A properly embedded holomorphic disc in the ball with finite area and dense boundary curve

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The main result

Notation:

$$\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\} = \mathbb{B}^1$$

$$\mathbb{B}^n = \Big\{(z_1, \ldots, z_n) \in \mathbb{C}^n : |z|^2 = \sum_{j=1}^{n} |z_j|^2 < 1\Big\}$$

We prove the following result which answers a question posed by Filippo Bracci (private communication, June 2017).

Theorem

Given $n > 1$ and $\epsilon > 0$, there exists a proper holomorphic embedding $F\colon \mathbb{D} \hookrightarrow \mathbb{B}^n$ which extends to an injective holomorphic immersion $F\colon \overline{\mathbb{D}} \setminus \{\pm 1\} \to \overline{\mathbb{B}}^n$ such that $\mathrm{Area}(F(\mathbb{D})) < \epsilon$ and the boundary curve $F(b\mathbb{D} \setminus \{\pm 1\})$ is everywhere dense in the sphere $b\mathbb{B}^n$.

The map F extends holomorphically to an open neighborhood of $\overline{\mathbb{D}} \setminus \{\pm 1\}$. A similar phenomenon exists on $\overline{\mathbb{D}} \setminus \{1\}$.

An embedded disc in $\mathbb{B}^2$ with bounded defining function and dense boundary curve

Berndtsson 1980 A properly embedded holomorphic curve of finite area in the ball $\mathbb{B}^2$ is the zero set of a bounded holomorphic function on $\mathbb{B}^2$.

Corollary

There is a bounded holomorphic function on $\mathbb{B}^2$ whose zero set is a smooth curve of finite area, biholomorphic to the disc, and whose real analytic boundary curve is dense in $b\mathbb{B}^2$.

Globevnik & Stout 1989 Every strongly pseudoconvex domain $D \subset \mathbb{C}^n$ $(n \geq 2)$ with real analytic boundary contains a proper holomorphic disc $F\colon \mathbb{D} \to D$ of arbitrarily small area such that $\overline{F(\mathbb{D})} = F(\mathbb{D}) \cup \overline{\omega}$, where ω is a given nonempty connected subset of bD.

The main new point is that we find properly **embedded** holomorphic discs with these properties, even in the lowest dimensional case $n = 2$.

A result from contact geometry

Let $n \geq 2$. The sphere $b\mathbb{B}^n = S^{2n-1}$ carries the contact structure ξ given by the distribution of complex tangent lines. Let $x, y \in \mathbb{R}^{n-1}$, $z \in \mathbb{R}$. Removing a point from $b\mathbb{B}^n$ we obtain the Euclidean space $(\mathbb{R}^{2n-1}_{(x,y,z)}, \xi_0)$ with its **standard contact structure**

$$\xi_0 = \ker(dz + xdy), \quad xdy = \sum_{j=1}^{n-1} x_j dy_j.$$

A smooth curve $f\colon \mathbb{R} \to b\mathbb{B}^n$ is said to be complex tangential, or ξ-Legendrian, if $\dot{f}(t) \in \xi_{f(t)}$ holds for every $t \in \mathbb{R}$.

Lemma (1)

Let $n \in \mathbb{N}$. Every continuous map $f\colon \mathbb{R} \to \mathbb{R}^{2n+1}$ can be approximated as closely as desired in the fine $\mathscr{C}^0$ topology by a real analytic injective Legendrian immersions $\tilde{f}\colon \mathbb{R} \hookrightarrow (\mathbb{R}^{2n+1}, \xi_0)$. In particular, we can ensure that the curves $\Lambda_\pm = \tilde{f}(\mathbb{R}_\pm)$ are everywhere dense in $b\mathbb{B}^n$.

Sketch of proof of Lemma 1

The contact form $\alpha_0 = dz + xdy$ on $\mathbb{R}^{2n+1}$ extends to a holomorphic contact form on $\mathbb{C}^{2n+1}$ given by the same formula.

A real analytic Legendrian map $f \colon I \to \mathbb{R}^{2n+1}$ on an interval $I \subset \mathbb{R}$ extends by complexification to a holomorphic Legendrian map from a neighborhood of I in $\mathbb{C}$.

