Every meromorphic function is the Gauss map of a conformal minimal surface

Franc Forstnerič

University of Ljubljana
Institute of Mathematics, Physics and Mechanics
The Slovenian Academy of Sciences and Arts

Indam Meeting Complex and symplectic geometry
Cortona, 14 June 2016

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- X is harmonic: $\triangle X = 4\partial\bar{\partial}X = 0$.
- $\Phi = \partial X = (\phi_1, \ldots, \phi_n)$ is a nowhere vanishing holomorphic 1-form satisfying the nullity condition

$$(\phi_1)^2 + (\phi_2)^2 + \cdots + (\phi_n)^2 = 0. \tag{1}$$

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Conversely, if a nowhere vanishing holomorphic 1-form $\Phi = (\phi_1, \ldots, \phi_n)$ on M satisfies (1) and has vanishing real periods:

$$\int_\gamma \Re(\Phi) = 0 \quad \text{for all} \quad \gamma \in H_1(M; \mathbb{Z}),$$

then the integral of $\Re\Phi$ is a conformal minimal immersion $M \to \mathbb{R}^n$:

$$X(p) = \int_*^p 2\Re\Phi, \quad p \in M.$$

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The above shows that every conformal minimal immersion $X\colon M \to \mathbb{R}^n$ is of the form

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where

$$f\colon M \to \mathcal{A}_*^{n-1} = \mathcal{A}^{n-1} \setminus \{0\} \subset \mathbb{C}^n$$

is a holomorphic map with values in the **null quadric**

$$\mathcal{A}^{n-1} = \{(z_1, \ldots, z_n) \in \mathbb{C}^n \colon z_1^2 + z_2^2 + \cdots + z_n^2 = 0\}$$

such that

the 1-form $f\theta$ has vanishing real periods.

Flux and holomorphic null curves

The **flux** of a conformal minimal immersion $X\colon M \to \mathbb{R}^n$ is the group homomorphism

$$\mathrm{Flux}_X\colon H_1(M;\mathbb{Z}) \to \mathbb{R}^n$$

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$$\mathrm{Flux}_X(\gamma) = \int_\gamma \Im(\partial X) = \int_\gamma d^c X \quad \text{for every closed curve } \gamma \subset M.$$

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This always holds if M is simply connected (the disk or the plane).

The generalized Gauss map

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Since the 1-form $\partial X = (\partial X_1, \ldots, \partial X_n)$ is holomorphic and nowhere vanishing, it determines the Kodaira type holomorphic map

$$G_X \colon M \to \mathbb{CP}^{n-1}, \quad G_X(p) = [\partial X_1(p) \colon \cdots \colon \partial X_n(p)] \quad (p \in M).$$

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Since $\sum_{j=1}^n (\partial X_j)^2 = 0$, G_X assumes values in the complex hyperquadric

$$Q^{n-2} = \{[z_1 : \ldots : z_n] \in \mathbb{CP}^{n-1} : z_1^2 + \cdots + z_n^2 = 0\} = \pi(\mathcal{A}_*^{n-1})$$

where $\pi \colon \mathbb{C}_*^n \to \mathbb{CP}^{n-1}$ denotes the canonical projection.

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For every holomorphic map $\mathscr{G} \colon M \to Q^{n-2} \subset \mathbb{CP}^{n-1}$ there exists a conformal minimal immersion $X \colon M \to \mathbb{R}^n$ with the generalized Gauss map $G_X = \mathscr{G}$ and with vanishing flux.

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If in addition the map $\mathscr{G}$ is full (i.e., its image is not contained in any proper projective subspace), then X can be chosen to have arbitrary flux and to be an embedding if $n \geq 5$ and an immersion with simple double points if $n = 4$.

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If M and $\mathscr{G}$ are algebraic (so $M = \overline{M} \setminus \{p_1, \ldots, p_m\}$ with $\overline{M}$ a closed Riemann surface), we don't know whether X can be chosen algebraic.

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To prove the theorem, we construct a holomorphic multiplier $h\colon M \to \mathbb{C}_*$ such that the $\mathbb{C}^n$-valued holomorphic 1-form

$$\Phi = hf\theta = h(f_1, \ldots, f_n)\theta$$

has vanishing periods, so it integrates to a conformal minimal immersion

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Then, $\partial X = \Phi$ and hence $G_X = \mathscr{G}$.

