
- Algebraic manifolds that are covered Zariski locally affine ($\cong \mathbb{C}^n$);
- certain modifications of such (blowing up points, removing subvarieties of codim. ≥ 2);
- $\mathbb{C}^n$ blown up at all points of a tame discrete sequence;
- complex torus of dim> 1 with finitely many points removed, or blown up at finitely many points;
- toric varieties $X = (\mathbb{C}^m \setminus Z)/G$, where Z is a union of coordinate subspaces of $\mathbb{C}^m$, and G is a subgroup of $(\mathbb{C}^*)^m$ acting on $\mathbb{C}^m \setminus Z$ by diagonal matrices.

Methods to prove CAP $\Longrightarrow$ POP

A nonlinear Cousin-I problem:

Let (A, B) be a *Cartain pair* in a Stein manifold S (compacts such that $A \cup B$, $A \cap B$ have Stein neighborhood bases).

Given $f \colon A \to X$, $g \colon B \to X$ holomorphic, with $f \approx g$ on $A \cap B$, find a holomorphic map $\widetilde{f} \colon A \cup B \to X$ such that $\widetilde{f}|_A \approx f|_A$.

- Extend f, g to holomorphic maps

$$F \colon A \times \mathbb{B}^k \to X, \;\; G \colon B \times \mathbb{B}^k \to X,$$

submersive in $z \in \mathbb{B}^k \subset \mathbb{C}^k$; $f = F(\cdot\,, 0)$, $g = G(\cdot\,, 0)$.

- Find a holomorphic transition map $\gamma(x, z) = (x, c(x, z))$ over $(A \cap B) \times r\mathbb{B}^k$ $(r < 1)$, $\gamma \approx \mathrm{Id}$, such that $F = G \circ \gamma$.

- Split

$$\gamma = \beta \circ \alpha^{-1}, \quad \alpha, \beta \approx \mathrm{Id}.$$

Then $F \circ \alpha = G \circ \beta \colon (A \cup B) \times r\mathbb{B}^k \to X$ solves the problem.

Passing a critical point

Passing a critical point p_0 of a strongly plurisubharmonic exhaustion function $\rho\colon S \to \mathbb{R}$:

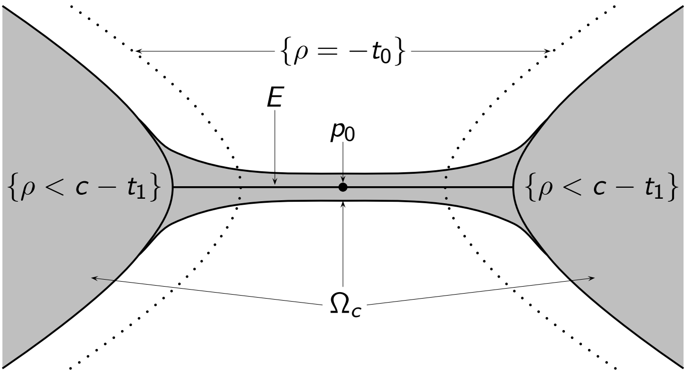


Figure: The set $\Omega_c = \{\tau < c\}$, $c > 0$.

From manifolds to maps

Grothendieck: Properties of objects (manifolds, varieties) should give rise to corresponding properties of maps (morphisms).

Oka properties of a holomorphic map $\pi\colon E \to B$ pertain to liftings in the following diagram, with Stein source S:

$$
\begin{tikzcd}
 & E \arrow[d, "\pi"] \\
P \times S \arrow[ur, "F"] \arrow[r, "f"'] & B
\end{tikzcd}
$$

For a given S-holo. map $f\colon P \times S \to B$ (with P a compact in $\mathbb{R}^m$), every continuous lifting F must be homotopic to an S-holo. lifting.

Theorem (F. 2010) Let $\pi\colon E \to B$ a stratified holo. submersion.
(a) BOP $\Longrightarrow$ POP, and these are local properties.
(b) A stratified holo. fiber bundle with Oka fibers enjoys POP.
(c) A stratified subelliptic submersion enjoys POP.