If a domain $D \subset \mathbb{C}$ is invariant under the conjugation $\tau(z) = \bar{z}$ and $f \colon D \to \mathbb{C}^{2n+1}$ is a holomorphic Legendrian map, then

$$F(z) = \frac{1}{2}\left(f(z) + \overline{f(\bar{z})}\right) \in \mathbb{C}^{2n+1}, \quad z \in D,$$

is a holomorphic Legendrian map satisfying $F(z) = \overline{F(\bar{z})}$; in particular, $F \colon D \cap \mathbb{R} \to \mathbb{R}^{2n+1}$ is a real analytic Legendrian map.

Using our recent work **Alarcón, F., López, Compositio Math. 2017** and the above symmetrization trick, we inductively construct an entire Legendrian map $F \colon \mathbb{C} \hookrightarrow \mathbb{C}^{2n+1}$ such that $\tilde{f} = F|_{\mathbb{R}^{2n+1}}$ is an injective immersion approximating f in the fine $\mathscr{C}^0$ topology on $\mathbb{R}^{2n+1}$.

The scheme of proof of the main theorem

A suitably chosen (thin) complexification of the embedded $\mathscr{C}^\omega$ Legendrian curve $\Lambda = f(\mathbb{R}) \subset b\mathbb{B}^n$ is an embedded complex disc $\Sigma_0 = f(S) \subset \mathbb{C}^n$ of arbitrarily small area such that $\Sigma_0 \cap \overline{\mathbb{B}}^n = \Lambda$. Here, $S \subset \mathbb{C}$ is a thin strip around $\mathbb{R}$.

Let $\sigma \colon \mathbb{C}^n_* \to \mathbb{CP}^{n-1}$ denote the canonical projection. If $f : \mathbb{R} \to b\mathbb{B}^n$ is a Legendrian immersion, then $\sigma \circ f : \mathbb{R} \to \mathbb{CP}^{n-1}$ is an immersion.

We shall find an embedded disc in $\mathbb{B}^n$ satisfying the theorem by pulling the strip $f(S)$ slightly into the ball along the curve $f(\mathbb{R}) \subset b\mathbb{B}^n$, where the amount of pulling decreases very fast near infinity in $\mathbb{R}$. When doing so, we shall pay special attention to ensure injectivity; this is a fairly delicate task since the curve $\Lambda = f(S) \cap b\mathbb{B}^n$ is dense in $b\mathbb{B}^n$.

The principal difficulty is that injectivity is not an open condition among immersions of noncompact manifolds in any fine topology. (However, immersions form an open set in the fine $\mathscr{C}^1$ topology.)

The scheme of proof of the main theorem

We shall be considering smoothly bounded, simply connected strips $D \subset \mathbb{C}$, symmetric with respect to both coordinate axes, with

$$\mathbb{R} \subset D \subset \overline{D} \subset S,$$

and functions $h = u + \mathrm{i} v \in \mathscr{A}^1(D) = \mathscr{C}^1(\overline{D}) \cap \mathcal{O}(D)$ such that

$$u > 0 \text{ on } D, \quad u(z) = u(\bar{z}), \quad \text{and} \quad e^{2u} < |f|^2 \text{ on } bD.$$

The map of class $\mathscr{A}^1(D)$ defined by

$$F = f_h := e^{-h} f : \overline{D} \to \mathbb{C}^n_*$$

then satisfies

$$|F|^2 = e^{-2u}|f|^2 < |f|^2 = 1 \text{ on } \mathbb{R}, \quad |F|^2 = e^{-2u}|f|^2 > 1 \text{ on } bD.$$

Hence,

$$F(\mathbb{R}) \subset \mathbb{B}^n \quad \text{and} \quad F(bD) \cap \overline{\mathbb{B}}^n = \varnothing.$$

The main lemma

Assume in addition that F **is transverse to the sphere** $b\mathbb{B}^n$. Let

$$(*) \qquad \Omega \subset \{z \in D : F(z) \in \mathbb{B}^n\} = F^{-1}(\mathbb{B}^n)$$

denote the connected component of $F^{-1}(\mathbb{B}^n)$ containing $\mathbb{R}$. Hence, $\overline{\Omega} \subset D$ **and** $F|_\Omega \colon \Omega \to \mathbb{B}^n$ **is a proper holomorphic map extending holomorphically to** $\overline{\Omega}$ **and mapping** $b\Omega$ **into** $b\mathbb{B}^n$**. Clearly,** Ω **is Runge in** $\mathbb{C}$ **and hence conformally equivalent to the disc.**

The main theorem is an immediate consequence of the following lemma.