The case $n = 3$

The quadric $Q^1 \subset \mathbb{CP}^2$ is an embedded rational curve $\mathbb{CP}^1 \hookrightarrow \mathbb{CP}^2$, and the *complex Gauss map* of a conformal minimal immersion $X = (X_1, X_2, X_3) \colon M \to \mathbb{R}^3$ is defined to be the holomorphic map

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If g is nonconstant, then we can find X with arbitrary given flux.

Metric and curvature

We can recover $\partial X = (\partial X_1, \partial X_2, \partial X_3)$ from the pair (g_X, ϕ_3) with $\phi_3 = \partial X_3$ by the classical Weierstrass formula

$$\partial X = \Phi = (\phi_1, \phi_2, \phi_3) = \left(\frac{1}{2}\left(\frac{1}{g_X} - g_X\right), \frac{i}{2}\left(\frac{1}{g_X} + g_X\right), 1\right)\phi_3.$$

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$$K = -\left(\frac{4|g|\cdot|dg|}{|\phi_3|(1+|g|^2)^2}\right)^2 \qquad \text{(the Gauss curvature)}.$$

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Fujimoto 1988 The Gauss map of a complete nonflat minimal surface cannot omit 5 points of the sphere. (**Xavier 1981:** It cannot omit 7 points. **López and Ros 1987:** it cannot omit 6 points.)

Stable minimal surfaces in $\mathbb{R}^3$

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Let $X\colon M \to \mathbb{R}^3$ be a conformal minimal immersion. The minimal surface $X(M) \subset \mathbb{R}^3$ is stable if the spherical image $g_X(M) \subset \mathbb{CP}^1$ of its Gauss map g_X has area less than $2\pi = \frac{1}{2}\mathrm{A}rea(\mathbb{CP}^1)$.

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Corollary

If M is an open Riemann surface and $g \colon M \to \mathbb{CP}^1$ is a holomorphic map whose image $g(M)$ has area less than 2π, then there exists a stable conformal minimal immersion $M \to \mathbb{R}^3$ with the complex Gauss map g.

Isotopies whose Gauss map omits two points

Theorem

Given an open Riemann surface M and a conformal minimal immersion $X\colon M \to \mathbb{R}^3$, there exists an isotopy

$$X_t\colon M \to \mathbb{R}^3, \quad t \in [0,1]$$

of conformal minimal immersions such that $X_0 = X$ and the complex Gauss map $g = g_{X_1}\colon M \to \mathbb{CP}^1$ of X_1 is nonconstant and avoids any two given points of the Riemann sphere.

Isotopies whose Gauss map omits two points

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There also exists an isotopy X_t as above such that $X_0 = X$ and X_1 is flat, so its Gauss map g is constant.

The main complex analytic result

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be a continuous family of group homomorphisms. Then, there exists a continuous family of holomorphic multipliers

$$h_t \colon M \to \mathbb{C}_*, \quad t \in [0,1]$$

such that

$$\int_\gamma h_t \Phi_t = \mathfrak{q}_t(\gamma) \quad \textit{for every closed curve } \gamma \subset M \textit{ and } t \in [0,1].$$

Proof

A desired family of holomorphic multipliers $h_t \colon M \to \mathbb{C}_* \ (t \in I = [0,1])$ is constructed by a recursive procedure with respect to an exhaustion of M by an increasing sequence of smoothly bounded Runge domains

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- Mergelyan's approximation theorem is used to approximate continuous families of multipliers on Runge domains with attached curves by continuous families of holomorphic multipliers.

Main Lemma 1 – multipliers on arcs

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Set $I = [0,1]$. *Given continuous maps*

$$f\colon I^2 = I \times I \to \mathbb{C}^n, \quad \alpha\colon I \to \mathbb{C}^n$$

such that $f_t := f(t,\cdot)\colon I \to \mathbb{C}^n$ *is nowhere flat for every* $t \in I$, *there exists a continuous function* $h\colon I^2 \to \mathbb{C}_*$ *such that*

$$\int_0^1 h(t,s) f(t,s)\, ds = \alpha(t), \quad t \in I$$

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In addition,

$$\int_0^1 f(0,s)\, ds = \alpha(0) \implies h(0,\cdot\,) \equiv 1.$$

Proof of Lemma 1, part 1

It suffices to prove that for any $\epsilon > 0$ there exists $h \colon I^2 \to \mathbb{C}_*$ such that

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Since f_t is nowhere flat and hence full for each fixed $t \in [0, 1]$, there is a division $0 = s_0 < s_1 < \cdots < s_N = 1$ of I such that