Application to the Gromov-Vaserstein Problem

Ivarsson and Kutzschebauch, Ann. Math., in press:
Let S be a Stein manifold and $f\colon S \to SL_m(\mathbb{C})$ a null-homotopic holomorphic mapping. There exist $k \in \mathbb{N}$ and holomorphic mappings $G_1, \ldots, G_k\colon S \to \mathbb{C}^{m(m-1)/2}$ such that

$$f(x) = \begin{pmatrix} 1 & 0 \\ G_1(x) & 1 \end{pmatrix} \begin{pmatrix} 1 & G_2(x) \\ 0 & 1 \end{pmatrix} \ldots \begin{pmatrix} 1 & G_k(x) \\ 0 & 1 \end{pmatrix}.$$

The proof uses Vaserstein's factorization of continuous maps (1988), together with the most advanced version of Oka principle for sections of stratified elliptic submersions over Stein spaces.

The algebraic case: Consider algebraic maps $\mathbb{C}^n \to SL_m(\mathbb{C}^)$.
Cohn (1966): The matrix

$$\begin{pmatrix} 1 - z_1 z_2 & z_1^2 \\ -z_2^2 & 1 + z_1 z_2 \end{pmatrix} \in SL_2(\mathbb{C}[z_1, z_2])$$

does not decompose as a finite product of unipotent matrices.
Suslin (1977): For $m \geq 3$ (and any n) any matrix in $SL_m(\mathbb{C}^{[n]})$ decomposes as a finite product of unipotent matrices.

Link with abstract homotopy theory

Lárusson, 2003–5: Being an Oka manifold is a homotopy-theoretic property.

The category of complex manifolds can be embedded into a model category such that

- a holomorphic map is acyclic iff it is topologically acyclic,
- a Stein inclusion is a cofibration, and
- a holomorphic map is a fibration iff it is a topological fibration and satisfies POP. Such a map is called an **Oka map**.

In this model category, a complex manifold is:

- cofibrant iff it is Stein, and
- fibrant iff it is an Oka manifold.

Open problems

- Find a geometric characterization of Oka manifolds.
- In particular, is every Oka manifold also elliptic (or at least subelliptic)? Is every subelliptic manifold elliptic?
- Which complex surfaces of non-general type are Oka?
- In particular, which K3 surfaces are Oka? Is every Kummer surface Oka?
- Which modifications (such as blow-ups or blow-downs) preserve Oka property?
- Is $\mathbb{C}^n \setminus$ (closed ball) Oka?
- Topological restrictions on the class of Oka manifolds?

The Soft Oka Principle

The Oka principle becomes a tautology if we allow a homotopic deformation of the Stein structure on the source manifold.

Slapar & F., 2007: Let (S, J) be a Stein manifold of dimension $\dim_{\mathbb{C}} S \neq 2$, and let X be an arbitrary complex manifold. For every continuous map $f : S \to X$ there exists a Stein complex structure $\widetilde{J}$ on S, homotopic to J, and a holomorphic map $\widetilde{f} : (S, \widetilde{J}) \to X$ that is homotopic to f.

If $\dim_{\mathbb{C}} S = 2$ then the above holds for a possibly *exotic Stein structure* $\widetilde{J}$ on S (we change the underlying $\mathcal{C}^\infty$ structure!).

These results are closely related to the Eliashberg-Gompf construction of Stein manifold structures on an oriented smooth manifold S^{2n} whose CW decomposition has no cells of index $> n$.

Example: The complex surface $S = \mathbb{CP}^1 \times \mathbb{C}$ does not admit a non-exotic Stein structure, but it admits exotic Stein structures.

Additional reading...

– F. Lárusson: What is an Oka manifold?
Notices Amer. Math. Soc. 57 (2010), no. 1, 50–52.

– F. Forstnerič and F. Lárusson: Survey of Oka theory.
New York J. Math., 17a (2011), 1–28.

– T. Ohsawa: Topics in complex analysis from the viewpoint of Oka principle. Preprint (2011), 30 pp.

– F. Forstnerič: *Stein Manifolds and Holomorphic Mappings.*
(The Homotopy Principle in Complex Analysis.)
Ergebnisse der Mathematik und ihrer Grenzgebiete, 3. Folge, vol. 56, Springer-Verlag, Berlin-Heidelberg (2011)