Lemma (2)

Let $n > 1$. Given $\epsilon > 0$, there exist a smoothly bounded, simply connected, symmetric domain $D \subset \mathbb{C}$ satisfying $\mathbb{R} \subset D \subset \overline{D} \subset S$ and a function $h = u + \imath v \in \mathscr{A}^1(D)$ such that the map $F = e^{-h} f \colon \overline{D} \to \mathbb{C}^n_*$ is an injective immersion transverse to $b\mathbb{B}^n$ and satisfying

(α) $\mathrm{Area}(F(\Omega)) < \epsilon$ *where Ω is given by (*), and*

(β) *the curve $F(b\Omega) \subset b\mathbb{B}^n$ is everywhere dense in the sphere $b\mathbb{B}^n$.*

The function space $\mathscr{H}(\mathbb{D})$

$$\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}, \quad \mathbb{T} = b\mathbb{D} = \{z \in \mathbb{C} : |z| = 1\},$$

$$\mathscr{H} = \{h = u + \imath v \in \mathscr{A}^\infty(\mathbb{D}) : h(\bar{z}) = \overline{h(z)}, \quad u|_{\mathbb{T}} = 0 \text{ near } \pm 1\}.$$

Note that every function $h = u + \imath v \in \mathscr{H}$ is determined by a smooth real function $u \in \mathscr{C}^\infty(\mathbb{T})$, supported away from the points ± 1 and satisfying $u(e^{\imath t}) = u(e^{-\imath t})$ for all $t \in \mathbb{R}$, by the formula

$$h(z) = T[u](z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{\imath\theta} + z}{e^{\imath\theta} - z}\, u(e^{\imath\theta})\, d\theta, \quad z \in \mathbb{D}.$$

We shall write

$$\mathscr{H}^\pm = \{h = u + \imath v \in \mathscr{H} : \pm u \geq 0\}, \quad \mathscr{H}^\pm_* = \mathscr{H}^\pm \setminus \{0\}.$$

The sets $\mathscr{H}^\pm$ and $\mathscr{H}^\pm_*$ are real cones, $\mathscr{H}^+ \cap \mathscr{H}^- = \{0\}$, and for every $h = u + \imath v \in \mathscr{H}^+_*$ we have that

$$u > 0 \text{ on } \mathbb{D}, \quad \frac{\partial u}{\partial x}(-1) > 0, \quad \frac{\partial u}{\partial x}(1) < 0, \quad v = 0 \text{ on } \mathbb{R} \cap \mathbb{D}.$$

A general position lemma

Lemma (3)

Let $f : \overline{\mathbb{D}} \to \mathbb{C}^n_*$ $(n > 1)$ *be a map of class* $\mathscr{A}^\infty(\mathbb{D})$ *such that*

(a) $|f|^2 := |f_1|^2 + \cdots + |f_n|^2 \le 1$ *on* $(-1, 1) = \mathbb{D} \cap \mathbb{R}$,

(b) $|f| > 1$ *on* $\mathbb{T} \setminus \{\pm 1\}$, *and*

(c) $\sigma \circ f = [f_1 : f_2 : \cdots : f_n] \colon \overline{\mathbb{D}} \to \mathbb{CP}^{n-1}$ *is an immersion.*

Assume that $E \subset b\mathbb{B}^n$ *is a compact smoothly embedded curve such that* $f(\pm 1) \notin E$. *Given a number* $\eta \in (0, 1)$, *there is a function* $h \in \mathscr{H}^+_*$ *arbitrarily close to* 0 *in* $\mathscr{C}^1(\overline{\mathbb{D}})$ *such that the* $\mathscr{A}^\infty(\mathbb{D})$ *immersion*

$$f_h := e^{-h} f : \overline{\mathbb{D}} \to \mathbb{C}^n_*$$

satisfies:

1. $|f_h| < 1$ *on* $(-1, 1)$ *and* $|f_h| > (1 - \eta)|f| + \eta > 1$ *on* $\mathbb{T} \setminus \{\pm 1\}$,
2. f_h *is transverse to the sphere* $b\mathbb{B}^n$, *and*
3. $f_h(\overline{\mathbb{D}}) \cap E = \varnothing$.