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Set

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By passing to a finer division of I, we may therefore assume that

$$\mathrm{span}\{V_1(t), \ldots, V_N(t)\} = \mathbb{C}^n, \quad t \in I.$$

Proof of Lemma 1, part 2

For each $t \in I$ we let $\Sigma_t \subset \mathbb{C}^N$ denote the affine complex hyperplane

$$\Sigma_t = \left\{ (g_1, \ldots, g_N) \in \mathbb{C}^N : \sum_{j=1}^{N} g_j V_j(t) = \alpha(t) \right\}.$$

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Hence, if $\int_0^1 f(0,s)\, ds = \alpha(0)$ then g can be chosen such that $g(0) = (1, \dots, 1) \in \mathbb{C}^N$. We assume in the sequel that this holds.

Proof of Lemma 1, part 3

By a small perturbation we may assume that $g_j(t) \in \mathbb{C}_*$ for every $t \in I$ and $j = 1, \ldots, N$. (We use general position and $\dim I = 1$.) This changes the exact condition in the above display to the approximate condition

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$$\left| \int_0^1 h(t,s) f(t,s)\, ds - \alpha(t) \right| < \epsilon, \quad t \in I.$$

This completes the proof of Lemma 1.

Period dominating families of multipliers

Lemma (2)

Let I' be a nontrivial closed subinterval of $I = [0, 1]$, let Q be a compact Hausdorff space (we may use $Q = I$), and let $n \in \mathbb{N}$.

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Given a continuous map $f\colon Q \times I \to \mathbb{C}^n$ such that $f(q,\cdot)$ is full on I' for every $q \in Q$, there exist finitely many continuous functions $g_1, \ldots, g_N\colon I \to \mathbb{C}$ $(N \geq n)$, supported on I', such that the function $h\colon \mathbb{C}^N \times I \to \mathbb{C}$ given by

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is a **period dominating multiplier of** f, *meaning that the map*

$$\frac{\partial}{\partial \zeta}\bigg|_{\zeta=0} \int_0^1 h(\zeta, s) f(q, s)\, ds\colon T_0\mathbb{C}^N \cong \mathbb{C}^N \to \mathbb{C}^n$$

is surjective for every $q \in Q$.

Proof of Lemma 2, part 1

Let $N \geq n$ be an integer and, for each $i \in \{1, \ldots, N\}$, let $g_i \colon I \to \mathbb{C}$ be a continuous function supported on I' (to be specified later).

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Let $\mathcal{P} \colon Q \times \mathbb{C}^N \to \mathbb{C}^n$ be the map given by

$$\mathcal{P}(q, \zeta) = \int_0^1 h(\zeta, s) f(q, s)\, ds, \quad (q, \zeta) \in Q \times \mathbb{C}^N.$$

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Proof of Lemma 2, part 2

Since $f(q,\cdot)$ is full on I' for every $q \in Q$, there are distinct points $s_1, \ldots, s_N$ in the interior of I' for a big $N \in \mathbb{N}$ such that

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for all $q \in Q$ and $i \in \{1, \ldots, N\}$. Therefore, if $\epsilon > 0$ is small enough,

$$\mathrm{span}\left\{ \left.\frac{\partial \mathcal{P}(q,\zeta)}{\partial \zeta_1}\right|_{\zeta=0}, \ldots, \left.\frac{\partial \mathcal{P}(q,\zeta)}{\partial \zeta_N}\right|_{\zeta=0} \right\} = \mathbb{C}^n \quad \text{for all } q \in Q.$$

This completes the proof of Lemma 2.

Construction of the multiplier on M

We exhaust M by smoothly bounded Runge domains $D_1 \subset D_2 \subset \cdots \cup_j D_j = M$ such that D_{j+1} is obtained from D_j by adding at most one handle, or a new connected component.

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The noncritical case: there is no change of topology for $D_j \subset D_{j+1}$. We embed the multiplier $h \colon D_j \to \mathbb{C}_*$ in a period dominating family $h_\zeta \colon D_j \to \mathbb{C}_*$ $(\zeta \in \mathbb{C}^N)$ with $h = h_0$.

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Step 2: Use Lemma 2 to embed the multiplier into a period dominating family over $D_j \cup E$. Now proceed as in the noncritical case.

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$\mathfrak{N}_*(M, \mathbb{C}^n)$: nonflat holomorphic null curves $M \to \mathbb{C}^n$

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Theorem (F. Lárusson and F. F., 2016)

Each of the maps ι, ϕ, ψ is a weak homotopy equivalence, and a strong homotopy equivalence if $H_1(M; \mathbb{Z})$ is finitely generated.