Proof of Lemma 3, #1

Given $h = u + \imath v \in \mathscr{H}$, we consoder the subharmonic functions

$$\rho = \log|f| \colon \overline{\mathbb{D}} \to \mathbb{R}, \quad \rho_h := \log|e^{-h}f| = -u + \rho : \overline{\mathbb{D}} \to \mathbb{R}.$$

Conditions (a) and (b) on f imply that

$$\rho \leq 0 \text{ on } [-1,1] = \overline{\mathbb{D}} \cap \mathbb{R}, \qquad \rho(\pm 1) = 0, \qquad \rho > 0 \text{ on } \mathbb{T} \setminus \{\pm 1\}.$$

It follows that

$$\frac{\partial \rho}{\partial x}(-1) \leq 0 \quad \text{and} \quad \frac{\partial \rho}{\partial x}(1) \geq 0.$$

It is obvious that for any $h \in \mathscr{H}_*^+$ with sufficiently small $\mathscr{C}^0(\overline{\mathbb{D}})$ norm the map f_h satisfies condition (1) of the lemma.

Proof of Lemma 3, #2

Note that the map $f_h = e^{-h} f$ intersects the sphere $b\mathbb{B}^n$ transversely if and only if 0 is a regular value of the function $\rho_h = \log |e^{-h} f|$.
Since $\frac{\partial\rho}{\partial x}(-1) \le 0$ and $\frac{\partial\rho}{\partial x}(1) \ge 0$, we have for every $h \in \mathscr{H}_*^+$ that

$$\frac{\partial\rho_h}{\partial x}(-1) = \frac{\partial\rho}{\partial x}(-1) - \frac{\partial u}{\partial x}(-1) < 0, \quad \frac{\partial\rho_h}{\partial x}(1) > 0.$$

Replacing f by $e^{-h} f$ for some such h close to 0, we may assume that $\rho = \log |f|$ satisfies these conditions. Hence, there are discs $U^{\pm} \subset \mathbb{C}$ around the points ± 1 such that $d\rho \neq 0$ on $\overline{\mathbb{D}} \cap (\overline{U}^+ \cup \overline{U}^-)$. It follows that for all $h \in \mathscr{H}$ with sufficiently small $\mathscr{C}^1(\overline{\mathbb{D}})$ norm we have

$$d\rho_h \neq 0 \ \text{ on } \ \overline{\mathbb{D}} \cap (\overline{U}^+ \cup \overline{U}^-).$$

Since $f(\pm 1) \notin E$, we may choose $U^{\pm}$ small enough such that

$$f_h\big(\overline{\mathbb{D}} \cap (\overline{U}^+ \cup \overline{U}^-)\big) \cap E = \varnothing$$

holds for all $h \in \mathscr{H}$ sufficiently close to 0 in $\mathscr{C}^1(\overline{\mathbb{D}})$.

Proof of Lemma 3, #3

Recall that $\rho = \log|f| < 0$ on $(-1,1) = \mathbb{D} \cap \mathbb{R}$. Hence, there is an open set $U_0 \Subset \mathbb{D}$ containing $(-1,1) \setminus (U^+ \cup U^-) \subset \mathbb{R}$ such that $\rho \le -c < 0$ on $\overline{U}_0$ for some $c > 0$.

Since $\rho > 0$ on $\mathbb{T} \setminus \{\pm 1\}$, there is an open set $U_1 \Subset \mathbb{C}$ containing $\mathbb{T} \setminus (U^+ \cup U^-)$ such that $\rho \ge c' > 0$ on $\overline{U}_1 \cap \overline{\mathbb{D}}$ for some $c' > 0$.

It follows that for all $h \in \mathscr{H}^+$ sufficiently close to 0 in $\mathscr{C}^1(\overline{\mathbb{D}})$ we have that $\rho_h < 0$ on $\overline{U}_0$, $\rho_h > 0$ on $\overline{U}_1 \cap \overline{\mathbb{D}}$, and hence

$$f_h\big(\overline{\mathbb{D}} \cap (\overline{U}_0 \cup \overline{U}_1)\big) \cap b\mathbb{B}^n = \varnothing.$$

For such h it follows that

$$\{z \in \overline{\mathbb{D}} : f_h(z) \in b\mathbb{B}^n,\ d\rho_h(z) = 0\} \subset K := \overline{\mathbb{D}} \setminus (U_0 \cup U_1 \cup U^+ \cup U^-).$$

Consider the family of functions $\rho_{th} = -tu + \rho \colon \overline{\mathbb{D}} \to \mathbb{R}$ for $t \in \mathbb{R}$. Since $\partial\rho_{th}/\partial t = -u < 0$ on $\mathbb{D}$, transversality theorem implies that for a generic choice of t, 0 is a regular value of $\rho_{th}|_{\mathbb{D}}$. By choosing $t > 0$ small enough, the map $f_{th} = e^{-th} f : \overline{\mathbb{D}} \to \mathbb{C}^2_*$ satisfies conditions (1) and (2). Replace f by f_h.

Proof of Lemma 3, #4

It remains to achieve also condition (3) in the lemma. We have

$$\{z \in \overline{\mathbb{D}} : f(z) \in E\} \subset K := \overline{\mathbb{D}} \setminus (U_0 \cup U_1 \cup U^+ \cup U^-).$$

It is elementary to see that **for every** $z \in \mathbb{D} \setminus (-1, 1)$ **there exist** $h_1, h_2 \in \mathscr{H}^+$ **such that** $h_1(z),\ h_2(z) \in \mathbb{C}$ **are** $\mathbb{R}$**-linearly independent.**

Since the set $K \subset \mathbb{D} \setminus (-1, 1)$ is compact, there are functions $h_1, \ldots, h_N \in \mathscr{H}_*^+$ such that for every point $z \in K$ the vectors $h_j(z) \in \mathbb{C}$ $(j = 1, \ldots, N)$ span $\mathbb{C}$ over $\mathbb{R}$. The corresponding family of maps

$$f_t(z) = \exp\Big(-\sum_{j=1}^{N} t_j h_j(z)\Big) f(z) \in \mathbb{C}^n, \quad z \in \overline{\mathbb{D}},\ t \in \mathbb{R}^N,$$

satisfies

$$\frac{\partial f_t(z)}{\partial t_j}\bigg|_{t=0} = -h_j(z) f(z), \quad j = 1, \ldots, l.$$

Proof of Lemma 3, #5

This means that **the spray of maps** $(z,t) \mapsto f_t(z) \in \mathbb{C}^n_*$ **for** $z \in K \subset \mathbb{D}$ **and** $t \in \mathbb{R}^N$ **is dominating in the radial direction** at $t = 0$.

Recall also that the natural projection $\sigma\colon \mathbb{C}^n_* \to \mathbb{CP}^{n-1}$ satisfies

$$\sigma \circ f_t = \sigma \circ f : \overline{\mathbb{D}} \to \mathbb{CP}^1 \text{ is an immersion.}$$

It follows that **the map**

$$\mathbb{R}^N \times K \ni (t,z) \longmapsto f_t(z) \in \mathbb{C}^n$$

is a submersion near $t = 0$ **onto a** 2**-dimensional submanifold** $W \subset \mathbb{C}^n$**.** (If $n = 2$ then this map is a submersion to $\mathbb{C}^2$.)

Since E is an arc, $E \cap W$ is a set of finite linear measure in W. Since $\dim K = 2$, the transversality theorem implies that for a generic $t \in \mathbb{R}^N$ near 0 the map $f_t\colon K \to \mathbb{C}^n_*$ misses E by dimension reasons, and hence

$$f_t(\overline{\mathbb{D}}) \cap E = \varnothing.$$

This proves Lemma 3.