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The theorem tells us that these mapping spaces all have the same weak homotopy type as the space $\mathfrak{H} = \mathscr{C}(M,\mathcal{A}_*^{n-1})$.

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Thus, the structure of $\mathfrak{H}$ can be understood in terms of spheres and their loop spaces. The homotopy groups of $\mathfrak{H}$ can be calculated in terms of homotopy groups of spheres.

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Today, we look at the path connected components of $\mathfrak{H}$ and leave the rest for another day.

Path components of the space $\mathfrak{M}_*(M, \mathbb{R}^n)$

Corollary

Let $H_1(M;\mathbb{Z}) = \mathbb{Z}^\ell$ *with* $\ell \in \mathbb{Z}_+ \cup \{\infty\}$. *Then*

$$\pi_0\left(\mathfrak{M}_*(M, \mathbb{R}^3)\right) = (\mathbb{Z}_2)^\ell, \quad \pi_0\left(\mathfrak{M}_*(M, \mathbb{R}^n)\right) = 0 \ \ \textit{if } n > 3.$$

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If $n > 3$ then $\mathscr{C}(M, \mathcal{A}_*^{n-1})$ is path connected since $\pi_1(\mathcal{A}_*^{n-1}) = 0$.

Path components of the space $\mathfrak{M}(M, \mathbb{R}^n)$

$\mathfrak{M}(M, \mathbb{R}^n) =$ the space of all conformal minimal immersions $M \to \mathbb{R}^n$.

Theorem (A. Alarcón, F.J. López, F.F., 2016)

Let M be an open connected Riemann surface. The natural inclusion

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induces a bijection of path components of the two spaces. In particular,

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The case $n > 3$ is a corollary to the following result.

Theorem (Deforming flat immersions to nonflat ones)

Given a flat conformal minimal immersion $X : M \to \mathbb{R}^n$ $(n \geq 3)$, there exists an isotopy $X_t : M \to \mathbb{R}^n$ $(t \in [0, 1])$ of conformal minimal immersions such that $X_0 = X$ and X_1 is nonflat.

Proof of the deformation theorem

We may assume that $n = 3$ and $\partial X = (1, \imath, 0)\phi_3$ where ϕ_3 is an exact holomorphic 1-form without zeros on M.

The gist of the proof is show that there is nonconstant holomorphic function $g\colon M \to \mathbb{C}_*$ such that $g\phi_3$ and $g^2\phi_3$ are exact 1-forms on M. This is similar to the proof of Lemma 1. Then,

$$\Phi_\lambda = \left(1 - \lambda^2 g^2, \imath(1 + \lambda^2 g^2), 2\lambda g\right) \phi_3, \quad \lambda \in \mathbb{C}$$

is an exact holomorphic 1-form and the map Φ_λ/ϕ_3 assumes values in the punctured null quadric $\mathcal{A}_* \subset \mathbb{C}^3$ for every $\lambda \in \mathbb{C}$. Thus, Φ_λ provides a conformal minimal immersion $X_\lambda\colon M \to \mathbb{R}^3$ by

$$X_\lambda(p) = X(p_0) + 2\int_{p_0}^{p} \Re(\Phi_\lambda), \quad p \in M.$$

Since $\Phi_0 = \partial X$, we have that $X_0 = X$. Furthermore, since g is nonconstant, X_1 is nonflat, and hence the isotopy $X_t\colon M \to \mathbb{R}^n$ $(t \in [0,1])$ satisfies the conclusion of the theorem.

Path components of the space $\mathfrak{M}(M, \mathbb{R}^3)$

In dimension $n = 3$, we obtain that

$$\pi_0(\mathfrak{M}(M, \mathbb{R}^3)) = \mathbb{Z}_2^\ell$$

by using the corresponding result for nonflat immersions,

$$\pi_0(\mathfrak{M}_*(M, \mathbb{R}^3)) = \mathbb{Z}_2^\ell,$$

along with the deformation theorem and the following result.

Theorem

Let θ be a nowhere vanishing holomorphic 1-form on M. For every group homomorphism

$$\mathfrak{p}\colon H_1(M;\mathbb{Z}) \cong \mathbb{Z}^\ell \to \mathbb{Z}_2$$

there exists a **flat conformal minimal immersion** $X\colon M \to \mathbb{R}^3$ *satisfying* $H_1(\partial X/\theta) = \mathfrak{p}$.

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- General position and Runge-Mergelyan approximation theorems for Legendrian curves in $\mathbb{C}^{2n+1}$.