Special domains in $\mathbb{C}$

Let $z = x + iy \in \mathbb{C}$. The antiholomorphic involutions

$$\tau_x(x + iy) = -x + iy, \quad \tau_y(x + iy) = x - iy$$

generate an abelian group $\Gamma = \langle \tau_x, \tau_y \rangle \cong \mathbb{Z}_2^2$.

A set $D \subset \mathbb{C}$ is said to be **Γ-invariant** if $\gamma(D) = D$ holds for all $\gamma \in \Gamma$. A map $\phi \colon D \to \mathbb{C}$ on a Γ-invariant set is said to be **Γ-equivariant** if

$$\phi = \gamma \circ \phi \circ \gamma \quad \text{holds for all} \quad \gamma \in \Gamma.$$

Definition (special domains)

A nonempty connected domain $D \subset \mathbb{C}$ is **special** if it is bounded with $\mathscr{C}^\infty$ smooth boundary, simply connected, and Γ-invariant.

If D is a special domain and $\phi \colon D \to D'$ is a biholomorphic map onto a bounded domain $D' = \phi(D)$ with smooth boundary satisfying $\phi(0) = 0$ and $\phi'(0) > 0$, then ϕ is Γ-equivariant if and only if D' is special.

A lemma on conformal maps

The following is a more precise version of a lemma proved by **Erlend Wold** and myself in **J. Math. Pures Appl. 2009.**

Lemma (4)

Assume that $D \subset \mathbb{C}$ is a special domain with the base $D \cap \mathbb{R} = (-a, a)$. Fix a number $b > a$ and set $I^+ = [a, b]$, $I^- = [-b, -a]$, $I = I^+ \cup I^-$.

Given an open neighborhood $V \subset \mathbb{C}$ of I, $\epsilon > 0$, and $\ell \in \mathbb{Z}_+$, there exists a Γ-equivariant biholomorphism $\phi \colon D \to \phi(D) = D'$ onto a special domain D' with the base $(-b, b)$ satisfying the following conditions:

(a) $\phi(0) = 0$, $\phi'(0) > 0$, $\phi(\pm a) = \pm b$,

(b) $D \subset D' \subset D \cup V$ *(hence $D \setminus V = D' \setminus V$), and*

(c) $\|\phi - \mathrm{Id}\|_{\mathscr{C}^\ell(\overline{D} \setminus V)} < \epsilon$.

Proof of the Main Lemma, #1

Recall that $S = \{x + iy \in \mathbb{C} : x \in \mathbb{R}, \ \ |y| < g(x)\}$ for some $g\colon \mathbb{R} \to (0, +\infty)$ and $f\colon S \hookrightarrow \mathbb{C}^n$ is a holomorphic embedding such that $f(S) \cap \overline{\mathbb{B}}^n = f(\mathbb{R})$ is a dense Legendrian curve in $b\mathbb{B}^n$. Hence

$$|f(x + iy)| \geq 1 + c(x)|y|^2$$

for a positive smooth function $c\colon \mathbb{R} \to (0, \infty)$. If the strip S is chosen thin enough, then $\mathrm{Area}(f(S)) = \int_S |f'|^2 dxdy$ is as small as desired.

Lemma (This is the Main Lemma we have stated before)

Let $n > 1$. Given $\epsilon > 0$, there exist a smoothly bounded, simply connected, symmetric domain $D \subset \mathbb{C}$ satisfying $\mathbb{R} \subset D \subset \overline{D} \subset S$ and a function $h = u + iv \in \mathscr{A}^1(D)$ such that the map $F = e^{-h} f\colon \overline{D} \to \mathbb{C}^n_*$ is an injective immersion transverse to $b\mathbb{B}^n$, satisfying

(α) $\mathrm{Area}(F(\Omega)) < \epsilon$ *where* $\Omega = F^{-1}(\mathbb{B}^n) \subset D$, *and*

(β) *the curve $F(b\Omega) \subset b\mathbb{B}^n$ is everywhere dense in the sphere $b\mathbb{B}^n$.*

Proof of the Main Lemma, #2

We construct an increasing sequence of special domains

$$\mathbb{D} = D_1 \subset D_2 \subset D_3 \subset \cdots D = \bigcup_{j=1}^{\infty} D_j \subset S$$

whose union D is a simply connected, smoothly bounded, Γ-invariant domain satisfying $\mathbb{R} \subset D \subset \overline{D} \subset S$.

For every $k \in \mathbb{N}$ we let $D_{k+1} = D_k \cup R_k$ be a special domain with the base $(-k-1, k+1) \subset \mathbb{R}$, furnished by Lemma 4. (Recall that R_k is a thin Γ-invariant strip around the interval $(-k-1, k+1)$.)

For each $k \geq 2$, let ψ_k be the biholomorphism

$$\psi_k \colon D_k \to D_{k-1}, \quad \psi_k(0) = 0, \quad \psi_k'(0) > 0.$$

Then ψ_k extends to a smooth Γ-equivariant diffeomorphism $\psi_k \colon \overline{D}_k \to \overline{D}_{k-1}$ satisfying $\psi_k([-k, k]) = [-k+1, k-1]$. Set

$$\Psi_1 = \mathrm{Id}|_{\overline{D}_1}, \qquad \Psi_k = \psi_2 \circ \cdots \psi_k \colon \overline{D}_k \to \overline{D}_1 \quad (k = 2, 3, \ldots).$$

Proof of the Main Lemma, #3

We also find a sequence of holomorphic multipliers

$$h_k = \tilde{h}_k \circ \Psi_k \in \mathscr{A}^\infty(D_k), \quad \tilde{h}_k \in \mathscr{H}_*^+, \quad k \in \mathbb{N}$$

such that the map

$$F_k = e^{-h_k} f \colon \overline{D}_k \hookrightarrow \mathbb{C}_*^n$$

is an embedding of class $\mathscr{A}^\infty(D_k)$ that is transverse to $b\mathbb{B}^n$, and F_{k+1} approximates F_k as closely as desired uniformly on $\overline{D}_k$ and in $\mathscr{C}^1(\overline{D}_k \setminus U_k)$, where $U_k = U_k^+ \cup U_k^-$ is a small neighborhood of $\{-k, k\}$.

In the induction step, we first apply Lemma 3 to find a small perturbation of F_k such that $F_k(\overline{D}_k)$ intersects the pair of arcs $E_k^+ = f([k, k+1]) \subset b\mathbb{B}^n$ and $E_k^- = f([-k-1, -k]) \subset b\mathbb{B}^n$ only at the points $f(\pm k)$. Lemma 4 then furnishes the next map

$$F_{k+1} = e^{-h_{k+1}} f \colon \overline{D}_{k+1} \hookrightarrow \mathbb{C}_*^n$$

which is an embedding mapping the strip $\overline{D_{k+1} \setminus D_k}$ into a small neighborhood of the arcs $E_k^+ \cup E_k^-$.

Proof of the Main Lemma, #4

The sequence h_k is chosen such that it converges on $\overline{D} = \bigcup_{j=1}^{\infty} \overline{D}_j \subset S$ to a multiplier

$$h = u + \imath v = \tilde{h} \circ \Psi \in \mathscr{A}^1(D),$$

with $\tilde{h} = \lim_{k\to\infty} \tilde{h}_k \in \mathscr{A}^\infty(\mathbb{D})$, satisfying

$$u > 0 \text{ on } \mathbb{R} \quad \text{and} \quad e^{2u} < |f|^2 \text{ on } bD.$$

If theh approximationsn are close enough, then the limit map

$$F = \lim_{k\to\infty} F_k = e^{-h} f \colon \overline{D} \to \mathbb{C}^n_*$$

is an injective immersion of class $\mathscr{A}^1(D)$ which is transverse to the sphere $b\mathbb{B}^n$. The connected component

$$\Omega \subset \{z \in D : F(z) \in \mathbb{B}^n\} = F^{-1}(\mathbb{B}^n)$$

containing $\mathbb{R}$ then satisfies the main lemma, and $F(\Omega) \subset \mathbb{B}^n$ is a properly embedded holomorphic disc satisfying the main theorem.

THANK YOU

FOR YOUR ATTENTION